Zero temperature dynamics of Ising model on a densely connected small world network

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The zero temperature quenching dynamics of the ferromagnetic Ising model on a densely connected small world network is studied where long range bonds are added randomly with a finite probability p. We find that in contrast to the sparsely connected networks and random graph, there is no freezing and an initial random configuration of the spins reaches the equilibrium configuration within a very few Monte Carlo time steps in the thermodynamic limit for any $p \neq 0$. The residual energy and the number of spins flipped at any time shows an exponential relaxation to equilibrium. The persistence probability is also studied and it shows a saturation within a few time steps, the saturation value being 0.5 in the thermodynamic limit. These results are explained in the light of the topological properties of the network which is highly clustered and has a novel small world behaviour.

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I. INTRODUCTION

It is well known that the one dimensional Ising model with nearest neighbour interaction does not have any non-trivial phase transition. However, a drastic change is observed in its critical behavior when even a few long range interactions are added randomly. Such a one dimensional lattice with extra random connections is known to behave as a small-world network (SWN) [1–3], which means that the average shortest distance between any two sites in this lattice scales with the logarithm of the number of sites. The Ising model on small world networks not only has a finite temperature phase transition [4,5], but the critical behavior is also mean-field like [6–11]. A network in which the distribution of the number of links follows a power-law is known as a scale free network (SFN). Here also a finite temperature phase transition of the Ising model with a diverging critical temperature [12] has been observed.

Recently, quite a few studies on the dynamical properties of the Ising model on random graphs as well as networks have been reported both at finite and zero temperatures [13–19]. Dynamics of Ising models is a much studied phenomenon and has emerged as a rich field of present-day research. Models having same static critical behavior may display different behavior when dynamic critical phenomena are considered [20]. An important dynamical feature commonly studied is the quenching phenomenon below the critical temperature. In a quenching process, the system has a disordered initial configuration corresponding to a high temperature and its temperature is suddenly dropped. This results in quite a few interesting phenomena like domain growth [21,22], persistence [23–27] etc.

In one dimension, a zero temperature quench of the Ising model ultimately leads to the equilibrium configuration, i.e., all spins point up (or down). The average domain size D increases in time t as $D(t) \sim t^{1/z}$, where z is the dynamical exponent. As the system coarsens, the magnetisation also grows in time as $m(t) \sim t^{1/2z}$. In two or higher dimensions, however, the system does not always reach equilibrium [27] although these scaling relations still hold good.

Apart from the domain growth phenomenon, another important dynamical behavior commonly studied is persistence. In Ising model, persistence is simply the probability that a spin has not flipped till time t and is given by $P(t) \sim t^{-\theta}$. θ is called the persistence exponent and is unrelated to any other static or dynamic exponents. Persistence can also be studied at finite temperatures and the exponent may change at the critical temperature [25,28].

We have studied the zero temperature quenching dynamics of the ferromagnetic Ising model on a SWN. It is important to carefully describe the type of network under consideration. There are two well-known methods of generating a small world network starting from a chain of nodes having connections with nearest neighbours only. These are (a) the addition type, where new long range (LR) bonds are added randomly keeping the nearest neighbour connections intact, and (b) the rewiring type where the existing nearest neighbour bonds are rewired to distant neighbours randomly. In the first case, when bonds are added with probability p, the total number of LR bonds is pN^2 for large N (N is the number of nodes). Even with $p \sim 1/N$, i.e., with a finite number of LR bonds, a phase transition has been observed in the Ising model [5]. While considering the dynamics of Ising models on addition type network, again $p = \gamma/N$ has been the usual choice, where γ is a finite quantity. The dynamical behaviour of such a network for any value of $\gamma > 1$ is comparable to that of the random graph in the sense that the system fails to reach its global minimum energy with zero temperature Glauber dynamics even in the thermodynamic limit [13,14]. Quenching dynamics of the Ising model on the scale-free network has also been considered recently [19] with an average connectivity $k = pN$ for each node. Here k was fixed such that $p \sim 1/N$. The results again show a freezing at zero temperature.

In our study, we have considered an addition type network generated from a one-dimensional chain of Ising spins with nearest neigbours. Here each node has pN number of LR bonds with p fixed (i.e., finite in the limit $N \to \infty$) such that the network is a densely connected network. In this network, we have shown that the freezing or blocking effect is removed for any $p > 0$ in the thermodynamic limit. This conclusion is reached by studying various quantities like the domain sizes, magnetisation, residual energy, number of flipped spins etc. as functions of time. We also find that the relaxation to the equilibrium state is exponential.

The study of persistence of the Ising model on this network shows that there is no algebraic decay as it reaches a constant finite value similar to what happens in lattices of dimension greater than three [25]. The behaviour of $P(t)$ with p is described in detail later in the paper.

In section II we describe the dynamical model under consideration and the physical quantities calculated. The topological properties of the network are described in section III where we find that there is a novel behaviour of the network as far as small world property is concerened. The results are discussed in section IV. Summary and some conclusive comments have been given in section V.

II. THE MODELS AND DYNAMIC PROPERTIES STUDIED

We have considered a one-dimensional ferromagnetic Ising model on a network, in which, apart from nearest neighbour links, there exist some random long range connections with probability p. The Hamiltonian for this system is

$$H = -\sum_i J_{ij} s_i s_j, \tag{1}$$

where $s_i = \pm 1$ is the state of the spin at the ith site, $J_{ij} = 1$ for nearest neighbours and for other neighbours equal to 1 with probability p. The Hamiltonian should be divided by a factor of pN, however, at $T = 0$, only the sign of the energy differences are required for the Glauber dynamics and therefore this factor has not been included. We have simulated this system with periodic boundary condition. The initial configuration is random and single spin flip Glauber dynamics has been used for subsequent updating, i.e., a spin is picked up at random and flipped if the resulting configuration has lower energy, never flipped if the energy is raised and flipped with probability 1/2 if there is no change in energy on flipping.

We have estimated the following quantities in the system.
(1) Average domain size $D(t)$ at time t.
(2) Magnetization $m(t)$ as a function of time (the average magnetisation being zero from symmetry, here $m(t)$ has been calculated by taking the average of the absolute values of the magnetisation).
(3) Residual energy $E_r(t) = E(t) - E_g$ where E_g is the energy of the ground state and $E(t)$ the energy at time t.
(4) $N_{flip}(t)$, the number of spin flips at time t per unit time.
(5) Persistence probability $P(t)$ defined as in section I.

When considering the domains, we have in mind the original one-dimensional lattice and measure the domain size along it.

In finite dimensional nearest neighbour Ising models, the dynamics of these quantities is governed by the exponents θ and z, i.e.,
$D(t) \sim t^{1/z}$
$m(t) \sim t^{1/2z}$
$E_r(t) \sim t^{-1/z}$
$N_{flip}(t) \sim t^{-1/z}$
$P(t) \sim t^{-\theta}$
with $\theta = 0.375$ and $z = 2$ in one dimension.
In principle, it is not essential to study all these quantities to determine the characteristics of the dynamical system. However, in a numerical study, it is better to check that the behavior of these different quantities is consistent with a single z and θ .

Since freezing is a key question here, we have also calculated the freezing probability $F(p)$ as a function of p for various system sizes. Several other quantities and exponents related to domain dynamics in the quenching phenomena can be defined [29], but here we have restricted our study to those mentioned above.

In the simulations, we have restricted the system sizes to $N \leq 1000$ as a large number of configurations is required to have accurate data. The results have been averaged out over 1000 initial configurations and network configurations (typically). As the network is densely connected, the updatings consume a lot of CPU time forcing us to restrict our study to rather small system sizes.

III. TOPOLOGICAL PROPERTIES OF THE NETWORK

The essential difference between a simple one dimensional lattice with nearest neighbour links and the present network lies in the topology of the networks. The topological properties of a network like the average shortest distances $\langle S \rangle$ (here distance means number of steps required to reach another node) and the clustering coefficient $\mathcal{C}$ may help in understanding the static and dynamical phenomena of Ising model on such network. In the nearest neighbour lattice, $\langle S \rangle$ is proportional to N and $\mathcal{C}$ is zero (as there are no loops).

Super small world effect: The average shortest distance behaves in the expected manner; $\langle S \rangle$ is very small ($O(10)$) for very small values of p and decreases to the exact value 1 in the $p \to 1$ limit. In a small world network, the shortest distance is supposed to scale as $\ln(N)$. Here, if p is kept fixed, $\langle S \rangle$ actually decreases with N showing that it approaches the behaviour corresponding to $p = 1$ in the thermodynamic limit. This is a novel behaviour which one can call a "super small world" effect as noted earlier in [31]. One can also compare data for different $N's$ by keeping the number of edges per site, pN, a constant rather than p. Therefore, in two networks of size N_1 and N_2, the values of p are kept p_1 and p_2 respectively such that $p_1 N_1 = p_2 N_2$. We then observe that $\langle S_{N_1,p_1} \rangle / \langle S_{N_2,p_2} \rangle = \ln(N_1)/\ln(N_2)$ which is true for the conventional small world networks. For example, for $N_1 = 100$ and $p_1 = 0.05$, $\langle S_{100,p_1} \rangle = 2.58$ and for $N_2 = 500$ and $p_2 = 0.01$, $\langle S_{500,0.01} \rangle = 3.43$, and the ratio of these two quantities is very close to that of $\ln(100)$ and $\ln(500)$.

The clustering coefficient is estimated in the standard way: the probability that two neighbours of a particular node are also connected to each other is a measure of the clustering tendency. $\mathcal{C}$ is also calculated for different p and N values. Obviously it increases with p and is equal to one at $p = 1$ when all the nodes are connected to each other. Since the nearest neighbours are already connected, clustering coefficient would increase if the long range bonds happen to be next nearest neighbours [32]. For small values of p, this is more probable in smaller lattices and therefore clustering coefficient shows a marked decrease with N. Fig. 1 shows the data for both $\langle S \rangle$ and $\mathcal{C}$.

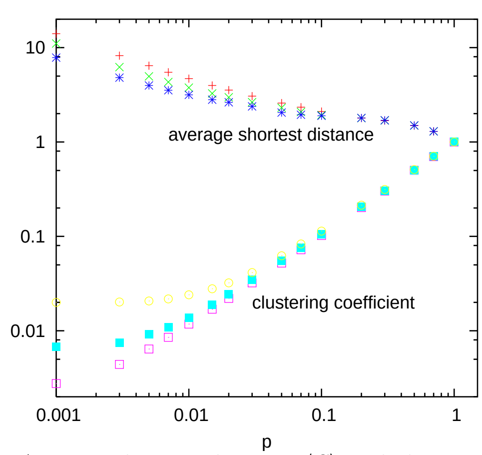

FIG. 1. Average shortest distance $\langle S \rangle$ and clustering coefficient $\mathcal{C}$ as functions of p for different system sizes. In both cases, the values are lesser for larger N values.

These results will be helpful to interpret the observations which we have made and will be referred to in the later sections.

IV. RESULTS AND DISCUSSIONS

We first discuss the results for the growth of the domains sizes and magnetisation. The domain sizes have been scaled by the system sizes N such that $D(t) \leq 1$. We have verified that for $p = 0$, all the physical quantities under consideration follow the known behaviour summarised in section II (with z and θ assuming the values corresponding to one dimension). As soon as a non-zero p value is introduced, both $m(t)$ and $D(t)$ quickly reach a saturation value such that a variation with time occurs only over a short initial period of time (Fig. 2).

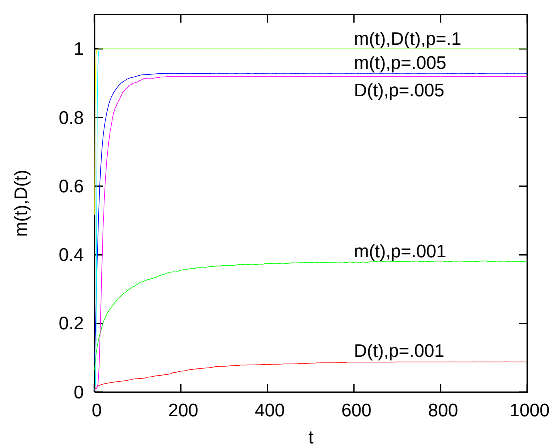

FIG. 2. Magnetisation $m(t)$ and domain size $D(t)$ as functions of time t for different values of p with the system size $N = 500$. Both saturate very quickly as p is increased.

For small system sizes and at low p, the saturation values of $m(t)$ and $D(t)$ are far from those of the equilibrium values (both unity at T=0) similar to the results on random graphs and sparsely connected small world networks. This apparently suggests that the system gets "frozen" in one of the metastable states. However, in the present model, blocking seems to be effective only for finite sizes, as the saturation values of both magnetisation and average domain size approach unity (i.e., the equilibrium value) when the system size is increased (Fig. 3). This is true for any finite value of $p \neq 0$. The apparent blocking effect is more prominent for small values of p.

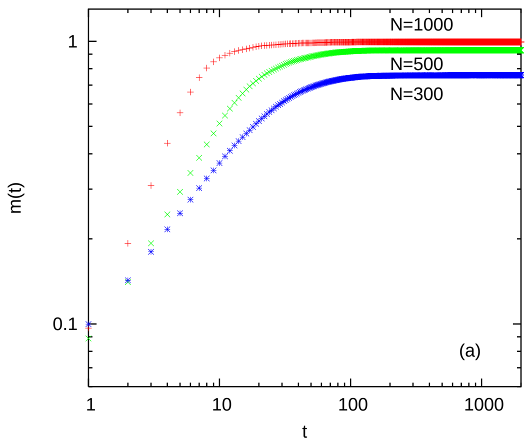

FIG. 3. Magnetization $m(t)$ as function of t for different system sizes at $p = .005$ is shown. Saturation value is higher for larger size at a particular p implying that blocking is a finite size effect here. This behaviour is true for all $p > 0$. The domain size $D(t)$ shows similar behaviour.

The comparison of $m(t)$ and $D(t)$ for different system sizes also shows that the initial growth becomes rapidly sharper with the system size, so that any time dependence in the initial period loses its significance. The period over which this growth takes place also shrinks in size in larger system sizes (Fig. 3) signifying a very fast growth in the magnetization.

The study of the distribution of magnetisation is also consistent with the fact that blocking occurs for finite sizes only. The distribution has finite values for all values of m ($-1 < m < 1$) for small N, but for larger sizes has non-zero values very close to $m = \pm 1$ only.

The fast growth in $m(t)$ and $D(t)$ is supported by the behaviour of $E_r(t)$ and $N_{flip}(t)$ as both show an exponential decay ($E_r(t)$, $N_{flip}(t) \sim \exp(-\alpha t)$) with $\alpha = 1$ for all p (Fig. 4).

Freezing probability

The above observations suggest that for finite sizes, the probability that the system goes to a frozen state is finite for small values of p. Since for the limiting cases $p = 1$ and $p = 0$, there is no freezing, it is expected that the freezing probability $F(p)$ will have a peak for a non-zero value of p. We study the freezing probability for fixed values of N and find out some interesting features. There is indeed a peak occurring at $p = p_m$ with $p_m \sim 1/N$. Except for very small values of p, $F(p)$ decreases with N signifying the disappearance of freezing in the thermodynamic limit. For small values of p, the behaviour may be different. Specifically, if $p = 1/N$, we find that the freezing probability *increases* with N, which is in consistency with the observation of [19]. But at this value of p, the network is a sparsely connected one and therefore it has a freezing tendency indicated by this increase. In fact, keeping p fixed at a certain value such that $p < p_m$ (for the system sizes concerned) we find that the freezing tendency gets enhanced with N. Also, we find that $F(p)$ shows an exponential decay beyond p_m: $F(p) \sim \exp(-\gamma p)$ for large values of p with $\gamma \sim N$.

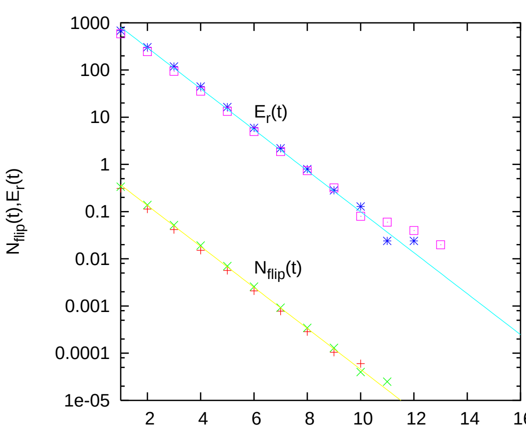

FIG. 4. $N_{flip}(t)$ and scaled residual energy $E_r(t)/p$ as function of t for two different values of p ($p = .1$ and $.8$) with the system size $N = 1000$. Both $N_{flip}(t)$ and $E_r(t)$ follow exponential decay as $\exp(-\alpha t)$ with $\alpha = 1$ for all p and all system sizes. The straight lines have slope $= 1$.

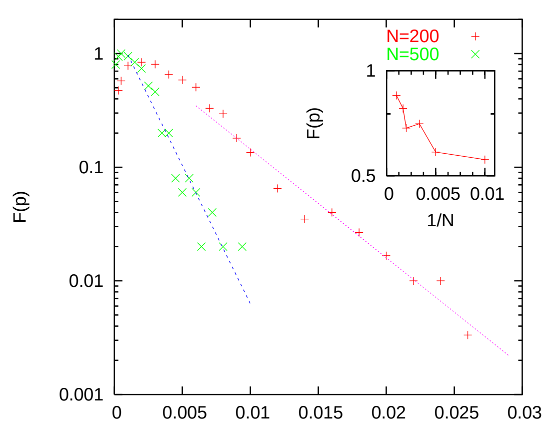

FIG. 5. Freezing probability $F(p)$ as a function of p for different system sizes. $F(p)$ has a peak which shifts towards zero for larger N. The tail of $F(p)$ follows exponential decay as $\exp(-\gamma p)$ with $\gamma \sim N$ shown by the dotted lines. Inset shows $F(p)$ against $1/N$ for $p = 1/N$.

These results (shown in Fig. 5) indicate again that there is a sharp discontinuity in the dynamical behaviour at $p = 0$ and that freezing disappears for any non-zero p for large values of N as long as p is a finite quantity.

In reference [15], it had been shown how the domain walls get pinned when the number of extra bonds is $O(N)$, i.e., $p \sim 1/N$ and the system freezes. The domain walls become mobile as soon as the number of long range interactions increase to $O(N^2)$ and ultimately they disappear by annihilating each other. Thus, in a densely connected infinite network, freezing disappears for any $p \neq 0$.

We next study the behaviour of the persistent probability $P(t)$ with time for different values of p (Fig. 6). $P(t)$ follows the well known power law decay for $p = 0$, but quickly falls to a finite saturation value P_{sat} for any non-zero value of p. The decay is sharper for higher values of p. The saturation behaviour is similar to that of $D(t)$ and $m(t)$.

The behaviour of P_{sat} with different system sizes however, shows an interesting feature. In a finite d-dimensional lattice of size L, when $P(t)$ decays algebraically in time with the exponent θ, the persistent probability as a function of t and L is given by

$$P(L,t) \sim t^{-\theta} f(t/L^z), \qquad (2)$$

where $f(x) =$ constant for small x and $f(x) = x^{\theta}$ for large x such that $P(L, t \to \infty) \sim L^{-z\theta}$ which indicates that the time independent persistence probability decreases with N, where $N = L^d$, the total number of lattice sites. For non-zero values of p, when there is no algebraic decay of the persistent probability with time, we find that (Fig. 7) there exists a value of $p = p^*$ below which the persistence probability actually increases with N, the increase with N being slower than a power law. Beyond p^*, P_{sat} decreases with N but the decrease is fairly weak. However for both $p < p^*$ and $p > p^*$, $P(N, t, p)$ approaches a constant close to 0.5 from below and above respectively. That the saturation value of the persistence probability is close to 0.5 can be justified: initially fifty persent spins are up/down, and spins of only one type are flipped only within the short time the system reaches the equilibrium state. The value of $p^* \simeq 0.25$.

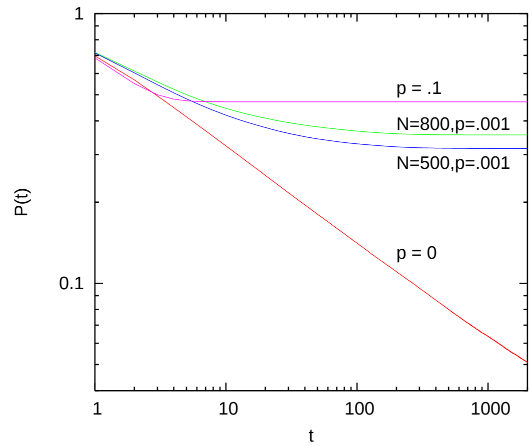

FIG. 6. Persistence $P(t)$ as a function of time t for different probability p for system size $N = 500$. $P(t)$ follows the well known power law decay as $t^{-\theta}$ with $\theta = 0.375$ at $p = 0$. At finite p, $P(t)$ decays to a constant value within a few time steps. For higher p, the decay is even faster and the dynamics stops earlier.

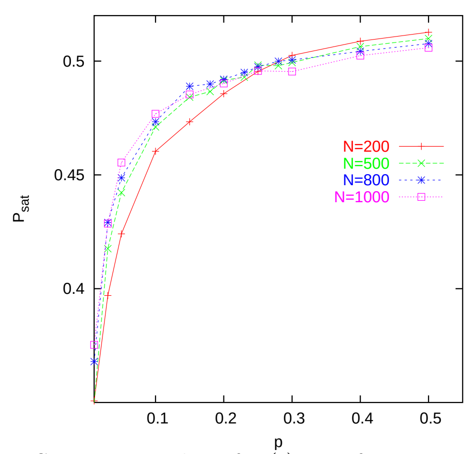

FIG. 7. Saturation value of $P(t)$ as a function of p for different system sizes.

V. SUMMARY AND CONCLUSIONS

We have studied the dynamics of the ferromagnetic Ising model on a small-world network at zero temperature. The network is densely connected in the sense that there is a finite number of extra bonds for each node. It is observed that addition of long range bonds with probability p brings the initial random configuration of Ising spins to the equilibrium ferromagnetic configuration within a very few time steps even if p is very small. Magnetisation and average domain size quickly reach the saturation values without showing any scaling behaviour with time. Consistent with this observation, the residual energy and the number of flippings at any time shows an exponential decay. The study of the freezing probability shows a peak occurring in the distribution at $p_m \sim 1/N$ and an exponential decay which becomes faster with the system size. Persistence probability also reaches a saturation value within a few time steps. These results obtained for the quenching dynamics indicate that the dynamical behaviour of this densely connected network is much different compared to that of random graphs and sparsely connected SWN where the same dynamics leads to a frozen state not equivalent to the equilibrium ground state. That there is no power law behaviour but exponential relaxation is consistent with the mean field behaviour of the network [30].

In order to explain our observations, we notice that the present model is highly clustered as the density of connections is large (see section III). In contrast, the addition type small world network (with two nearest neighbour only), generated with a $p \sim 1/N$ has a vanishing clustering coefficient even though it has small world property [32]. This is in fact the reason for its behaviour as a random graph which also has a small world effect but vanishing clustering coefficient. The large clustering in our model is effective in making the domains entirely non-local and therefore the system can reach the global equilibrium state very fast. The super small world effect, by which we mean that the average shortest distance decreases with N (tending to unity), also helps in understanding the results. For any $p \neq 0$, the network flows towards the $p = 1$ fixed point (for large sized networks) for which one does not expect any freezing.

Our results are consistent with some very recent observations [19] where the dynamics of Ising model on a scale free network with increasing number of connectivity $k = pN$ has been studied. From the data shown, one can obtain the freezing probability as a function of a fixed p (e.g., $p = 0.1$) as well, which shows that freezing will *disappear* for large system sizes.

Usually persistence probability has a unique behaviour and is governed by an independent exponent θ not related to the dynamic exponent z which dictates the behaviour of $D(t)$, $m(t), E_r(t)$ and $N_{flip}(t)$. Here one cannot make any statement about the independence of persistence and the domain growth phenomena. The only distinctive behaviour of persistence seems to be a difference in behaviour with finite sizes occurring for $p < p^*$ where $p^* \simeq 0.25$. At present our understanding of the network and the dynamics is not enough to explain the significance of p^*, although the limiting value of the persistence probability being 0.5 is easily explained from the mean field nature of the network.

From the results reported in the present paper, we conclude that for quenching dynamics, for any $p \leq 1/N$ the time evolution leads to a frozen state far from equilibrium whereas with a finite p freezing is overcome. The finite density of connections thus acts as a driving force, like a finite temperature, which drives the system out of the frozen state. This may appear as a dynamical phase transition in finite systems, however, in the thermodynamic limit ($N \to \infty$) of course, there is no such transition.

We finally remark that as far as statics is concerned, a finite value of p is not required to get a phase transition but the finiteness of p is essential to remove the dynamic frustration when dynamics is considered.

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