An exactly solvable $\mathcal{PT}$ symmetric potential from the Natanzon class

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Abstract. The $\mathcal{PT}$ symmetric version of the generalised Ginocchio potential, a member of the general exactly solvable Natanzon potential class is analysed and its properties are compared with those of $\mathcal{PT}$ symmetric potentials from the more restricted shape-invariant class. It is found that the $\mathcal{PT}$ symmetric generalised Ginocchio potential has a number of properties in common with the latter potentials: it can be generated by an imaginary coordinate shift $x \to x + \mathrm{i}\varepsilon$; its states are characterised by the quasi-parity quantum number; the spontaneous breakdown of $\mathcal{PT}$ symmetry occurs at the same time for all the energy levels; and it has two supersymmetric partners which cease to be $\mathcal{PT}$ symmetric when the $\mathcal{PT}$ symmetry of the original potential is spontaneously broken.

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1. Introduction

Exactly solvable examples played an important role in the understanding of $\mathcal{PT}$ symmetric quantum mechanics and its unusual features. The first $\mathcal{PT}$ symmetric potentials, i.e. potentials which are invariant under the *simultaneous* action of the $\mathcal{P}$ space- and $\mathcal{T}$ time reflection operations were found numerically [1]. The most surprising result was that these one-dimensional complex potentials possessed *real* energy eigenvalues, which however, turned pairwise into complex conjugated pairs as some potential parameter was tuned. This mechanism was interpreted as the spontaneous breakdown of $\mathcal{PT}$ symmetry, since the potential remained $\mathcal{PT}$ invariant throughout, but the eigenfunctions associated with the discrete spectrum lost this property as the spectrum turned into complex gradually with the tuning of the potential parameter. Further examples have been found in numerical [2], semiclassical [3] and perturbative [4] studies, but a number of quasi-exactly solvable (QES) [5] and exactly solvable [6, 7, 8] $\mathcal{PT}$ symmetric potentials have also been identified. These latter potentials were analogues of Hermitian exactly solvable potentials with the property that their real and imaginary components were even and odd functions of the coordinate x, respectively.

More recently $\mathcal{PT}$ symmetric problems have been analysed in terms of pseudo-Hermiticity [9], and their unusual features have been interpreted in terms of this more general context. A Hamiltonian is said to be η-pseudo-Hermitian if there exists a linear, invertible, Hermitian operator for which $H^\dagger = \eta H \eta^{-1}$ holds. It has been shown that for systems with Hamiltonians of the type $H = p^2 + V(x)$ $\mathcal{PT}$ symmetry is equivalent with $\mathcal{P}$-pseudo-Hermiticity. With other choices of η further complexified Hamiltonians can be generated, which might also have real energy eigenvalues but do not fulfill $\mathcal{PT}$ symmetry [10, 11]. Furthermore, with $\eta = 1$ η-pseudo-Hermiticity reduces to conventional Hermiticity.

Here we restrict our analysis to $\mathcal{PT}$ symmetric problems, and in particular, to exactly solvable ones. A number of peculiar features of $\mathcal{PT}$ symmetric potentials became apparent only during the analysis of these potentials.

- It has been observed that several exactly solvable $\mathcal{PT}$ symmetric potentials possess *two* sets of normalisable solutions [6, 10, 12] in the sense that there can be two normalisable states with the same principal quantum number n. The second set of solutions can appear in several ways. For potentials which are singular at the origin the problem can be redefined on various trajectories of the complex x plane such that the integration path avoids the origin and the solutions remain asymptotically normalisable. (This latter feature is similar to some numerically solvable $\mathcal{PT}$ symmetric problems [1].) A special case of this scenario is obtained when an *imaginary* coordinate shift $x \to x + \mathrm{i}\varepsilon$ is employed [7]. The advantage of this scenario is that the discussion of the $\mathcal{PT}$ symmetric potential remains rather similar to that of its Hermitian counterpart, and by suitable and straightforward modification of the formalism it can also be interpreted as a conventional complex potential defined on the x axis. This imaginary coordinate shift can be employed [7] to all the shape-invariant [13] potentials (e.g. the $\mathcal{PT}$ symmetric harmonic oscillator [6] or the generalised Pöschl–Teller potential [14]) with the exception of the $\mathcal{PT}$ symmetric Morse [15] and Coulomb [16] potentials. The cancellation of the singularity then regularizes the solution which would be irregular at the origin in the Hermitian setting. A different mechanism appears for potentials which are not singular in their Hermitian version, such as the $\mathcal{PT}$ symmetric Scarf II potential, which is defined on the full x axis. In this case the second set of normalisable solutions originates from states which have complex eigen-energy in the Hermitian case, but which turn into normalisable states with real energy when the potential is forced to become $\mathcal{PT}$ symmetric and the $\mathcal{PT}$ symmetry is not broken spontaneously [12, 17, 14, 18]. The two set of solutions are distinguished by the quasi-parity quantum number [19].

- In the process of generating the spontaneous breakdown of $\mathcal{PT}$ symmetry by tuning the potential parameters it was found that the pairwise merging of the energy eigevalues and their re-emergence as complex conjugated pairs occurs at the *same* value of the potential parameter [20, 17]. In other words, the spontaneous breakdown of $\mathcal{PT}$ symmetry is realised suddenly in the case of shape-invariant potentials, as opposed to a gradual process observed in the case of numerical examples [1, 21].
- For certain exactly solvable (and shape-invariant) examples, such as the $\mathcal{PT}$ symmetric harmonic oscillator [22] and the Scarf II potential [23] it was found that there are *two* 'fermionic' SUSY partners of the original 'bosonic' potential, and they are distinguished by the quasi-parity quantum number carried by the 'bosonic' bound states. (In the latter case this has also been found in other realisations of SUSYQM [24].) This doubling of the partner potentials is an obvious consequence of the fact that there are *two* nodeless normalisable solutions corresponding to the 'ground state' in the two segments of the spectrum with quasi-parity $q = +1$ and -1. Furthermore, it was also established that the 'fermionic' partner potentials are $\mathcal{PT}$ symmetric themselves too, in case the 'bosonic' potential has unbroken $\mathcal{PT}$ symmetry, while they cease to be $\mathcal{PT}$ symmetric if the $\mathcal{PT}$ symmetry of the 'bosonic' potential is spontaneously broken.

The peculiar features mentioned above have been observed until now for the $\mathcal{PT}$ symmetric version of shape-invariant potentials, while potentials beyond this class often behaved in a different way. This naturally raises the question whether these features also characterise non-shape-invariant, but exactly solvable examples. Natural candidates for this analysis are Natanzon-class potentials [25] which have the property that their bound-state solutions are written in terms of a *single* hypergeometric or confluent hypergeometric function. This potential class depends on six parameters, but it is prohibitively complicated in its general form, so its subclasses with two to four parameters have been analysed in detail until now [26, 27, 28, 29, 30, 31].

Perhaps the most well-known member of the Natanzon-class is the Ginocchio potential, which has a one-dimesional version defined on the x axis [26] and a radial one, which has an r^{-2}-like singularity at the origin [27]. An important feature of this potential is that for a special choice of a potential parameter it reduces to a shape-invariant potential, namely to the Pöschl–Teller hole (in one dimension) and to the generalised Pöschl–Teller potential (in the radial case). The one-dimensional version of the Ginocchio potential has been analysed in an algebraic framework, and an su(1,1) algebra has been associated with it [32], the discrete non-unitary irreducible representations of which correspond to resonances in the transmission coefficients. (These states have been identified as quasi-bound states in an independent study [33].) Furthermore, it was also shown that this algebra reduces in two different shape-invariant limits to an su(1,1) potential algebra and to an su(2) spectrum generating algebra [34]. The phase-equivalent supersymmetric partners of the generalised Ginocchio potential have also been derived in a completely analytic form [35].

Although the generalised Ginocchio potential is an 'implicit' potential, i.e. the $z(r)$ function which is used to transform the Schrödinger equation into the differential equation of the hypergeometric function $F(a, b; c; z)$ is known only in an implicit form as $r(z)$, nevertheless, similarly to a number of other 'implicit' potentials [29], this does not restrict the applicability of the formulae, because $V(r)$ and the wavefunctions can be determined to any desired accuracy, and all the calculations involving these quantities (matrix elements, etc.) can be evaluated analytically [35]. Furthermore, we shall see that the imaginary coordinate shift which is essential to impose $\mathcal{PT}$ symmetry on the generalised Ginocchio potential can also be implemented without complications.

We note that a Natanzon-class potential, the so-called DKV potential has already been analysed in the $\mathcal{PT}$ symmetric setting by the point canonical transformation of a shape-invariant potential [36], but it was found that it has to be defined on a curved integration path. Nevertheless, it also showed similarities with $\mathcal{PT}$ symmetric shape-invariant potentials, as its spectrum was also richer that that of its Hermitian counterpart.

In section 2 we present the Hermitian version of the generalised Ginocchio potential for reference, and in section 3 we construct its $\mathcal{PT}$ symmetric version. Section 4 deals with the supersymmetric aspects of this potential, while in section 5 a summary of the results is presented.

2. The generalised Ginocchio potential

The first version of the Ginocchio potential was introduced as a one-dimensional quantum mechanical problem which is symmetric with respect to the $x \to -x$ transformation [26]. Later it was generalised to a radial problem [27], which also contains an r^{-2}-like singular term at the origin. (This latter version of the potential also allows a particular functional form of an effective mass, but it can be reduced to a constant value by setting one of the parameters (a) to zero.) Following the notation of Ref. [35]

we define the generalised Ginocchio potential as

$$V(r) = -\frac{\gamma^4(s(s+1)+1-\gamma^2)}{\gamma^2+\sinh^2 u} + \gamma^4\lambda(\lambda-1)\frac{\coth^2 u}{\gamma^2+\sinh^2 u} - \frac{3\gamma^4(\gamma^2-1)(3\gamma^2-1)}{4(\gamma^2+\sinh^2 u)^2} + \frac{5\gamma^6(\gamma^2-1)^2}{4(\gamma^2+\sinh^2 u)^3}\ , \qquad (1)$$

where we changed the notation of Ref. [27] to make it more suitable for our purposes. This form can be obtained from the original formulae by setting $a=0$, $\alpha_l=\lambda-\frac{1}{2}$, $\nu_l=s$, $\beta_{nl}=\mu$, $\lambda=\gamma$ and $y=\sinh u(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}$.

The (generalised) Ginocchio potential is an example for 'implicit' potentials, because it is expressed in terms of a function $u(r)$ which is known only in the implicit $r(u)$ form:

$$r = \frac{1}{\gamma^2}\Big[\tanh^{-1}\Big((\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\Big) + (\gamma^2-1)^{\frac{1}{2}}\tan^{-1}\Big((\gamma^2-1)^{\frac{1}{2}}(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\Big)\Big]\ . \qquad (2)$$

r can take values from the positive half axis, which is mapped by the monotonously increasing implicit $u(r)$ function onto itself. This function is, actually, the solution of an ordinary first-order differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\gamma^2\cosh u}{(\gamma^2+\sinh^2 u)^{\frac{1}{2}}} \qquad (3)$$

defining a variable transformation connecting the Schrödinger equation with the differential equation of the Jacobi (and Gegenbauer) polynomials [37]. It can be seen from Eqs. (2) and (3) that $u(r)$ behaves approximately as γr near the origin, and as $\gamma^2 r$ for large values of r. In the $\gamma\to 1$ limit u becomes identical with r, and (1) reduces to the generalised Pöschl–Teller potential.

Bound states are located at

$$E_n = -\gamma^4\mu_n^2\ , \qquad (4)$$

where n varies from 0 to $n_{\max}$ defined below and

$$\mu_n = \frac{1}{\gamma^2}\Big[-\Big(2n+\lambda+\frac{1}{2}\Big) + \Big[\Big(2n+\lambda+\frac{1}{2}\Big)^2(1-\gamma^2)+\gamma^2\Big(s+\frac{1}{2}\Big)^2\Big]^{\frac{1}{2}}\Big]\ . \qquad (5)$$

All the terms in (1) are finite at the origin, with the exception of the last one, which shows r^{-2}-like singularity there, and can be considered either as an approximation of the centrifugal term with $l=\lambda-1$ (λ integer), or as a part of a singular potential with arbitrary $l\neq\lambda-1$. Setting $\lambda=1$ or 0 we get the 'simple' Ginocchio potential [26] defined on the line.

The bound-state wavefunctions are expressed in terms of Jacobi polynomials

$$\psi_n(r) = \mathcal{N}_n(\gamma^2+\sinh^2 u)^{\frac{1}{4}}(\sinh u)^{\lambda}(\cosh u)^{-\mu_n-\lambda-\frac{1}{2}}P_n^{(\mu_n,\lambda-\frac{1}{2})}(2\tanh^2 u-1) \qquad (6)$$

which reduce to Gegenbauer polynomials [37] for $\lambda=1$. The normalisation is given by

$$\mathcal{N}_n = \Big[\frac{2\gamma^2 n!\,\Gamma(\mu_n+\lambda+n+\frac{1}{2})\mu_n(\mu_n+\lambda+2n+\frac{1}{2})}{\Gamma(\mu_n+n+1)\Gamma(\lambda+n+\frac{1}{2})(\mu_n\gamma^2+\lambda+2n+\frac{1}{2})}\Big]^{\frac{1}{2}}. \qquad (7)$$

Considering that the $r \to \infty$ asymptotical limit corresponds to $u \to \infty$ (see Eq. (2)), the wavefunctions become zero asymptotically if $\mu_n > 0$ holds. Applying this condition to Eq. (5) we find that the number of bound states is set by $n_{\max} < \frac{1}{2}(s - \lambda)$.

3. $\mathcal{PT}$ symmetrisation of the generalised Ginocchio potential

The first step in the $\mathcal{PT}$ symmetrisation of the generalised Ginocchio potential is performing the imaginary coordinate shift which allows its extension to the full x axis by cancelling the singularity at the origin. This imaginary coordinate shift is a constant of integration from (3), and it modifies (2) such that $r \to x + \mathrm{i}\varepsilon$. Here we also switched to x instead of r to indicate that the original radial potential is extended also to the negative x axis, following the standard treatment of $\mathcal{PT}$ symmetric potentials. Similarly to the Hermitian case, the variable transformation is determined by an implicit formula,

$$x + \mathrm{i}\varepsilon = \frac{1}{\gamma^2}\Big[\tanh^{-1}\left((\gamma^2 + \sinh^2 u)^{-\frac{1}{2}} \sinh u\right) + (\gamma^2 - 1)^{\frac{1}{2}} \tan^{-1}\left((\gamma^2 - 1)^{\frac{1}{2}}(\gamma^2 + \sinh^2 u)^{-\frac{1}{2}} \sinh u\right)\Big] \quad (8)$$

however, now u takes on complex values. Figure 1 shows the $u(x)$ function for a particular value of γ and ε. This function varies smoothly and it is odd under the $\mathcal{PT}$ transformation: $\mathcal{PT}u(x) = -u(x)$, i.e. its real and imaginary components are odd and even functions of x, respectively. Asymptotically the relation $u(x) \to_{x\to\pm\infty} \gamma^2(x + \mathrm{i}\varepsilon)$ holds, while near $x = 0$ there is a 'kink' in both the real and the imaginary component of $u(x)$.

The $\mathcal{PT}$ transform of $\sinh u(x{+}\mathrm{i}\varepsilon)$ is $-\sinh u(x{+}\mathrm{i}\varepsilon)$ (which can be seen analytically too by series expansion), and this also determines the $\mathcal{PT}$ transform of the potential (1). In particular, we may note that it contains $\sinh u$ everywhere as $\sinh^2 u$ (including also the term with $\coth^2$), so (1) is $\mathcal{PT}$ symmetric if all the coupling coefficients are *real*. This restricts γ^2, $s(s+1)$ and $\lambda(\lambda-1)$ to real values. The latter two requirements allow the following values of s and λ:

$$s = \begin{cases} \text{real} & \text{when } s(s+1) \geq -\frac{1}{4} \\ -\dfrac{1}{2} + \mathrm{i}\sigma & \text{when } s(s+1) \leq -\frac{1}{4} \end{cases} \qquad \lambda = \begin{cases} \text{real} & \text{when } \lambda(\lambda-1) \geq -\frac{1}{4} \\ \dfrac{1}{2} + \mathrm{i}l & \text{when } \lambda(\lambda-1) \leq -\frac{1}{4} \end{cases} \quad (9)$$

We are going to discuss these possibilities later.

Let us now analyse the two independent solutions of the generalised Ginocchio potential, written in the form hypergeometric functions. From among the possible linear combinations we chose the ones which have different behaviour at the origin in the $\varepsilon \to 0$ limit:

$$\psi_1(x) \sim (\gamma^2{+}\sinh^2 u)^{1/4}(\cosh u)^{a-b}(\sinh u)^{a+b-c+\frac{1}{2}} F(a, a{-}c{+}1; a{+}b{-}c{+}1; -\sinh^2 u)\,, \quad (10)$$

$$\psi_2(x) \sim (\gamma^2{+}\sinh^2 u)^{1/4}(\cosh u)^{a-b}(\sinh u)^{c-a-b+\frac{1}{2}} F(1{-}b, c{-}b; c{-}a{-}b{+}1; -\sinh^2 u)\,, \quad (11)$$

where a, b and c have to satisfy the following relations:

$$c = 1 \pm \mu \qquad a + b - c = \pm(\lambda - \frac{1}{2}) \qquad a - b = \pm\left[(s + \frac{1}{2})^2 - (\gamma^2 - 1)\mu^2\right]^{1/2} \equiv \pm\omega\ . \quad (12)$$

With the conditions (12) the two independent solutions can be written as

$$\psi_1(x) \sim (\gamma^2 + \sinh^2 u)^{1/4} (\cosh u)^{\pm\omega} (\sinh u)^{\lambda} \times F(\frac{1}{2}(\mu + \lambda + \frac{1}{2} \pm \omega), \frac{1}{2}(-\mu + \lambda + \frac{1}{2} \pm \omega); \lambda + \frac{1}{2}; -\sinh^2 u) \ , \tag{13}$$

$$\psi_2(x) \sim (\gamma^2 + \sinh^2 u)^{1/4} (\cosh u)^{\pm\omega} (\sinh u)^{1-\lambda} \times F(\frac{1}{2}(-\mu - \lambda + \frac{3}{2} \pm \omega), \frac{1}{2}(\mu - \lambda + \frac{3}{2} \pm \omega); \frac{3}{2} - \lambda; -\sinh^2 u) \ . \tag{14}$$

Note that the same two functions are obtained irrespective of the signs chosen in the first two equations in (12), while the sign of ω remains to be determined from the normalisability conditions of the wavefunctions.

Up to this point the functions (13) and (14) supply the general solutions for the energy eigenvalue $E = -\gamma^4 \mu^2$. In order to obtain solutions belonging to discrete energy eigenvalues one has to set one of the first two arguments of the hypergeometric functions to the non-positive integer value $-n$, reducing them to Jacobi polynomials [37]. We find that in contrast with the Hermitian case, normalisable solutions can be obtained in *two* different ways, corresponding to the condition

$$2n + 1 + \mu_{nq} + q(\lambda - \frac{1}{2}) - \omega = 0 \ , \tag{15}$$

where $q = 1$ and $q = -1$ holds for (13) and (14), respectively. In this case the two solutions can be written in a compact form as

$$\psi_{nq}(x) \sim (\gamma^2 + \sinh^2 u)^{1/4} (\cosh u)^{-2n-1-\mu_{nq}-q(\lambda-\frac{1}{2})} (\sinh u)^{\frac{1}{2}+q(\lambda-\frac{1}{2})} \times P_n^{(q(\lambda-\frac{1}{2}), -2n-1-\mu_{nq}-q(\lambda-\frac{1}{2}))}(\cosh(2u)) \ . \tag{16}$$

Here n is the principal quantum number labelling the bound states and q is the quasi-parity $q = \pm 1$ [19]. This quantum number characterises the solutions of $\mathcal{PT}$ symmetric potentials, but the potential itself does not depend on it. Its name originates from the analysis of the $\mathcal{PT}$ symmetric version of the one-dimensional harmonic oscillator: in the Hermitian limit of this potential (i.e. for $\varepsilon \to 0$ it essentially reduces to the parity quantum number.

We see that the two solutions are distinguished by the $q = \pm 1$ quasi-parity quantum number, similarly to potentials belonging to the shape-invariant class. Actually, the corresponding solutions of the $\mathcal{PT}$ symmetric generalised Pöschl–Teller potential [14] can be obtained by setting $\gamma = 1$. It is also obvious that normalisability requires $\mathrm{Re}(\mu_{nq}) > 0$.

In order to obtain explicit expression for μ_{nq} one has to combine (15) with (12) and to solve a quadratic algebraic equation for $\mu = \mu_{nq}$:

$$\mu_{nq} = \frac{1}{\gamma^2} \left[-\left(2n + 1 + q(\lambda - \frac{1}{2})\right) + \left[\gamma^2 (s + \frac{1}{2})^2 + (1 - \gamma^2) \left(2n + 1 + q(\lambda - \frac{1}{2})\right)^2 \right]^{1/2} \right] \tag{17}$$

This expression recovers the corresponding formula for the Hermitian generalised Ginocchio potential for $q = 1$.

Similarly to the case of the Hermitian version of the generalised Ginocchio potential the energy eigenvalues are written as $E_{nq} = -\gamma^4\mu_{nq}^2$, and they are independent of ε. Actually, we find that for the $q = 1$ choice the expressions for the Hermitian problem are recovered formally. However, the forthcoming analysis will show that despite the similar form, some quantities can be chosen complex for the $\mathcal{PT}$ symmetric case. Before going on we may note that μ_{nq}, and consequently E_{nq} depends on the $2n+1+q(\lambda-\frac{1}{2})$ combination, and this leads to a degeneracy between levels with $q = 1$ and $q = -1$ whenever λ is a real half-integer number. Furthermore, for $\lambda = \frac{1}{2}$ states with opposite quasi-parity and with the same n become degenerate: in fact, this is the point where the spontaneous breakdown of $\mathcal{PT}$ symmetry sets in if the λ parameter is continued to complex values allowed by (9).

Let us now analyse the conditions for having real and complex energy eigenvalues E_{nq}, which corresponds to inspecting the nature of μ_{nq} (17) in terms of the allowed values of s and λ displayed in (9). These also have to be combined with the condition $\mathrm{Re}(\mu_{nq}) > 0$ which guarantees normalisability of the solutions (16). The key element of the analysis is the term containing the square root in (17), so it is useful to inspect separately the cases when it is real, imaginary or complex, which corresponds to $A \geq 0$, $A < 0$ and complex A, where

$$A \equiv \gamma^2(s+\frac{1}{2})^2 + (1-\gamma^2)\left(2n+1+q(\lambda-\frac{1}{2})\right)^2 \ . \tag{18}$$

We restrict our analysis to $\gamma^2 > 1$: the alternative choice, $\gamma^2 < 1$ would change the nature of the $r(u)$ function in (2). We can note that λ occurs everywhere in the combination $q(\lambda-\frac{1}{2})$, so when λ is real, we can assume that $\lambda \geq \frac{1}{2}$, because the $\lambda \leq \frac{1}{2}$ cases can be obtained simply by switching $q = +1$ to $q = -1$. Also, when $\lambda = \frac{1}{2} + \mathrm{i}l$, it is enough to assume $l > 0$ for the same reason.

- $A \geq 0$. This can happen only if λ is real, while from $A \geq 0$ in (18) and $\gamma^2 > 1$ it follows that s also has to be real. Inspecting the allowed values of n for various parameter domains we find the following. Normalizable states can be obtained for μ_{nq} in (17) when
$$-\frac{1}{2}\left(1+q(\lambda-\frac{1}{2})+\left(\frac{\gamma^2}{\gamma^2-1}\right)^{1/2}|s+\frac{1}{2}|\right) \leq n \leq -\frac{1}{2}\left(1+q(\lambda-\frac{1}{2})-|s+\frac{1}{2}|\right) \tag{19}$$
holds. If the upper boundary of this domain is negative, then there are no normalizable solutions. This depends on the relative magnitude of s and λ.
- $A < 0$. Here again λ has to be real, while s can take both real and complex values allowed in (9). The $\mathrm{Re}(\mu_{nq}) > 0$ condition now reduces to $2n+1+q(\lambda-\frac{1}{2}) < 0$, which has to be combined with $A < 0$. The resulting condition is then
$$n \leq -\frac{1}{2}\left(1+q(\lambda-\frac{1}{2})+\left(\frac{\gamma^2}{\gamma^2-1}\right)^{1/2}|s+\frac{1}{2}|\right) \tag{20}$$
for real values of s and
$$n \leq -\frac{1}{2}\left(1+q(\lambda-\frac{1}{2})\right) \tag{21}$$

for $s=-\frac{1}{2}+\mathrm{i}\sigma$. Note that these conditions can be met only for $q=-1$ if λ is large enough (and positive, as we assumed before).

- A is complex. For this $\lambda=\frac{1}{2}+\mathrm{i}l$ is required, while s can be both real and $s=-\frac{1}{2}+\mathrm{i}\sigma$. This situation corresponds to the spontaneous breakdown of $\mathcal{PT}$ symmetry, and the energy eigenvalues appear in complex conjugated pairs due to $(\mu_{nq})^*=\mu_{n-q}$, which leads to $(E_{nq})^*=E_{n-q}$. At the same time the $\mathrm{Re}(\mu_{nq})>0$ condition turns out to be the same for $q=+1$ and -1, in accordance with the expectation that the number of normalisable states has to be the same for both quasi-parities. The detailed analysis is more complicated for complex values of A (and λ) than for real A, so we can resort only to numerical calculations in this respect. The outcome depends on the relative magnitude of $|s+\frac{1}{2}|$ (which is $|\sigma|$ for complex values of s) and l.

We can now address the question whether with the tuning of the potential parameters the spontaneous breakdown of $\mathcal{PT}$ symmetry (i.e. the appearance of complex conjugate pairs of eigenvalues) happens at the same time for *all* the bound states as in the case of shape-invariant potentials [20], or gradually, as for some non-shape-invariant potentials, such as the $\mathcal{PT}$ symmetric square well [21]. From the analysis above we find that when this mechanism is realised via setting λ to the complex value $\lambda=\frac{1}{2}+\mathrm{i}l$, then all the energy eigenvalues turn to complex at the same time. The spectrum can also be changed to complex by tuning s from real to imaginary values and keeping λ real. Since s is contained in the formulae in the combination $s(s+1)$ or $(s+\frac{1}{2})^2=s(s+1)+\frac{1}{4}$ which is always real, this possibility is more limited. In this case again all the energy eigenvalues turn into complex at the same time, but the character of the potential also changes, as its leading term in (1) changes sign. In the Hermitian setting this would correspond to replacing the potential well with a barrier, which obviously changes the nature of the problem. Although in the $\mathcal{PT}$ symmetric version of (1) the situation is less transparent, the complexification of the spectrum via tuning s to complex values and keeping λ real is clearly different from the situation when s is kept real and the spontaneous breakdown of $\mathcal{PT}$ symmetry is induced by tuning λ to complex values.

In Figures 2, 3 and 4 the real and imaginary components of (1) are plotted for fixed values of ε, γ and s and for various values of λ corresponding to unbroken and spontaneously broken $\mathcal{PT}$ symmetry. The position of the energy eigenvalues are also indicated.

A similar analysis can be performed for $\gamma^2<1$ too. In this case the $s=-\frac{1}{2}+\mathrm{i}\sigma$ choice plays a more important role, but otherwise the results are qualitatively the same. Note that in this case $r(u)$ in (2) changes, and this also modifies the nature of the potential.

Before closing this section we mention briefly some aspects of the one-dimesional version of the Ginocchio potential [26] which is obtained from (1) by the $\lambda=0$ or 1 substitution. This limit is analogous to the one-dimensional version of the harmonic oscillator, which is also obtained from the radial harmonic oscillator after cancelling the singular centrifugal term, allowing the extension of the potential to the full x axis. Another similarity between the two systems is that the two choices of λ correspond to the even and odd solutions, and this can clearly be seen from the structure of the bound-state solutions (6), in which the Jacobi polynomial reduces to an even and odd Gegenbauer polynomial for $\lambda = 0$ and 1, respectively [37]. Losing the λ parameter means that in the case of the $\mathcal{PT}$ symmetric one-dimensional Ginocchio potential the spontaneous breakdown of $\mathcal{PT}$ symmetry cannot be implemented as in the general case. It is also interesting to note that the quasi-parity quantum number occurs only in the combination $q(\lambda - \frac{1}{2})$, which means that the $q = +1$, $\lambda = 0$ combination is equivalent with $q = -1$, $\lambda = 1$, and $q = +1$, $\lambda = 1$ is equivalent with $q = -1$, $\lambda = 0$, and this reflects the relation of the quasi-parity quantum number with ordinary parity, similarly to the case of the one-dimensional harmonic oscillator [19]. It is also worthwhile to note that the Hermitian version of the one-dimensional Ginocchio potential possesses a number of complex-energy solutions (resonances) [26], and since the energy eigenvalues are not sensitive to the ε parameter appearing in the complex coordinate shift in (8), these remain unchanged after the $\mathcal{PT}$ symmetrisation of the potential. However, these are unbound solutions, so their character is different from that of the (normalisable) complex-energy solutions which appear when the $\mathcal{PT}$ symmetry is spontaneously broken.

4. Supersymmetric aspects of the $\mathcal{PT}$ symmetric generalised Ginocchio potential

According to Ref. [23] the supersymmetric partner of a $\mathcal{PT}$ symmetric potential depends on the quasi-parity q, and thus corresponds to *two distinct* potentials. For this, the partner potentials have to be constructed by using a q-dependent factorization energy such that [23]

$$V_{\pm}^{(q)}(x) = U_{\pm}^{(q)}(x) + \epsilon^{(q)} \equiv [W^{(q)}(x)]^2 \pm \frac{\mathrm{d}W^{(q)}}{\mathrm{d}x} + \epsilon^{(q)} \ , \tag{22}$$

where $\epsilon^{(q)} = E_{0,-}^{(q)}$ is the ground-state energy of the 'bosonic' potential , which, due to this construction is independent from q, i.e. $V_{-}^{(q)}(x) = V(x)$. The superpotential $W^{(q)}(x)$ is expressed in terms of the ground-state ($n = 0$) wavefunction of $V(x)$ (1)

$$\begin{aligned} W^{(q)}(x) = & -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_{0q}(x) \\ = & \frac{\gamma^2(\gamma^2-1)\sinh u}{2(\gamma^2+\sinh^2 u)^{3/2}} + \frac{\gamma^2 \mu_{0q} \sinh u}{(\gamma^2+\sinh^2 u)^{1/2}} - \frac{\gamma^2(\frac{1}{2}+q(\lambda-\frac{1}{2}))}{(\gamma^2+\sinh^2 u)^{1/2}\sinh u} \ , \end{aligned} \tag{23}$$

and it clearly depends on q explicitly (in third term) and implicitly via μ_{0q} (in the second term). The 'fermionic' partner potentials of the generalised Ginocchio potential contain the same terms as (1), but the coupling coefficients are different, and pick up q-dependence, as expected [23]:

$$V_{+}^{(q)}(x) = -\frac{A_q}{\gamma^2+\sinh^2 u} + B_q \frac{\coth^2 u}{\gamma^2+\sinh^2 u} + \frac{C_q}{(\gamma^2+\sinh^2 u)^2} - \frac{7\gamma^6(\gamma^2-1)^2}{4(\gamma^2+\sinh^2 u)^3} \ , \tag{24}$$

$$A_q = \gamma^4[s(s+1)+\gamma^2-2-2\gamma^2\mu_{0q}-q(2\lambda-1)]\ , \tag{25}$$

$$B_q = \gamma^4[\lambda(\lambda-1)+1+q(2\lambda-1)]\ , \tag{26}$$

$$C_q = \gamma^4(\gamma^2-1)\left(\frac{11\gamma^2-9}{4}-2\gamma^2\mu_{0q}-q(2\lambda-1)\right) \tag{27}$$

These partner potentials are $\mathcal{PT}$ symmetric if μ_{0q} (and thus λ too) are real, which, under rather general conditions, coincides with the requirement of the (unbroken) $\mathcal{PT}$ symmetry of (1) itself. This was the case for some $\mathcal{PT}$ symmetric shape-invariant potentials too [23, 22]. When $\lambda = \frac{1}{2} + \mathrm{i}l$, which happens when the $\mathcal{PT}$ symmetry of (1) is spontaneously broken, (24) ceases to be $\mathcal{PT}$ symmetric, which is again a result similar to those obtained for shape-invariant potentials [23, 22].

Figures 2, 3 and 4 display also the 'fermionic' partners of the respective 'bosonic' potentials. Due to the SUSYQM construction the energy eigenvalues of the 'fermionic' partners are the same with the exception that the levels with $n = 0$ and $q = \pm 1$ are missing from the spectrum of $V_+^{(\pm 1)}(x)$. The example in figure 4 corresponds to the spontaneous breakdown of the $\mathcal{PT}$ symmetry of the 'bosonic' potential, and thus the $\mathcal{PT}$ symmetry of the 'fermionic' potentials is manifestly broken. This is indicated by the fact that the real and imaginary component of the potential ceases to have definite parity under space reflexion. However, the two 'fermionic' potentials are the $\mathcal{PT}$ transforms of each other: $V_+^{(+1)}(x) = [V_+^{(-1)}(-x)]^*$. Note that for $\lambda = \frac{1}{2}$, i.e. for the point of the spontaneous breakdown of $\mathcal{PT}$ symmetry the two 'fermionic' partners with $q = +1$ and $q = -1$ coincide.

5. Summary and conlcusions

We analysed a Natanzon-class potential, the generalised Ginocchio potential in a $\mathcal{PT}$ symmetric setting in order to explore similarities and differences with the more restricted shape-invariant potential class. This work was inspired by the fact that up to now the exactly solvable $\mathcal{PT}$ symmetric potentials were almost exclusively members of the shape-invariant class, and they showed marked differences compared to examples outside this class, e.g. those which have been solved numerically or had quasi-exactly solvable character. Our analysis showed that the $\mathcal{PT}$ symmetric generalised Ginocchio potential shares all the specific properties of shape-invariant potentials. In particular, its states can also be characterised by the quasi-parity quantum number, and the spontaneous breakdown of its $\mathcal{PT}$ symmetry takes place suddenly, i.e. by tuning a potential parameter (λ) all its real energy eigenvalues turn into complex conjugate pairs at the same value of this parameter. These results seem to originate from the 'robust' structure of the normalisable solutions of Natanzon-class potentials, which allows the implementation of $\mathcal{PT}$ symmetry to these technically non-trivial problems. Another similarity with the shape-invariant potentials is that the $\mathcal{PT}$ symmetric generalised Ginocchio potential has two 'fermionic' supersymmetric partners (generated by eliminating the lowest state of the original 'bosonic' potential with quasi-parity

$q = +1$ and -1), and the partner potentials also possess $\mathcal{PT}$ symmetry as long as the $\mathcal{PT}$ symmetry of the 'bosonic' potential is unbroken, but they cease to be $\mathcal{PT}$ symmetric when the $\mathcal{PT}$ symmetry of the 'bosonic' potential is spontaneously broken. This seems to indicate that the properties thought to be specific to $\mathcal{PT}$ symmetric shape-invariant potentials might be valid to the much larger Natanzon potential class too, and perhaps also beyond that. Further studies should be made to check the validity of this conjecture.

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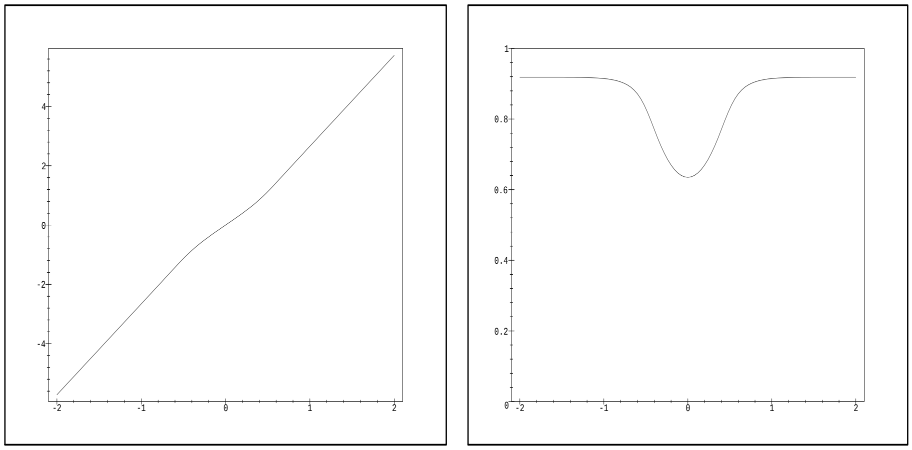

Figure 1. The real (left panel) and imaginary (right panel) component of the $u(x)$ function for $\gamma = 1.75$ and $\varepsilon = 0.3$. Note the different vertical scales.

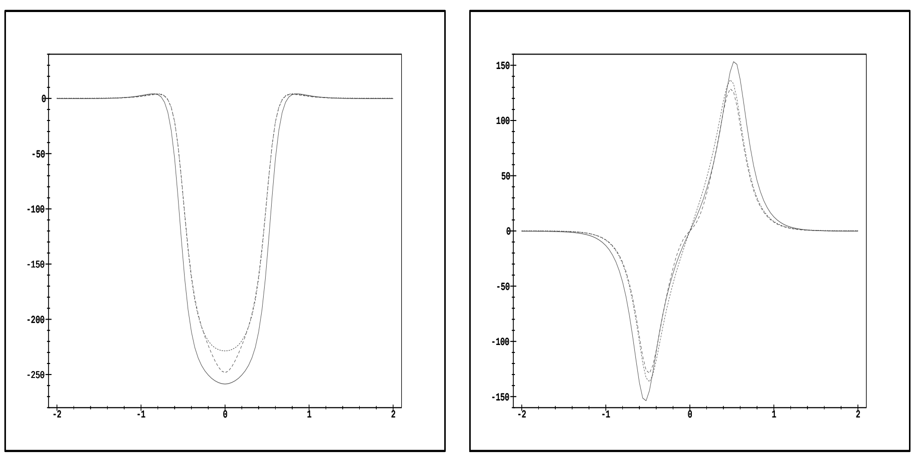

Figure 2. The real (left panel) and imaginary (right panel) component of the potential (1) for $\varepsilon = 0.3$, $\gamma = 1.75$, $s = 8.1$ and $\lambda = 1.25$ (solid line) and its supersymmetric partners $V_+^{(+1)}(x)$ (dashed line) and $V_+^{(-1)}(x)$ (dotted line) in (24). Normalisable states of (1) are found at $E_{0\ +1} = -171.313$, $E_{1\ +1} = -106.160$, $E_{2\ +1} = -46.679$, $E_{3\ +1} = -5.666$; $E_{0\ -1} = -218.913$,$E_{1\ -1} = -154.978$, $E_{2\ -1} = -90.379$, $E_{3\ -1} = -33.993$ and $E_{4\ -1} = -1.061$. The spectrum of $V_+^{(q)}(x)$ is the same, with the exception of the $E_{0\ q}$ level, which is missing from its spectrum.

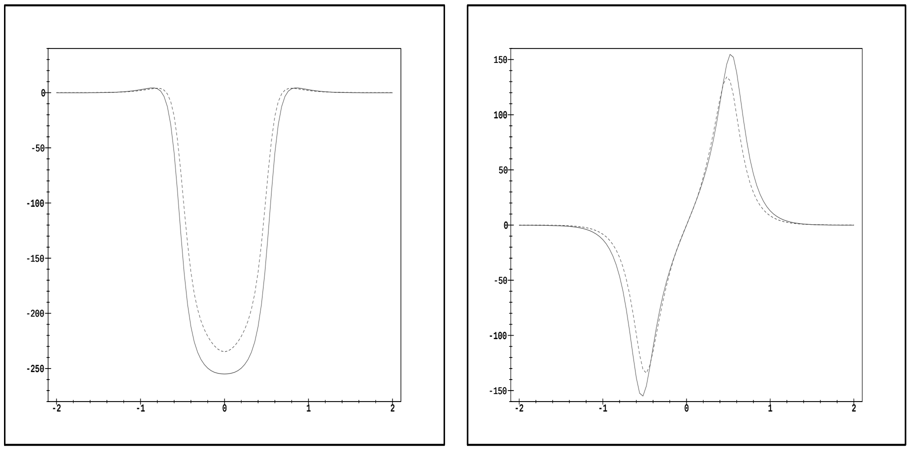

Figure 3. The same as Figure 2 with $\lambda = 0.5$. Normalisable states are found at $E_{0\ +1} = E_{0\ -1} = -195.477$, $E_{1\ +1} = E_{1\ -1} = -130.419$, $E_{2\ +1} = E_{2\ -1} = -67.675$ and $E_{3\ +1} = E_{3\ -1} = -17.640$. The supersymmetric partners $V_+^{(+1)}(x)$ and $V_+^{(-1)}(x)$ coincide in this case.

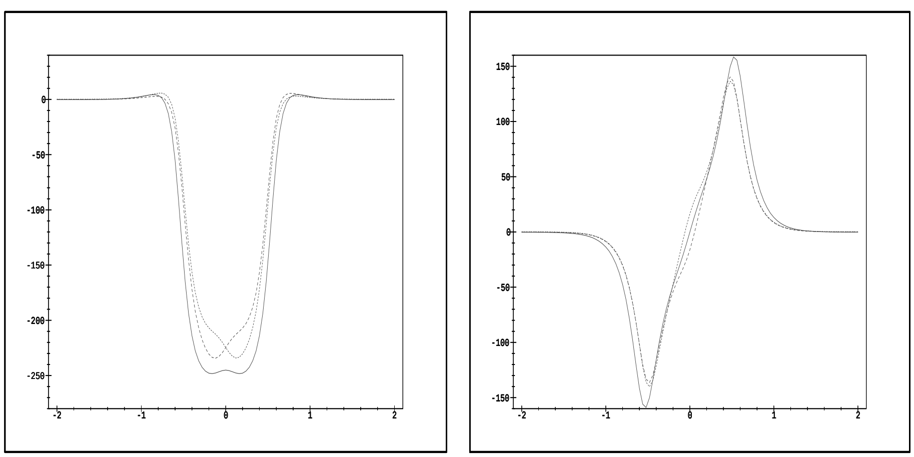

Figure 4. The same as Figure 2 with $\lambda = 0.5 + 1.25\mathrm{i}$ corresponding to spontaneously broken $\mathcal{PT}$ symmetry. Normalisable states are found at $E_{0\ +1} = (E_{0\ -1})^* = -196.494 + \mathrm{i}\ 40.038$, $E_{1\ +1} = (E_{1\ -1})^* = -130.023 + \mathrm{i}\ 41.130$, $E_{2\ +1} = (E_{2\ -1})^* = -65.367 + \mathrm{i}\ 37.105$ and $E_{3\ +1} = (E_{3\ -1})^* = -11.833 + \mathrm{i}\ 25.161$. The supersymmetric partners $V_+^{(+1)}(x)$ and $V_+^{(-1)}(x)$ cease to be $\mathcal{PT}$ symmetric in this case.