A NOTE ON TRANS-SASAKIAN MANIFOLDS

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Abstract. In this paper, we obtain some sufficient conditions for a 3-dimensional compact trans-Sasakian manifold of type (α, β) to be homothetic to a Sasakian manifold. A characterization of a 3-dimensional cosymplectic manifold is also obtained.

1. Introduction

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold (cf. [2]). Then the product $\overline{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\overline{M}, J, G)$ dictates the geometry of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ and gives different structures on M like Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and others (cf. [2], [3], [8]). It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ (cf. [6]) and using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$, a structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ on M called trans-Sasakian structure, was introduced (cf. [13]) that generalizes Sasakian and Kenmotsu structures on a contact metric manifold (cf. [3], [8]), where α, β are smooth functions defined on M. Since the introduction of trans-Sasakian manifolds, very important contributions of Blair and Oubiña [3] and Marrero [11] have appeared, studying the geometry of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). Trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian, and β-Kenmotsu manifolds respectively. Marrero [11] has shown that a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic, or α-Sasakian, or β-Kenmotsu. Since then, there is a concentration on studying geometry of 3-dimensional trans-Sasakian manifolds only (cf. [1], [4], [5], [9], [10]),

2010 *Mathematics Subject Classification.* Primary 53C15; Secondary (optional) 53D10.

Key words and phrases. Almost contact metric manifold, Sasakian manifold, Trans-Sasakian manifold.

This work was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

putting some restrictions on the smooth functions α, β appearing in the definition of trans-Sasakian manifolds. There are several examples of trans-Sasakian manifolds constructed mostly on 3-dimensional Riemannian manifolds (cf. [3], [11], [13]). Moreover, as the geometry of Sasakian manifolds is very rich, and is derived from contact geometry, the question of finding conditions under which a 3-dimensional trans-Sasakian manifold is homothetic to a Sasakian manifold becomes more interesting. In this paper we consider this question and obtain two different sufficient conditions for a trans-Sasakian manifold to be homothetic to a Sasakian manifold. One of them is expressed in terms of the smooth functions α, β and a bound on certain Ricci curvature, and the other requires that the Reeb vector should be an eigenvector of the Ricci operator (cf. Theorems 3.1, 3.2). We also find a characterization of cosymplectic manifolds (cf. Theorem 4.1).

Acknowledgement. *The authors wish to express their sincere thanks to the referee for many corrections. The first author also wishes to thank the DST-CIMS at Banaras Hindu University, Varanasi for the hospitality during his visit to the center.*

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold, where φ is a $(1,1)$-tensor field, ξ a unit vector field and η a smooth 1-form dual to ξ with respect to the Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi,\ \varphi(\xi) = 0,\ \eta \circ \varphi = 0,\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M (cf. [2]). If there are smooth functions α, β on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying

$$(\nabla \varphi)(X,Y) = \alpha \left(g(X,Y)\xi - \eta(Y)X \right) + \beta \left(g(\varphi X, Y)\xi - \eta(Y)\varphi X \right), \tag{2.2}$$

then this is said to be a trans-Sasakian manifold, where $(\nabla \varphi)(X,Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y)$, $X, Y \in \mathfrak{X}(M)$ and ∇ is the Levi-Civita connection with respect to the metric g (cf. [3], [11], [13]). We shall denote this trans-Sasakian manifold by $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ and it is called trans-Sasakian manifold of type (α, β). From equations (2.1) and (2.2), it follows that

$$\nabla_X \xi = -\alpha \varphi(X) + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M). \tag{2.3}$$

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold (cf. [2]) and a trans-Sasakian manifold of type $(0,1)$ is a Kenmotsu manifold (cf. [8]). A trans-Sasakian manifold of type $(0,0)$ is called a cosymplectic manifold (cf. [7]).

Let Ric be the Ricci tensor of a Riemannian manifold (M, g). Then the Ricci operator Q is a symmetric tensor field of type $(1,1)$ defined by $Ric(X,Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$. We prepare some tools for trans-Sasakian manifolds.

Lemma 2.1. *Let* $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ *be a* 3*-dimensional trans-Sasakian manifold. Then* $\xi(\alpha) = -2\alpha\beta$.

Proof. Using (2.3), we get that

$$d\eta(X, Y) = -2\alpha g(\varphi X, Y), \qquad X, Y \in \mathfrak{X}(M)$$

and as a consequence, the 2-form Ω defined by $\Omega(X, Y) = \alpha g(\varphi X, Y)$ is closed. Using (2.1), (2.2), and (2.3) in $d\Omega = 0$ after some trivial calculations, we arrive at

$$\varphi \left\{X(\alpha)Y - Y(\alpha)X - 2\alpha\beta\eta(Y)X + 2\alpha\beta\eta(X)Y\right\} + g(\varphi X, Y)\left(\nabla\alpha + 2\alpha\beta\xi\right) = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Operating φ on the equation above, we get

$$\begin{aligned} &Y(\alpha)X - X(\alpha)Y + 2\alpha\beta\eta(Y)X - 2\alpha\beta\eta(X)Y + X(\alpha)\eta(Y)\xi \\ &-Y(\alpha)\eta(X)\xi + g(\varphi X, Y)\varphi\left(\nabla\alpha\right) \;=\; 0. \end{aligned}$$

For a local orthonormal frame $\{e_1, e_2, e_3\}$ on M, taking $X = e_i$ in the equation above, taking the inner product with e_i and adding the resulting equations, we get

$$\left(2\alpha\beta + \xi(\alpha)\right)\eta(Y) = 0, \quad Y \in \mathfrak{X}(M)$$

which gives

$$\left(2\alpha\beta + \xi(\alpha)\right)\xi = 0$$

and we obtain the result. □

Lemma 2.2. *Let* $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ *be a* 3*-dimensional trans-Sasakian manifold. Then its Ricci operator satisfies*

$$Q(\xi) = \varphi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi$$

where $\nabla\alpha$, $\nabla\beta$ *are gradients of the smooth functions* α, β.

Proof. We use (2.1), (2.2), and (2.3) to calculate

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi$$

and after some easy computations we arrive at

$$\begin{aligned} R(X, Y)\xi \;&=\; Y(\alpha)\varphi X - X(\alpha)\varphi Y + X(\beta)(Y - \eta(Y)\xi) - Y(\beta)(X - \eta(X)\xi) \\ &\quad +(\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y). \end{aligned}$$

The above equation gives

$$\begin{aligned} Ric(Y, \xi) \;&=\; g(\varphi(\nabla\alpha), Y) - g(\nabla\beta, Y) - g(\nabla\beta, \xi)\eta(Y) \\ &\quad +2(\alpha^2 - \beta^2)\eta(Y), \end{aligned}$$

which proves the result. □

Next, we state the following result of [12], which we shall use in the sequel.

Theorem 2.1. [12] *Let (M, g) be a Riemannian manifold. If M admits a Killing vector field ξ of constant length satisfying*

$$k^2 \left(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi\right) = g(Y, \xi)X - g(X, Y)\xi$$

for a nonzero constant k and any vector fields X and Y, then M is homothetic to a Sasakian manifold.

3. Trans-Sasakian manifolds homothetic to Sasakian manifolds

In this section we study compact and connected 3-dimensional trans-Sasakian manifolds and obtain conditions under which they are homothetic to Sasakian manifolds. Our first result uses a bound on the Ricci curvature of the trans-Sasakian manifold in the direction of the vector field ξ.

Theorem 3.1. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold. If the Ricci curvature $Ric(\xi, \xi)$ satisfies*

$$0 < Ric(\xi, \xi) \leq 2\left(\alpha^2 + \beta^2\right),$$

then M is homothetic to a Sasakian manifold.

Proof. Using (2.3) we immediately compute

$$\delta\eta = \mathrm{div}\xi = 2\beta. \tag{3.1}$$

Also, since $d\eta(X, Y) = -2\alpha g(\varphi X, Y)$, we obtain

$$\|d\eta\|^2 = 8\alpha^2. \tag{3.2}$$

Now using (2.3), after some obvious calculations, we get

$$\|\nabla\xi\|^2 = 2(\alpha^2 + \beta^2). \tag{3.3}$$

Now, using (3.1)-(3.3) in the integral formula (cf. [14])

$$\int_M \left\{ Ric(\xi, \xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla\xi\|^2 - (\delta\eta)^2 \right\} = 0$$

and the hypothesis of the theorem, we deduce that

$$Ric(\xi, \xi) = 2\left(\alpha^2 + \beta^2\right). \tag{3.4}$$

Using Lemma 2.2, we have

$$Ric(\xi, \xi) = -2\xi(\beta) + 2(\alpha^2 - \beta^2)$$

which together with (3.4) gives

$$\xi(\beta) = -2\beta^2. \tag{3.5}$$

We claim that β must be a constant. If β is not a constant, then on the compact M it has a local maximum at some $p \in M$. We have $(\nabla\beta)(p) = 0$ and the Hessian H_β is negative definite at this point p. However, using the equation (3.5), we have $\xi(\beta)(p) = -2(\beta(p))^2 = 0$ and $H_\beta(\xi, \xi)(p) = \xi\xi(\beta)(p) =$

$4\,(\beta(p))^3 = 0$, (where we used $\nabla_\xi \xi = 0$), which yields a contradiction (as the Hessian is negative definite at p). Hence, β is a constant and this, combined with Stokes' theorem applied to $\operatorname{div}(\xi) = 2\beta$, proves that $\beta = 0$.

Since $\beta = 0$, the Lemma 2.1 gives $\xi(\alpha) = 0$. We claim that α is a constant. If not, on compact M the smooth function α attains a local maximum at some point $p \in M$. At this point, the Hessian H_α is negative definite. However, for the unit vector field ξ, we have $H_\alpha(\xi, \xi) = 0$, which fails to be negative definite at point p, which is a contradiction. Now, that α is a non-zero constant follows from the condition in the hypothesis. Thus, using (2.3), we compute

$$\alpha^{-2}\left(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi\right) = g(Y, \xi)X - g(X, Y)\xi,$$

and this implies by Theorem 2.1 that M is homothetic to a Sasakian manifold. □

As a direct consequence of the above theorem we have the following result, which has motivation from the fact that on a $(2n+1)$-dimensional Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the Ricci operator satisfies $Q(\xi) = 2n\xi$.

Corollary 3.1. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold. If the vector field ξ satisfies $Q(\xi) = 2\alpha^2\xi \neq 0$, then M is homothetic to a Sasakian manifold.*

As pointed out earlier, on a $(2n+1)$-dimensional Sasakian manifold $(M, \varphi, \xi, \eta, g)$, the Ricci operator satisfies $Q(\xi) = 2n\xi$, that is, the Reeb vector field ξ is an eigenvector of the Ricci operator. This motivates the question of whether a 3-dimensional trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ satisfying $Q(\xi) = \lambda\xi$ for a non-zero constant λ, is necessarily homothetic to a Sasakian manifold. We answer this question for compact connected 3-dimensional trans-Sasakian manifolds and show that they are homothetic to Sasakian manifolds.

Theorem 3.2. *Let $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold. Then M is homothetic to a Sasakian manifold if and only if the vector field ξ satisfies $Q(\xi) = \lambda\xi$ for a non-zero constant λ.*

Proof. Using $Q(\xi) = \lambda\xi$ in Lemma 2.2, we have

$$\varphi(\nabla\alpha) - \nabla\beta = \left(\lambda + \xi(\beta) - 2(\alpha^2 - \beta^2)\right)\xi. \tag{3.7}$$

Taking the inner product with ξ in the above equation, we obtain

$$\xi(\beta) = -\frac{\lambda}{2} + (\alpha^2 - \beta^2). \tag{3.8}$$

Inserting this value in (3.7), we have

$$\varphi(\nabla\alpha) - \nabla\beta = \left(\frac{\lambda}{2} - (\alpha^2 - \beta^2)\right)\xi \tag{3.9}$$

and applying φ to the above equation, we obtain

$$\nabla\alpha = -2\alpha\beta\xi - \varphi(\nabla\beta). \tag{3.10}$$

If A is a symmetric operator on the trans-Sasakian manifold M, we can choose a local orthonormal frame that diagonalizes A and consequently, we have

$$\sum g(\varphi(Ae_i), e_i) = 0. \tag{3.11}$$

Now for $X \in \mathfrak{X}(M)$, we compute

$$\nabla_X\left(\varphi(\nabla\beta) + 2\alpha\beta\xi\right) = \left(\nabla_X\varphi\right)(\nabla\beta) + \varphi\left(A_\beta X\right) + 2X(\alpha\beta)\xi + 2\alpha\beta\nabla_X\xi$$

where $A_\beta X = \nabla_X\nabla\beta$ is a symmetric operator $A_\beta : \mathfrak{X}(M) \to \mathfrak{X}(M)$. Taking the inner product with X in above equation and using equations (2.2) and (2.3), after some easy calculations we arrive at

$$\begin{aligned} g\left(\nabla_X\left(\varphi(\nabla\beta) + 2\alpha\beta\xi\right), X\right) &= \alpha X(\beta)\eta(X) - \alpha\xi(\beta)g(X,X) \\ &\quad + \beta g\left(\varphi X, \nabla\beta\right)\eta(X) \\ &\quad + g\left(\varphi\left(A_\beta X\right), X\right) + 2X(\alpha\beta)\eta(X) \\ &\quad + 2\alpha\beta^2 g(X,X) - 2\alpha\beta^2\left(\eta(X)\right)^2. \end{aligned}$$

Taking trace in the equation above, in view of the equation (3.11), we get

$$\operatorname{div}\left(\varphi(\nabla\beta) + 2\alpha\beta\xi\right) = -2\alpha\xi(\beta) + 2\xi(\alpha\beta) + 4\alpha\beta^2 = 0, \tag{3.12}$$

where we used the fact that $\xi(\alpha) = -2\alpha\beta$. Thus using (3.12) in the equation (3.10), we conclude that $\Delta\alpha = \operatorname{div}(\nabla\alpha) = 0$ on compact M, which proves that α is a constant. Using the constant α in the equation (3.9), we get

$$-\nabla\beta = \left(\frac{\lambda}{2} - (\alpha^2 - \beta^2)\right)\xi$$

which together with the equation (3.8) gives

$$\begin{aligned} \Delta\beta &= -2\beta\xi(\beta) - \left(\frac{\lambda}{2} - (\alpha^2 - \beta^2)\right)\operatorname{div}\xi \\ &= -2\beta\left(-\frac{\lambda}{2} + (\alpha^2 - \beta^2)\right) - 2\beta\left(\frac{\lambda}{2} - (\alpha^2 - \beta^2)\right) \\ &= 0. \end{aligned}$$

Here we used the fact that $\operatorname{div}\xi = 2\beta$. Thus β is a constant, which together with Stokes' theorem and $\operatorname{div}\xi = 2\beta$ proves that $\beta = 0$. If $\alpha = 0$, then (3.7) would imply $\lambda = 0$, which is a contradiction. Consequently, α is a non-zero constant which by the equation (3.1) satisfies

$$\alpha^{-2}\left(\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi\right) = g(Y,\xi) - g(X,Y)\xi.$$

This proves that M is homothetic to a Sasakian manifold. The converse is obvious. $\square$

4. A characterization of cosymplectic manifolds

In this section, we study 3-dimensional compact trans-Sasakian manifolds, and obtain a characterization of cosymplectic manifolds. Let $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ be a 3-dimensional trans-Sasakian manifold. Then for each point $p\in M$ there is a neighbourhood U of p, where we have a local orthonormal frame $\{e,\varphi e,\xi\}$ for a unit vector field e on U called an adapted frame. Using the equations (2.1), (2.2) and (2.3), we obtain the following local structure equations defined on U

$$\nabla_e\xi=\beta e-\alpha\varphi e,\quad \nabla_{\varphi e}\xi=\alpha e+\beta\varphi e,\quad \nabla_\xi\xi=0, \tag{4.1}$$

$$\nabla_e e=\gamma\varphi e-\beta\xi,\quad \nabla_{\varphi e}e=-\delta\varphi e-\alpha\xi,\quad \nabla_\xi e=\lambda\varphi e, \tag{4.2}$$

$$\nabla_e\varphi e=-\gamma e+\alpha\xi,\quad \nabla_{\varphi e}\varphi e=\delta e-\beta\xi,\quad \nabla_\xi\varphi e=-\lambda e, \tag{4.3}$$

where γ, δ, λ are smooth functions defined on U. Using the above equations, we compute

$$R(e,\varphi e)\xi=(e(\alpha)-\varphi e(\beta))\,e+(e(\beta)+\varphi e(\alpha))\,\varphi e$$

$$R(\varphi e,\xi)e=(\varphi e(\lambda)+\xi(\delta)+\beta\delta-\gamma\alpha-\gamma\lambda)\,\varphi e+(\xi(\alpha)+2\alpha\beta)\,\xi$$

$$R(\xi,e)\varphi e=(e(\lambda)-\xi(\gamma)-\beta\gamma-\delta\alpha-\delta\lambda)\,e+(\xi(\alpha)+2\alpha\beta)\,\xi.$$

Adding these three equations, we conclude that

$$e(\alpha)-\varphi e(\beta)+e(\lambda)-\xi(\gamma)=\beta\gamma+\delta\alpha+\delta\lambda, \tag{4.4}$$

$$e(\beta)+\varphi e(\alpha)+\varphi e(\lambda)+\xi(\delta)=\gamma\alpha+\gamma\lambda-\beta\delta, \tag{4.5}$$

and the third component gives the result in the Lemma 2.1. Also, we have

$$R(\xi,e)e=(\xi(\gamma)-e(\lambda)+\beta\gamma+\alpha\delta+\lambda\delta)\,\varphi e+\left(-\xi(\beta)+\alpha^2-\beta^2\right)\xi$$

and

$$R(\xi,\varphi e)\varphi e=(\xi(\delta)+\varphi e(\lambda)+\beta\delta-\alpha\gamma-\lambda\gamma)\,e+\left(-\xi(\beta)+\alpha^2-\beta^2\right)\xi$$

Using the two equations above in $Q(\xi)=R(\xi,e)e+R(\xi,\varphi e)\varphi e$, we obtain

$$\begin{aligned} Q(\xi) \;=\;& (\xi(\delta)+\varphi e(\lambda)+\beta\delta-\alpha\gamma-\lambda\gamma)\,e\\ &+(\xi(\gamma)-e(\lambda)+\beta\gamma+\alpha\delta+\lambda\delta)\,\varphi e\\ &+2\left(-\xi(\beta)+\alpha^2-\beta^2\right)\xi. \end{aligned}$$

This together with the equations (4.4) and (4.5) gives

$$Q(\xi)=-\left(e(\beta)+\varphi e(\alpha)\right)e+\left(e(\alpha)-\varphi e(\beta)\right)\varphi e+2\left(-\xi(\beta)+\alpha^2-\beta^2\right)\xi. \tag{4.6}$$

Recall that in Theorem 3.2, the vector field ξ being an eigenvector of the Ricci operator corresponding to a non-zero eigenvalue makes the trans-Sasakian manifold homothetic to a Sasakian manifold. Similarly, we have the following characterization of cosymplectic manifolds.

Theorem 4.1. *Let $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ be a 3-dimensional compact and connected trans-Sasakian manifold. Then M is a cosymplectic manifold if and only if the Ricci operator Q annihilates the vector field ξ.*

Proof. Suppose that $Q(\xi)=0$ holds. Then (4.6) gives

$$e(\beta)=-\varphi e(\alpha),\quad e(\alpha)=\varphi e(\beta),\text{ and }\xi(\beta)=\alpha^2-\beta^2. \tag{4.7}$$

Applying Lemma 2.1 and the equations (2.2), (2.3), (4.1)-(4.3) and (4.7), we obtain

$$\begin{aligned}
\Delta\alpha &= ee(\alpha)+\varphi e\varphi e(\alpha)+\xi\xi(\alpha)-\nabla_e e(\alpha)-\nabla_{\varphi e}\varphi e(\alpha)-\nabla_\xi\xi(\alpha)\\
&= [e,\varphi e](\beta)-2\xi(\alpha\beta)+\gamma e(\beta)-\delta\varphi e(\beta)-4\alpha\beta^2\\
&= \left(\nabla_e\varphi e\right)(\beta)-\left(\nabla_{\varphi e}e\right)(\beta)-2\xi(\alpha\beta)+\gamma e(\beta)-\delta\varphi e(\beta)-4\alpha\beta^2\\
&= \left(-\gamma e+\alpha\xi\right)(\beta)-\left(-\delta\varphi e-\alpha\xi\right)-2\xi(\alpha\beta)+\gamma e(\beta)-\delta\varphi e(\beta)-4\alpha\beta^2\\
&= 2\alpha\xi(\beta)-2\xi(\alpha\beta)-4\alpha\beta^2=0.
\end{aligned}$$

Thus thanks to compactness of M we have proved that α is a constant. If $\alpha\neq 0$, then Lemma 2.1, implies that $\beta=0$ and consequently the equation (4.7) gives $\alpha=0$, which is a contradiction. Hence $\alpha=0$ and the equation (4.7) gives $\xi(\beta)=-\beta^2$, that is, $\operatorname{div}(\beta\xi)=\beta^2$, where we used $\operatorname{div}\xi=2\beta$, which follows from the equation (2.3). Using Stokes' theorem in $\operatorname{div}(\beta\xi)=\beta^2$, we obtain $\beta=0$. That is, M is a cosymplectic manifold. Conversely, if M is a cosymplectic manifold, then the equation (4.6) gives that $Q(\xi)=0$. □

References

[1] AL-SOLAMY, F. R.—KIM, J.-S.—TRIPATHI, M. M.: *On η-Einstein trans-Sasakian manifolds*, An. Stiint. Univ. "Al.I.Cuza" din Iasi **57** (2011), no. 2, 417–440.

[2] BLAIR, D. E.: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics **509**, Springer (1976).
AARTS, J. M.—LUTZER, D. J.: *Pseudo-completeness and the product of Baire spaces*, Pacific J. Math. **48** (1973), 1–10.

[3] BLAIR, D. E.—OUBINA, J. A.: *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mat. **34** (1990), no. 1, 199–207.

[4] DE, U. C.—SARKAR, A.: *On three-dimensional trans-Sasakian manifolds*, Extracta Math. **23** (2008), no. 3, 265–277.

[5] DE, U. C.—TRIPATHI, M. M.: *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J. **43** (2003), no. 2, 247–255.

[6] GRAY, A.—HERVELLA, L. M.: *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), 35–58.

[7] FUJIMOTO, A.—MUTO, H.: *On cosymplectic manifolds*, Tensor **28** (1974), 43–52.

[8] KENMOTSU, K.: *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93-103.

[9] KIM, J.-S.—PRASAD, R—TRIPATHI, M. M.: *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc. **39** (2002), no. 6, 953–961.

[10] KIRICHENKO, V. F.: *On the geometry of nearly trans-Sasakian manifolds*, (Russian) Dokl. Akad. Nauk **397** (2004), no. 6, 733–736.

[11] MARRERO, J. C.: *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. (4) **162** (1992), 77–86.
[12] OKUMURA, M.: *Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures*, Tôhoku Math. J. (2) **16** (1964) 270–284.
[13] OUBINA, J. A.: *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.
[14] YANO, K.: *Integral formulas in Riemannian Geometry*, Marcel Dekker (1970).

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