A perturbative treatment of a generalized $\mathcal{PT}$ -Symmetric Quartic Anharmonic Oscillator

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ABSTRACT

We examine a generalized $\mathcal{PT}$ -symmetric quartic anharmonic oscillator model to determine the various physical variables perturbatively in powers of a small quantity ε.We make use of the Bender-Dunne operator basis elements and exploit the properties of the toatally symmetric operator $T_{m,n}$.

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1. Introduction:

During recent years $\mathcal{PT}$ -symmetric quantum mechanics has emerged as an area of high theoretical interest (e.g.,[1-10] and references therein). For one thing, $\mathcal{PT}$ -symmetry is a weaker condition compared to the usual Hermiticity but exhibits all the essential properties of a Hermitian quantum Hamiltonian. For another, $\mathcal{PT}$ -symmetry opens up the window to the non-Hermitian world, thus enabling one to address a much broader class of Hamiltonians.

Although the current interest in $\mathcal{PT}$ -symmetry stems from the 1998 seminal paper of Bender and Boettcher [1] where it was shown that for a certain class of $\mathcal{PT}$ -symmetric Hamiltonians the spectrum remained entirely real, discrete and bounded below, the concept of $\mathcal{PT}$ -symmetry had its roots in some earlier independent works as well. These include the ones of Caliceti et al [11] and Bessis and Zinn-Justin who studied a cubic anharmonic oscillator model with an imaginery coupling and that of Buslaev and Greechi [12] who analysed the spectra of certain non-Hermitian versions of the quartic anharmonic oscillator.

Recently Mostafazadeh [13] has revisited the question of observables for the $\mathcal{PT}$ -symmetric cubic anharmonic oscillator problem and, in this regard, has performed a perturbative calculation of the physical observables including investigation of the classical limit. Motivated by Mostafazadeh's work, we examine,in this note, the $\mathcal{PT}$ -symmetric version of a generalized quartic anharmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\mu^2}{2}x^2 + i\epsilon x^3 - m\hbar^2\epsilon^2 x^4 \tag{1}$$

with $(\mu, \nu \in R)$ that includes a cubic anharmonicity as well.

Noting that a $\mathcal{C}$ -operator can be introduced [14] in the physical Hilbert space $\mathcal{H}_{phys}$ subject to a $\mathcal{CPT}$ -inner product [15], and that it commutes with both H and $\mathcal{PT}$, we show that for the above H an equivalent Hermitian Hamiltonian h can be set up. The classical Hamiltonian H_c is then obtained in the limit $\hbar \to 0$.The physical position and momentum operators $\mathbf{X}$ and $\mathbf{P}$,which are actually η_+ -pseudo-Hermitian for the metric operator η_+ and related to the conventional position (x) and momentum (p) operators by the same similarity transformation that links H and h,clearly turns out to be $\mathcal{PT}$ -symmetric,a result similar to the $\mathcal{PT}$ -symmetric cubic oscillator. We also calculate the eigenvalues of H based on the first-order Rayleigh-Schrödinger perturbation theory upto and including terms of order ϵ^3. Further we determine the conserved probability density for a given state vector $\psi \in \mathcal{H}_{phys}$. It should be mentioned that our calculations are somewhat different from Mostafazadeh's in that we have exploited the symmetrized objects $T_{m,n}$ [17] satisfying commutation (lowering type) and anti-commutation (raising type) relations to write down the perturbative expansion of the $\mathcal{C}$ -operator.

2. Basic equations:

To ensure the reality of the spectrum of a diagonalizable operator it is necessary that the Hamiltonian H must be Hermitian with respect to a positive definite inner product $< .,. >_{+}$. The latter can be expressed in terms of a positive definite metric operator $\eta_{+} : \mathcal{H} \to \mathcal{H}$ of the reference Hilbert space $\mathcal{H}$ in which H acts:

$$< .,. >_{+} = < ., \eta_{+}. > \tag{2}$$

where η_{+} belongs to the set of all Hermitian invertiable operators $\eta : \mathcal{H} \to \mathcal{H}$ satisfying $H^{\dagger} = \eta H \eta^{-1}$ [18] and can be expressed as

$$\eta_{+} = e^{-Q} \tag{3}$$

where Q is Hermitian. In terms of η_{+}, $\mathcal{C}$ admits a representation

$$\mathcal{C} = \mathcal{P}\eta_{+} = \eta_{+}^{-1}\mathcal{P} \tag{4}$$

The operator $\mathcal{C}$ commutes with both H and $\mathcal{PT}$: $[\mathcal{C}, H] = 0$,$[\mathcal{C}, \mathcal{PT}] = 0$ and mimicks the charge conjugation operator in particle theory.

Any Hermitian physical observable $\mathcal{O} \in \mathcal{H}_{phys}$ can be converted to a Hermitian operator $o \in \mathcal{H}$ by the transformation

$$\mathcal{O} = \rho^{-1} o \rho \tag{5}$$

where $\rho = \sqrt{\eta_{+}}$ is a unitary operator and because of (3) may be given by

$$\rho = e^{-Q/2} \tag{6}$$

In view of (5) we can write

$$H = \rho^{-1} h \rho \tag{7}$$

where h is the corresponding Hermitian Hamiltonian. The classical Hamiltonian $H_c(\mathbf{x_c}, \mathbf{p_c})$ is obtained from $h(\mathbf{x_c}, \mathbf{p_c})$ by the relation

$$H_c(\mathbf{x_c}, \mathbf{p_c}) = \lim_{\hbar \to 0} h(\mathbf{x_c}, \mathbf{p_c}) \tag{8}$$

where the limit is assumed to exist.

For the sake of convenience let us introduce a set of new variables

$$X := \hbar^{-1}\mathbf{x}, \quad P := \mathbf{p}, \quad \mathcal{M} := m^{1/2}\hbar\mu, \quad \varepsilon := m\hbar^{3}\epsilon \tag{9}$$

In terms of X and P, $H(\mathbf{x}, \mathbf{p}) \to H(X, P)$ with

$$H(X, P) = H_0(X, P) + \varepsilon H_1(X, P) + \varepsilon^2 H_2(X, P) \tag{10}$$

$$H_0(X,P) = \tfrac{1}{2}P^2 + \tfrac{1}{2}\mathcal{M}^2X^2 \quad (11)$$

$$H_1(X,P) = iX^3 \quad (12)$$

$$H_2(X,P) = -X^4 \quad (13)$$

$$H(X,P) = mH(\mathbf{x},\mathbf{p}) \quad (14)$$

3. Determining the Q and $\mathcal{C}$-operators:

Using (3)in (4), we consider the general form of $\mathcal{C}$ as

$$\mathcal{C} = e^{Q(X,P)}\mathcal{P} \tag{15}$$

It has the following properties:

$$[\mathcal{C}, \mathcal{PT}] = 0 \tag{16}$$

$$\mathcal{C}^2 = 1 \tag{17}$$

$$[\mathcal{C}, H(X,P)] = 0 \tag{18}$$

but $[\mathcal{C}, \mathcal{P}] \neq 0$ and $[\mathcal{C}, \mathcal{T}] \neq 0$

Substitution of $\mathcal{C}$ from (15) into (16)implies

$$e^{Q(X,P)}\mathcal{P}\mathcal{P}\mathcal{T} = \mathcal{P}\mathcal{T}e^{Q(X,P)}\mathcal{P} \tag{19}$$

showing $Q(X,P)$ to be an even function of X: $e^{Q(X,P)} = e^{Q(-X,P)}$ .That $Q(X,P)$ is an odd function of P follows from the consideration (17):

$$e^{Q(X,P)}\mathcal{P}e^{Q(X,P)}\mathcal{P} = 1 \tag{20}$$

which yields $e^{Q(X,P)} = e^{-Q(-X,-P)}$.

We now expand of $Q(X,P)$ in a series of odd powers of ε , namely

$$Q(X,P) = \varepsilon Q_1(X,P) + \varepsilon^3 Q_3(X,P) + \varepsilon^5 Q_5(X,P) + \varepsilon^7 Q_7(X,P) + O(\varepsilon^8) \tag{21}$$

Using(18),it follows that

$$e^{Q(X,P)}H_0 - H_0e^{Q(X,P)} = \varepsilon(e^{Q(X,P)}H_1 + H_1e^{Q(X,P)}) - \varepsilon^2(e^{Q(X,P)}H_2 - H_2e^{Q(X,P)})$$

Left multiplying both sides leads to,

$$H_0 - e^{-Q(X,P)}H_0e^{Q(X,P)} = \varepsilon(H_1 + e^{-Q(X,P)}H_1e^{Q(X,P)}) - \varepsilon^2(H_2 - e^{-Q(X,P)}H_2e^{Q(X,P)}) \tag{22}$$

Using Baker-Campbell-Hausdorff identity, i.e,

$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A],A],A] + ... \tag{23}$$

we can arrange the above expression as

$$\begin{aligned}
&-[H_0,Q] - \tfrac{1}{2!}[[H_0,Q],Q] - \tfrac{1}{3!}[[[H_0,Q],Q],Q] - \tfrac{1}{4!}[[[[H_0,Q],Q],Q],Q] \\
&-\tfrac{1}{5!}[[[[[H_0,Q],Q],Q],Q],Q] - \tfrac{1}{6!}[[[[[[H_0,Q],Q],Q],Q],Q],Q] \\
&-\tfrac{1}{7!}[[[[[[[H_0,Q],Q],Q],Q],Q],Q],Q] \\
= \;& 2\varepsilon H_1 + \varepsilon[H_1,Q] + \tfrac{\varepsilon}{2!}[[H_1,Q],Q] + \tfrac{\varepsilon}{3!}[[[H_1,Q],Q],Q] + \tfrac{\varepsilon}{4!}[[[[H_1,Q],Q],Q],Q] \\
&+\tfrac{\varepsilon}{5!}[[[[[H_1,Q],Q],Q],Q],Q] + \tfrac{\varepsilon}{6!}[[[[[[H_1,Q],Q],Q],Q],Q],Q] \\
&+\tfrac{\varepsilon}{7!}[[[[[[[H_1,Q],Q],Q],Q],Q],Q],Q] + \varepsilon^2[H_2,Q] + \tfrac{\varepsilon^2}{2!}[[H_2,Q],Q] + \tfrac{\varepsilon^2}{3!}[[[H_2,Q],Q],Q] \\
&+\tfrac{\varepsilon^2}{4!}[[[[H_2,Q],Q],Q],Q] + \tfrac{\varepsilon^2}{5!}[[[[[H_2,Q],Q],Q],Q],Q] \\
&+\tfrac{\varepsilon^2}{6!}[[[[[[H_2,Q],Q],Q],Q],Q],Q] + \tfrac{\varepsilon^2}{7!}[[[[[[[H_2,Q],Q],Q],Q],Q],Q],Q] \qquad (24)
\end{aligned}$$

where we have taken the terms upto order ε^7

Substituting (21) into (24) and equating terms of order $\varepsilon, \varepsilon^3, \varepsilon^5, \varepsilon^7$ we get,

$$[H_0,Q_1] = -2H_1 \tag{25}$$

$$[H_0,Q_3] = -\tfrac{1}{6}[Q_1,[Q_1,H_1]] + [Q_1,H_2] \tag{26}$$

$$[H_0,Q_5] = -\tfrac{1}{6}([Q_3,[Q_1,H_1]] + [Q_1,[Q_3,H_1]]) + \tfrac{1}{360}[Q_1,[Q_1,[Q_1,[Q_1,H_1]]]] + [Q_3,H_2] \tag{27}$$

$$\begin{aligned}
[H_0,Q_7] = \;& -\tfrac{1}{6}([Q_5,[Q_1,H_1]] + [Q_3,[Q_3,H_1]] + [Q_1,[Q_5,H_1]]) \\
&+\tfrac{1}{360}([Q_3,[Q_1,[Q_1,[Q_1,H_1]]]] + [Q_1,[Q_3,[Q_1,[Q_1,H_1]]]] \\
&+[Q_1,[Q_1,[Q_3,[Q_1,H_1]]]] + [Q_1,[Q_1,[Q_1,[Q_3,H_1]]]]) \\
&-\tfrac{1}{15120}[Q_1,[Q_1,[Q_1,[Q_1,[Q_1,[Q_1,H_1]]]]]] + [Q_5,H_2] \qquad (28)
\end{aligned}$$

note that terms of order $\varepsilon^2, \varepsilon^4, \varepsilon^6$, i.e, even powers of ε give no new results.

To solve for (25),(26),(27)and (28) we introduce, following Bender and Dunne [16] the totally symmetrized sum $T_{r,s}$ over all terms containing r-factor of P and s-factor of X. For example,we have

$$\begin{aligned}
T_{0,0} &= 1 \\
T_{1,0} &= P \\
T_{1,2} &= \tfrac{1}{3}(PX^2 + XPX + X^2P) \\
T_{0,3} &= X^3 \\
T_{3,1} &= \tfrac{1}{4}(XP^3 + PXP^2 + P^2XP + P^3X)
\end{aligned}$$

and so on.

We thus get

$$Q_1 = -\frac{4}{3}\mathcal{M}^{-4}T_{3,0} - 2\mathcal{M}^{-2}T_{1,2} \quad (28a)$$

$$\begin{aligned} Q_3 &= (\tfrac{128}{15}\mathcal{M}^{-10} - \tfrac{32}{5}\mathcal{M}^{-8})T_{5,0} + (\tfrac{40}{3}\mathcal{M}^{-8} \\ &\quad -16\mathcal{M}^{-6})T_{3,2} + (8\mathcal{M}^{-6} - 8\mathcal{M}^{-4})T_{1,4} - (12\mathcal{M}^{-8} - 8\mathcal{M}^{-6})T_{1,0} \end{aligned} \quad (28b)$$

$$\begin{aligned} Q_5 &= (\tfrac{6368}{15}\mathcal{M}^{-12} - 128\mathcal{M}^{-10} + 128\mathcal{M}^{-8})T_{1,2} + (-64\mathcal{M}^{-10} + 32\mathcal{M}^{-8} - 32\mathcal{M}^{-6})T_{1,6} \\ &\quad +(\tfrac{24736}{45}\mathcal{M}^{-14} - 256\mathcal{M}^{-12} + \tfrac{640}{3}\mathcal{M}^{-10})T_{3,0} + (-\tfrac{512}{3}\mathcal{M}^{-12} + \tfrac{352}{3}\mathcal{M}^{-10} - 128\mathcal{M}^{-8})T_{3,4} \\ &\quad +(-\tfrac{544}{3}\mathcal{M}^{-14} + 128\mathcal{M}^{-12} - 128\mathcal{M}^{-10})T_{5,2} + (-\tfrac{320}{3}\mathcal{M}^{-16} + \tfrac{256}{7}\mathcal{M}^{-14} - \tfrac{256}{7}\mathcal{M}^{-12})T_{7,0} \end{aligned} \quad (28c)$$

$$\begin{aligned} Q_7 &= (\tfrac{553984}{315}\mathcal{M}^{-22} - \tfrac{124416}{315}\mathcal{M}^{-20} + \tfrac{69632}{315}\mathcal{M}^{-18} - \tfrac{2048}{9}\mathcal{M}^{-16})T_{9,0} \\ &\quad +(\tfrac{97792}{35}\mathcal{M}^{-20} - \tfrac{62208}{35}\mathcal{M}^{-18} + \tfrac{34816}{35}\mathcal{M}^{-16} - 1024\mathcal{M}^{-14})T_{7,2} \\ &\quad +(\tfrac{377344}{105}\mathcal{M}^{-18} - \tfrac{35456}{15}\mathcal{M}^{-16} + \tfrac{7424}{5}\mathcal{M}^{-14} - 1536\mathcal{M}^{-12})T_{5,4} \\ &\quad +(\tfrac{721024}{315}\mathcal{M}^{-16} - \tfrac{4096}{3}\mathcal{M}^{-14} + \tfrac{2432}{3}\mathcal{M}^{-12} - \tfrac{2560}{3}\mathcal{M}^{-10})T_{3,6} \\ &\quad +(\tfrac{1792}{3}\mathcal{M}^{-14} - 256\mathcal{M}^{-12} + 128\mathcal{M}^{-10} - 128\mathcal{M}^{-8})T_{1,8} \\ &\quad +(-\tfrac{2209024}{105}\mathcal{M}^{-20} + \tfrac{619648}{75}\mathcal{M}^{-18} - \tfrac{54272}{15}\mathcal{M}^{-16} + 3584\mathcal{M}^{-14})T_{5,0} \\ &\quad +(-\tfrac{2875648}{105}\mathcal{M}^{-18} + \tfrac{141824}{15}\mathcal{M}^{-16} - \tfrac{15616}{3}\mathcal{M}^{-14} + 5120\mathcal{M}^{-12})T_{3,2} \\ &\quad +(-\tfrac{390336}{35}\mathcal{M}^{-16} + \tfrac{40832}{15}\mathcal{M}^{-14} - 1216\mathcal{M}^{-12} + 1280\mathcal{M}^{-10})T_{1,4} \\ &\quad +(\tfrac{46976}{5}\mathcal{M}^{-18} - \tfrac{49472}{15}\mathcal{M}^{-16} + 1536\mathcal{M}^{-14} - 1280\mathcal{M}^{-12})T_{1,0} \end{aligned} \quad (28d)$$

Explicit forms of $Q(X, P)$ and $\mathcal{C}(X, P)$ are then obtained by substituting the above expressions for Q_1, Q_3, Q_5, Q_7 into (21) and (15).

4. Determining the X and P operators:

Now we calculate the physical position and momentum operator **X** and **P** from the previously introduced variables X,P using the similarity transformation (5):

$$\mathbf{X} = \rho^{-1}X\rho = e^{\frac{Q}{2}}Xe^{-\frac{Q}{2}} \quad (29)$$

$$\mathbf{P} = \rho^{-1}P\rho = e^{\frac{Q}{2}}Pe^{-\frac{Q}{2}} \quad (30)$$

Using (23),we obtain for $\mathbf{X}$ and $\mathbf{P}$

$$\begin{aligned}
\mathbf{X} &= X+\varepsilon(2i\mathcal{M}^{-4}P^2+i\mathcal{M}^{-2}X^2)+\varepsilon^2(2\mathcal{M}^{-6}XP^2-2i\mathcal{M}^{-6}P-\mathcal{M}^{-4}X^3) \\
&+\varepsilon^3[(-\tfrac{172}{15}\mathcal{M}^{-10}+16\mathcal{M}^{-8})iP^4-(5\mathcal{M}^{-6}-4\mathcal{M}^{-4})iX^4 \\
&-(\tfrac{128}{3}\mathcal{M}^{-8}-48\mathcal{M}^{-6})XP+(\tfrac{64}{3}\mathcal{M}^{-8}-24\mathcal{M}^{-6})iX^2P^2 \\
&+(\tfrac{50}{3}\mathcal{M}^{-8}-16\mathcal{M}^{-6})i]+O(\varepsilon^4) \qquad (31) \\
\mathbf{P} &= P-\varepsilon(2i\mathcal{M}^{-2}(XP-\tfrac{i}{2}))+\varepsilon^2(2\mathcal{M}^{-6}P^3-\mathcal{M}^{-4}(X^2P-iX)) \\
&-i\varepsilon^3[(16\mathcal{M}^{-8}-16\mathcal{M}^{-6})(XP^3-\tfrac{3}{2}iP^2) \\
&+(16\mathcal{M}^{-6}-16\mathcal{M}^{-4})(X^3P-\tfrac{3}{2}iX^2)]+O(\varepsilon^4) \qquad (32)
\end{aligned}$$

where we retained terms of order of ε^3.

From (31) and (32),we easily see that

$$\mathcal{P}\mathbf{X}\mathcal{P}\neq-\mathbf{X}$$

$$\mathcal{P}\mathbf{P}\mathcal{P}\neq-\mathbf{P}$$

$$\mathcal{T}\mathbf{X}\mathcal{T}\neq\mathbf{X}$$

$$\mathcal{T}\mathbf{P}\mathcal{T}\neq-\mathbf{P}$$

but

$$\mathcal{PT}\mathbf{X}\mathcal{PT}=-\mathbf{X}$$

$$\mathcal{PT}\mathbf{P}\mathcal{PT}=\mathbf{P} \qquad (33)$$

From (33) we conclude that the physical position and momentum operator,i.e, $\mathbf{X}$ and $\mathbf{P}\in\mathcal{H}_{phys}$ are $\mathcal{PT}$-symmetric.

' 5. The equivalent Hermitian Hamiltonian:

For the operator ρ, the corresponding Hermitian Hamiltonian $\mathbf{h}(X,P)$ is given according to (7)with previously introduced variables X,P as

$$\mathbf{h}(X,P)=e^{-Q/2}H(X,P)e^{Q/2} \qquad (34)$$

Introducing the perturbative expansion for $\mathbf{h}(X,P)$ as

$$\mathbf{h}=\sum_{i=0}^{\infty}h^{(i)}\varepsilon^i \qquad (35)$$

and using (10),(21),(25)-(28) and (34) along with the Baker-Campbell-Hausdorff identity(23), we obtain for the various coefficients $h^{(i)}$,i=0,1,2,3,...., the results

$$\begin{aligned} h^{(0)} &= H_0 \\ h^{(2)} &= H_2 + \tfrac{1}{4}[H_1, Q_1] \\ h^{(4)} &= \tfrac{1}{4}[H_1, Q_3] - \tfrac{1}{192}[[[H_1, Q_1], Q_1], Q_1] \\ h^{(6)} &= \tfrac{1}{4}[H_1, Q_5] - \tfrac{1}{192}([[[H_1, Q_1], Q_1], Q_3] + [[[H_1, Q_1], Q_3], Q_1] \\ &\quad +[[[H_1, Q_3], Q_1], Q_1]) + \tfrac{1}{7680}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1] \end{aligned} \qquad (36)$$

with the odd ones vanishing: $h^{(i)} = 0$,i=1,3,5,7,....

Keeping terms up to the order ε^5 ; $\mathbf{h}(X, P)$ can thus be expressed as

$$\begin{aligned} \mathbf{h}(X, P) &= \tfrac{1}{2}(T_{2,0} + \mathcal{M}^2 T_{0,2}) + \varepsilon^2[-\tfrac{1}{2}\mathcal{M}^{-4} + (\tfrac{3}{2}\mathcal{M}^{-2} - 1)T_{0,4} + 3\mathcal{M}^{-4}T_{2,2}] \\ &\quad +\varepsilon^4[-(36\mathcal{M}^{-10} - 24\mathcal{M}^{-8})T_{4,2} + (27\mathcal{M}^{-10} - 24\mathcal{M}^{-8})T_{2,0} - (\tfrac{51}{2}\mathcal{M}^{-8} - 36\mathcal{M}^{-6})T_{2,4} \\ &\quad +(\tfrac{179}{24}\mathcal{M}^{-8} - 12\mathcal{M}^{-6})T_{0,2} - (\tfrac{7}{2}\mathcal{M}^{-6} - 6\mathcal{M}^{-4})T_{0,6} + 2\mathcal{M}^{-12}T_{6,0}] + O(\varepsilon^6) \end{aligned} \qquad (37)$$

If now we consider the normalized eigen vector $|n>$ of the conventional Harmonic oscillator H_0, then we can easily calculate E_n for $\mathbf{H}$ by the first order Rayleigh-Schrödinger perturbation theory. We obtain upto the terms of order ε^3

$$\begin{aligned} E_n &= \mathcal{M}(n + \tfrac{1}{2}) + \varepsilon^2 < n|h^{(2)}|n > + O(\varepsilon^4) \\ &= \mathcal{M}(n + \tfrac{1}{2}) + \tfrac{\varepsilon^2}{4}[\tfrac{1}{2\mathcal{M}^4}(30n^2 + 30n + 11) - (6n^2 + 6n + 3)] + O(\varepsilon^4) \end{aligned} \qquad (38)$$

6. Classical Hamiltonian:

Employing (9) , we have,

$$T_{r,s} = \hbar^{-s} S_{r,s} \qquad (39)$$

where $S_{r,s}$ be the totally symmetrized sum of all terms containing r-factor of p and s-factor of x. Specifically

$$\begin{aligned} S_{0,0} &= 1 \\ S_{0,1} &= x \\ S_{3,0} &= p^3 \\ S_{1,3} &= \tfrac{1}{4}(x^3 p + xpx^2 + x^2 px + px^3) \end{aligned}$$

and so on.

Now from (14),

Keeping terms up to the terms of order of ε^3 in (37) and using the relations (9),(39),(40) we finally obtain

$$h(x,p)=\frac{p^2}{2m}+\frac{1}{2}\mu^2x^2+\epsilon^2m[(\frac{3}{2}m^{-1}\mu^{-2}-\hbar^2)x^4-2m^{-2}\hbar^2\mu^{-4}-6im^{-2}\hbar\mu^{-4}xp-3m^{-2}\mu^{-4}x^2p^2]+O(\epsilon^4) \quad (41)$$

The corresponding classical Hamiltonian can be read off from (8):

$$\begin{aligned} H_c(x_c,p_c) &= \lim_{\hbar\to 0} h(x_c,p_c) \\ &= \frac{p_c^2}{2M(x_c)}+\frac{1}{2}\mu^2x_c^2+\frac{3\epsilon^2}{2\mu^2}x_c^4+O(\epsilon^4) \end{aligned} \quad (42)$$

where

$$M(x_c)=\frac{m}{1-6\mu^{-4}\epsilon^2x_c^2} \quad (43)$$

A position-dependent mass $M(x_c)$ is implied by (43) for the classical particle whose dynamics is dictated by the Hamiltonian $H_c(x_c,p_c)$.

7. Conserved probability density:

For a given state vector $\psi\in\mathcal{H}_{phys}$, the perturbation expansion for the corresponding physical wave function is

$$\begin{aligned} \Psi(x) &= <x\mid e^{-\frac{Q}{2}}\mid\psi> \\ &= <x\mid \textstyle\sum_{k=0}^{\infty}\frac{(-1)^kQ^k}{2^kk!}\mid\psi> \\ &= \psi(x)+<x\mid -\frac{Q_1}{2}\mid\psi>\varepsilon+<x\mid \frac{Q_1^2}{8}\mid\psi>\varepsilon^2+<x\mid(-\frac{Q_3}{2}-\frac{Q_1^3}{48})\mid\psi>\varepsilon^3+O(\varepsilon^4) \end{aligned} \quad (44)$$

Using (9),(28a) and (28b) we obtain from (44)

$$\Psi(x)=(1+\epsilon L_1+\epsilon^2L_2+\epsilon^3L_3)\psi(x)+O(\epsilon^4) \quad (45)$$

where

$$L_1 = -\frac{m}{2}\hbar^3\hat{Q}_1 \quad (46)$$

$$L_2 = \frac{m^2}{8}\hbar^6\hat{Q}_1{}^2 \quad (47)$$

$$L_3 = -\frac{m^3}{2}\hbar^9\hat{Q}_3-\frac{m^3}{48}\hbar^9\hat{Q}_1{}^3 \quad (48)$$

and,

$$\hat{Q}_1 = -\frac{4}{3}m^{-2}\hbar^{-4}\mu^{-4}S_{3,0}-2m^{-1}\hbar^{-2}\mu^{-2}S_{1,2} \quad (49)$$

$$\hat{Q}_3 = (\frac{128}{15}m^{-5}\hbar^{-10}\mu^{-10}-\frac{32}{5}m^{-4}\hbar^{-8}\mu^{-8})S_{5,0}+(\frac{40}{3}m^{-4}\hbar^{-10}\mu^{-8}-16m^{-3}\hbar^{-8}\mu^{-6})S_{3,2}$$

Employing (46)-(50) into (45), we find the conserved probability density ϱ associated with a given state vector $\psi \in \mathcal{H}_{phys}$ as

$$\varrho(x) = N^{-1} \mid \Psi(x) \mid^2 \tag{51}$$

where

$$N = \int_{-\infty}^{\infty} \mid \Psi(x) \mid^2 dx \tag{52}$$

8. Conclusion:

We have carried out a perturbative treatment to study a $\mathcal{PT}$ -symmetric quartic anharmonic oscillator model. We have shown possible to set up an equivalent Hermitian Hamiltonian by employing a similarity transformation. Such a Hamiltonian has a classical limit too. Physical position and momentum operators have been determined perturbatively and energy eigenvalues are obtained in the framework of first order Rayleigh-Schrödinger perturbation theory. In all these calculations we have kept terms up to and including those of order ε^3. The conserved probability density is also determined. Finally,let us mention that in the absence of the quartic term in (1) our results essentially reduce to those of Mostafazadeh's [13] for the physical observables **X** and **P**, equivalent Hermitian Hamiltonian $h(X, P)$ and the energy spectrum derived from the first order Rayleigh-Schrödinger perturbation theory.

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References:

1. Bender C M and Boettcher S 1998 Phys.Rev.Lett. **24** 5243

   See also

   Bender C M -Introduction to $\mathcal{PT}$-symmetric quantum theory,preprint quant-ph/0501052

2. Delabaere E and Pham F 1998 Phys.Lett.A **250** 25

   Fernandez F M, Guardiola R, Ros J and Znojil M 1998 J.Phys.A: Math.Gen. **31** 10105

   Bender C M, Boettcher S and Meisinger P N 1999 J.Math.Phys. **40** 2201

Znojil M 2000 J.Phys.A: Math.Gen **33** 4561

Ramirez A and Mielnit B 2003 Rev.Fis.Mex. 49S2 130

Bender C M, Brody D C and Jones H F 2003 Am.J.Phys. **71** 1095

3. Bagchi B and Roychoudhury R 2000 J.Phys.A: Math.Gen. **33** L1

   Bagchi B and Quesne C 2000 Phys.Lett.A **273** 285

   Bagchi B and Quesne C 2002 A**300** 18

   Bagchi B, Cannata F and Quesene C 2000 Phys.Lett.A **209** 79

   Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 Phys.Lett.B **483** 284

   Bagchi B, Quesne C and Znojil M 2001 Mod.Phys.Lett.A **16** 2047

4. Scholtz F G, Geyer H B and Hahne F J W 1992 Ann. Phys.(NY) **213** 74

5. Dorey P, Dunning C and Tateo R 2001 J.Phys.A: Math.Gen. **34** 5679

6. Levai G and Znojil M 2000 J.Phys.A: Math.Gen. **33** 7165

7. Kaushal R S 2001 J.Phys.A: Math.Gen. **34** L709

8. Ahmed Z 2001 Phys.Lett.A **282** 343

9. Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1999 Int.J.Mod.Phys.A **14** 2675

10. Kretschmer R and Szymanowski L 2004 Phys.Lett.A **325** 112

11. Caliceti E, Graffi S and Maioli M 1980 Commun.Math.Phys. **75** 51

12. Buslaev V and Grecchi V 1993 J.Phys.A: Math.Gen. **26** 5541

13. Mostafazadeh A - $\mathcal{PT}$ -symmetric cubic anharmonic oscillator as a physical model, preprint quant-ph/0411137

14. Bender C M, Brody D C and Jones H F 2002 Phys.Rev.Lett. **89** 270401

15. Bender C M, Meisinger P N and Wang Q 2003 J.Phys.A: Math.Gen. **36** 1973

16. Bagchi B, Banerjee A, Caliceti E, Cannata F, Geyer H B, Quesne C and Znojil M - $\mathcal{CPT}$ -conserving Hamiltonians and their nonlinear supersymmetrization using differential charge-operators $\mathcal{C}$,preprint hep-th/0412211

17. Bender C M and Dunne G 1989 Phys.Rev.D **40** 2739

18. Mostafazadeh A 2002 J.Math.Phys. **43** 205

    Mostafazadeh A 2002 J.Math.Phys. **43** 2814

19. Mostafazadeh A,Batal A 2004 J.Phys.A: Math.Gen.**37** 11645

20. Mostafazadeh A - preprint quant-ph/0310164