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Wave propagation in a rotating randomly varying granular generalized thermoelastic medium

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ABSTRACT

Elastic wave propagation has been explored in an infinite granular thermoelastic medium rotating with constant speed. The elastic and thermal parameters of the granular medium are taken to be randomly fluctuated so that the medium represents the randomly fluctuating inhomogeneous medium. The method of smooth perturbation has been used, which requires the inversion of a deterministic differential operator to find the solution of governing equations in the relevant media. The analysis is based on the dynamics of granular medium as propounded by N. Oshima. All field parameters are functions of space vector and time. A general dispersion equation for waves propagating in the rotating random granular generalized thermal elastic medium has been obtained. The compression and shear wave propagations have been studied. It has been pointed out that in the case of compression waves, the mean and auto-correlation function of the thermo-mechanical coupling parameter greatly influence the mean wave propagation. For shear waves, however, randomness has no effect on wave propagation. Effects of non-random granular elastic medium, randomness and rotation of the frame of reference are discernible from analyses of dispersion equations. The study may find applications in soil mechanics, seismology and oil-prospecting. Computational results have been shown.

1. Introduction

The study of randomness in relation to wave propagation in different media has been steadily drawing attention of applied mathematicians from the time of Chernov [1]. Various aspects of wave propagation phenomena are being investigated by authors with a view to characterizing random variation of properties of medium. The propagation of elastic waves, among others, in a random medium was treated by Keller [2] by using the method of smooth perturbation. Keller's smooth perturbation technique requires an inversion (L_0^{-1}) of the deterministic differential operator L_0 to be performed in order to enable the evaluation of the mean field quantity. Accordingly, in every application of this technique, the computation of an appropriate Green's tensor becomes unavoidable. Ryzhik, Papanicolaou and Keller [3] derived and analysed transport equations for energy density of waves of any kind in random medium. Beran and McCoy [4] enunciated iterative perturbation theory in discussing mean field variation in random dielectric and other media. The mean field equation, unlike Keller's smooth perturbation technique, was presented in terms of an infinite sequence of correlation functions associated with the field parameter(s). In another paper Beran and McCoy [5] discussed in detail the method of determination of the mean field quantities in a statistical sample of heterogeneous linearly elastic solids. Beran, Frankenthal, Deshmukh and Whitman [6] studied propagation of radiation in time-dependent three-dimensional random media. Many other methods pertaining to

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Nomenclature

$\vec{u}$	Linear displacement vector
$\vec{\xi}$	Rotation vector
θ	Generalized temperature
λ	Lame' parameter
μ	Lame' parameter
A	Coefficient of friction between the individual grains
B	Third elastic parameter
ν	Thermal diffusivity
ρ	Density
α	Thermal expansion coefficient
C_s	Specific heat
m	Thermomechanical coupling parameter
t_0, t_1	Generalized thermal parameters
θ_0	Reference temperature
$\vec{F}$	Constant linear body force
$\vec{\psi}$	Angular body force
$\vec{Q}$	Heat source
$\vec{k}$	Wave vector
δ_{lk}	Kronecker delta.

(Graphs were drawn taking $\delta_{lk} = 1 \cdot 0$ *conforming to L–S theory*).

random character of inhomogeneities of media have been developed. A detailed mathematical discussion can be found in treatises of Bharucha Reid [7], Uscinski [8] and others. Sobczyk [9a] studied elastic wave propagation in a discrete random medium. He developed a general formalism of the analysis of mean (or coherent) elastic waves in terms of scatterers. Sobczyk, Wedrychowicz and Spencer Jr [9b] analysed the dynamics of structural systems with randomly varying parameters. Wenzel [10] made a theoretical study on radiation and attenuation of waves in a random medium. Chow [11] employed smooth perturbation in studying thermoelastic waves in a random medium. Bera [12] investigated wave propagation in a rotating magneto-thermoelastic medium under generalized thermoelasticity involving two relaxation times, t_0, t_1, studying randomness of interacting media employing smooth perturbation method. Bhattacharyya [13a] investigated the problem of reflection of waves, incident on the plane boundary of a semi-infinite elastic medium with randomly varying inhomogeneities. Mitra and Bhattacharyya [13b] recently studied wave propagation in a random micropolar thermal elastic medium, second moments and associated Green's tensor.

In this paper we shall adopt Keller's method to analyse the mean generalized thermo-elastic wave in a slightly random, inhomogeneous granular elastic medium under a rotating frame of reference. By this we mean that the density, the *Lame*' constants, the thermal expansion coefficient, the specific heat, the thermal diffusivity and parameters defining granular character of the elastic medium are all random functions of positions. The medium is assumed to rotate under a frame of reference defined in Schöenberg and Censor [14]. Many authors have studied effect of rotation for different types of problems of wave propagation in different media. Recently Tomar and Ogden [15] studied wave propagation in an isothermal linear isotropic elastic material with voids rotating with constant angular velocity about the z-axis. On the other hand, Bakshi, Bera and Debnath [16] investigated effects of rotation in studying magneto-thermoelastic problems with thermal relaxation and heat sources in a three-dimensional infinite elastic medium. Earlier, Roy Chaudhuri and Debnath [17] investigated magneto-thermoelastic plane waves in rotating media; effects of rotation, though small, were computed for the phase speed, attenuation and specific energy loss, etc.

The granular character of the medium has been defined under the dynamics of granular medium described by Oshima [18] as a Cosserat-continuum. The study of the granular character of the medium finds importance in soil mechanics, earthquake source mechanism in the crustal layer of the earth and in petroleum engineering. The effects though small of the order of second order perturbation, nonetheless, come to influence computational results. The medium under consideration is a discontinuous one being composed of numerous large and small grains. Oshima visualizes that 'unlike a continuous body each element or grain not only translates but also rotates about its centre of gravity. The later motion affects the equation of motion by producing internal friction. It is assumed that the medium contains so many grains that they will never be separated from each other during the deformation and the grain has perfect elasticity.' He uses the concept of averages to reduce the granular medium to a continuous medium by introducing additional physical constants [19]. The stress tensor is not symmetrical. Besides the non-symmetrical stress-tensor there is also the non-symmetric stress couple which is related to a non-symmetric strain-tensor. Also the couple stress theory developed by Eringen [20,21] comprises granular as also composite fibrous materials. As such micropolar or microstructure materials and granular materials present an inclusive model of composite materials.

In our case the medium is considered under generalized thermo-elastic coupling. Bhattacharyya [22] studied Rayleigh wave propagation in a granular medium. Ahmed [23] discussed Rayleigh waves in a thermo-elastic granular medium under initial stress. In a subsequent paper Ahmed [24] investigated propagation of Stoneley waves in a non-homogeneous orthotropic granular elastic medium. Various other models of granularity were formulated by many other authors. A statistical approach to wave propagation in granular medium will be found in Hudson [25]. Beckman et al. [26] used methods of statistical mechanics to model the arrangements and behaviour of granular material while Kitamura [27] used Markov process to study the mechanics of granular materials. In a two part article, Suiker, Borst and Chang [28] discussed the micro-mechanical modelling of granular medium. A second gradient micropolar constitutive theory was developed and plane wave propagation in an infinite non-random medium was discussed. Recently Kruyt and Rothenburg [29] studied quasi-static deformation of granular materials applying DEM simulations, considering granular materials as assemblies of semi-rigid particles where particles interact through contact forces at small contact areas. In one of the latest additions to the literature Jiang and Liu [30] have discussed the unified-continuum-mechanical theory of the granular character of media based on hydrodynamic approach and termed as Granular Solid Hydrodynamics (GSH). Even then it may be recorded here that not much research, theoretical or numerical was reported in the literature based on the model of granularity propounded by Oshima [18] during the last more than half centuries. The present paper endeavours to revisit Oshima's model of the dynamics of granular medium theoretically as also computationally with a view to exploring possibilities of applications in wave propagation in the crustal surface of the earth, soil mechanics and oil-prospecting.

The specific objective of this paper is to study and compute the effect which is small of random variation of inhomogeneities of the coupled thermo-granular-elastic medium. With this end in view we have chosen to employ the generalized heat conduction equation involving two relaxation times. The details of the theory can be found in [12,31–33]. The two-temperature Green–Naghdi thermoelasticity theories or the Fractional order thermoelasticity theory [34–36] can be applied in subsequent studies.

The study may find application in investigating seismic surface wave propagation which has become a subject of intense interest these days amongst seismologists, physicists and natural hazard mitigation scientists. Random effects may help them fine-tune their models for prediction of earthquakes and tsunamis by producing synthetic seismograms.

2. The problem

Let $\vec{u}(u, v, w)$ be the displacement vector in a random granular generalized thermo-elastic medium, θ the generalized temperature and $\vec{\xi}(\xi, \eta, \zeta)$ the angular displacement vector of grains. The stress–strain relationships for the granular elastic medium as propounded by Oshima have been derived in [18] in detail. These results have subsequently been used by Paria [19], Bhattacharyya [22] and others. The stress tensor τ_{ij} is not symmetrical and is written as the sum of a symmetrical and anti-symmetrical tensor as

$$\tau_{ij} = \sigma_{ij} + \sigma'_{ij}$$

where

$$\sigma_{ij} = \frac{1}{2}\left(\tau_{ij} + \tau_{ji}\right)$$
$$\sigma'_{ij} = \frac{1}{2}\left(\tau_{ij} - \tau_{ji}\right).$$

The symmetric tensor $\sigma_{ij} = \sigma_{ji}$ is related to the symmetric strain tensor

$$e_{ij} = e_{ji} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

by the Hooke's law and thus in isotropic medium we have

$$\begin{aligned} \sigma_{xx} &= \lambda e + 2\mu e_{xx}, \qquad \sigma_{yz} = 2\mu e_{yz} \\ \sigma_{yy} &= \lambda e + 2\mu e_{yy}, \qquad \sigma_{zx} = 2\mu e_{zx} \\ \sigma_{zz} &= \lambda e + 2\mu e_{zz}, \qquad \sigma_{xy} = 2\mu e_{xy} \end{aligned}$$

where $e = e_{xx} + e_{yy} + e_{zz}$.

The anti-symmetric stresses σ'_{ij} are related to the rotation components (ξ, η, ζ) as

$$\sigma'_{yz} = -A\frac{\partial \xi}{\partial t}, \qquad \sigma'_{zx} = -A\frac{\partial \eta}{\partial t}, \qquad \sigma'_{xy} = -A\frac{\partial \zeta}{\partial t}$$
$$\sigma'_{xx} = \sigma'_{yy} = \sigma'_{zz}$$

where A is the coefficient of friction between individual grains and t denotes time. Besides the non-symmetric stress-tensor τ_{ij} there is also the non-symmetric stress-couple M_{ij} which is related to the non-symmetric strain-tensor γ_{ij} as $M_{ij} = B\gamma_{ij}$

where B is a third elastic constant. The tensor γ_{ij} is defined as

$$\gamma_{xx} = \frac{\partial}{\partial x}(\varpi_x + \xi), \qquad \gamma_{yy} = \frac{\partial}{\partial y}(\varpi_y + \eta), \qquad \gamma_{zz} = \frac{\partial}{\partial z}(\varpi_z + \zeta)$$
$$\gamma_{yz} = \frac{\partial}{\partial x}(\varpi_z + \zeta), \qquad \gamma_{zx} = \frac{\partial}{\partial z}(\varpi_x + \xi), \qquad \gamma_{xy} = \frac{\partial}{\partial x}(\varpi_y + \eta)$$
$$\gamma_{zy} = \frac{\partial}{\partial z}(\varpi_y + \eta), \qquad \gamma_{xz} = \frac{\partial}{\partial y}(\varpi_x + \zeta), \qquad \gamma_{yx} = \frac{\partial}{\partial y}(\varpi_x + \xi)$$

where

$$\varpi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \varpi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \varpi_z = \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y}.$$

Then the displacement equation of motion for the rotating thermo-granular elastic medium can be written following Schöenberg and Censor [14] and Chow [11] as,

$$\begin{aligned}
&\rho(\vec{x})\ddot{\vec{u}}(\vec{x},t) + 2\vec{\Omega} \times \dot{\vec{u}}(\vec{x},t) + \vec{\Omega} \times (\vec{\Omega} \times \vec{u}(\vec{x},t)) \\
&\quad = (\lambda(\vec{x}) + \mu(\vec{x}))\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}(\vec{x},t)\right) + \mu(\vec{x})\nabla^2\vec{u}(\vec{x},t) + \vec{\nabla}\lambda(\vec{x})\left(\vec{\nabla}\cdot\vec{u}(\vec{x},t)\right) \\
&\qquad + \left(\vec{\nabla}\mu(\vec{x})\right) \times \left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right) + 2\left(\vec{\nabla}\mu(\vec{x})\cdot\vec{\nabla}\right)\vec{u}(\vec{x},t) + A(\vec{x})\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\xi}(\vec{x},t)\right) \\
&\qquad + \frac{\partial}{\partial t}\left(\vec{\nabla}A(\vec{x}) \times \vec{\xi}(\vec{x},t)\right) - \vec{\nabla}\left(m(\vec{x})\left\{\theta(\vec{x},t) + t_1(\vec{x})\dot{\theta}(\vec{x},t)\right\}\right) + \vec{F}.
\end{aligned} \tag{1}$$

The micro-rotation equation of motion of the granular elastic medium can be taken as

$$\begin{aligned}
&B(\vec{x})\nabla^2\left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right) + \left(\vec{\nabla}B(\vec{x})\cdot\vec{\nabla}\right)\left(\vec{\nabla}\times\vec{u}(\vec{x},t)\right) \\
&\quad + \left[-2A(\vec{x})\frac{\partial\vec{\xi}(\vec{x},t)}{\partial t} + B(\vec{x})\nabla^2\vec{\xi}(\vec{x},t) + \left(\vec{\nabla}B(\vec{x})\cdot\vec{\nabla}\right)\vec{\xi}(\vec{x},t)\right] + \vec{\psi} = \mathbf{0}.
\end{aligned} \tag{2}$$

Also the heat conduction equation is represented by

$$\gamma(\vec{x})\left\{\dot{\theta}(\vec{x},t) + t_0\ddot{\theta}(\vec{x},t)\right\} - \vec{\nabla}\cdot\left\{\nu(\vec{x})\vec{\nabla}\left[\theta(\vec{x},t)\right]\right\} + \theta_0 m(\vec{x})\vec{\nabla}\cdot\left\{\dot{\vec{u}}(\vec{x},t) + \delta_{lk}t_0\ddot{\vec{u}}(\vec{x},t)\right\} = \Theta \tag{3}$$

where $\dot{\theta} = \frac{\partial\theta(\vec{x},t)}{\partial t}$, $\ddot{\theta} = \frac{\partial^2\theta(\vec{x},t)}{\partial t^2}$, etc.

Also, λ, μ are *Lame*' parameters, ρ is the density. Here $B(\vec{x})$ represents the new, i.e., the third elastic parameter and $A(\vec{x})$ is the coefficient of friction between individual grains. $m(\vec{x})$ is the thermomechanical coupling parameter, ν is the thermal diffusivity and m, γ are defined to be

$$m = (\lambda + \mu)\alpha, \qquad \gamma = \rho C_s$$

where α, C_s are the thermal expansion coefficient and the specific heat respectively. $\vec{u}$, $\vec{\xi}$, θ are functions of space vector $\vec{x}(x, y, z)$ and time t. The functions $\vec{F}$, $\vec{\psi}$, Θ are constant linear body force, angular body force, and heat source respectively. All parameters including the generalized thermal parameters t_0 and t_1 and A, B are functions of the space vector $\vec{x}$. Also δ_{lk} stands for Kronecker delta such that $t_1 \geq t_0 \geq 0$, $\delta_{lk} = 1$, $t_1 = 0$ for the Lord–Shulman (L–S) theory and $\delta_{lk} = 0$, $t_1 > 0$ for the Green–Lindsay (G–L) theory [12,37,38].

The entire frame of reference is assumed to be rotating with uniform angular velocity $\vec{\Omega} = \Omega\vec{n}$ where $\vec{n}$ is a unit vector representing direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms [14]:

(i) Centripetal acceleration $\vec{\Omega} \times (\vec{\Omega} \times \vec{u}(\vec{x}, t))$ due to the time varying motion only and
(ii) the Coriolis acceleration $2\vec{\Omega} \times \vec{u}(\vec{x}, t)$.

Let us now take for Eqs. (1)–(3)

$$\left\{\vec{u}(\vec{x},t), \vec{\xi}(\vec{x},t), \theta(\vec{x},t)\right\} = e^{-i\omega t}\left\{\vec{u}'(\vec{x}), \vec{\xi}'(\vec{x}), \theta'(\vec{x})\right\} \tag{4}$$

where ω stands for the frequency of thermo-elastic waves propagating in the coupled medium. Dropping the dashes we write the system of equations (1)–(3) in the linear differential operator form as

$$LV = f \tag{5}$$

where

$$L\left(\vec{x}\right)=\begin{pmatrix} M\left(\vec{x}\right) & P\left(\vec{x}\right) & K\left(\vec{x}\right)\\ N\left(\vec{x}\right) & Q\left(\vec{x}\right) & 0\\ R\left(\vec{x}\right) & 0 & S(\vec{x})\end{pmatrix}$$

$$V\left(\vec{x}\right)=\begin{pmatrix}\vec{u}\left(\vec{x}\right)\\ \vec{\xi}\left(\vec{x}\right)\\ \theta\left(\vec{x}\right)\end{pmatrix}$$

$$f\left(\vec{x}\right)=\begin{pmatrix}-\vec{F}\left(\vec{x}\right)\\ -\vec{\psi}\left(\vec{x}\right)\\ -\Theta\left(\vec{x}\right)\end{pmatrix}.$$

The linear differential operators $\{M, P, K, N, Q, R, S\}$, are defined below:

$$M = \rho\left[\omega^2-\vec{\Omega}\times\left(\vec{\Omega}\times\right)+2i\omega\left(\vec{\Omega}\times\right)\right]+(\lambda+\mu)\,\vec{\nabla}\left(\vec{\nabla}\cdot\right)+\mu\left(\nabla^2\right)+\left(\vec{\nabla}\lambda\right)\left(\vec{\nabla}\cdot\right)+\left(\vec{\nabla}\mu\right)\times\left(\vec{\nabla}\times\right)+2\left(\vec{\nabla}\mu\right)\cdot\vec{\nabla}$$

$$P=-i\omega\left[A\left(\vec{\nabla}\times\right)+\left(\vec{\nabla}A\times\right)\right]$$

$$K=-\vec{\nabla}\left\{m\left(1-i\omega t_1\right)\right\}$$

$$N=B\nabla^2\left(\vec{\nabla}\times\right)+\left(\vec{\nabla}B\cdot\vec{\nabla}\right)\left(\vec{\nabla}\times\right)$$

$$Q=2iA\omega+B\nabla^2+(\vec{\nabla}B\cdot\vec{\nabla})$$

$$R=\theta_0 m\left(i\omega+t_0\omega^2\delta_{lk}\right)\left(\vec{\nabla}\cdot\right)$$

$$S=\gamma\left(i\omega+\omega^2 t_0\right)+\vec{\nabla}\cdot\left(\nu\vec{\nabla}\right).$$

It is now assumed that the physical parameters characterizing the medium under consideration are random functions of the space variable $\vec{x}$ the statistics of which are known, viz, the statistical mean, auto- and cross-correlation functions and the variance. Following Keller [2], ([39a], [39b]) the linear operator L is now represented in perturbed form as

$$L=L_0+\epsilon L_1\left(\vec{x}\right)+\epsilon^2 L_2\left(\vec{x}\right), \tag{6}$$

where ϵ measures the scale of fluctuation of random inhomogeneities of the medium and L_0 and (L_1, L_2) are the constant and randomly fluctuating parts of L respectively. Under the circumstances it can be shown that the mean field quantity $\left\langle V\left(\vec{x}\right)\right\rangle$ satisfies the integrodifferential equation

$$\left[L_0+\epsilon\langle L_1\rangle+\epsilon^2\langle L_2\rangle+\epsilon^2\langle L_1\rangle L_0^{-1}\langle L_1\rangle-\epsilon^2\langle L_1L_0^{-1}L_1\rangle\right]\left\langle V\left(\vec{x}\right)\right\rangle=f. \tag{7}$$

The procedure for the derivation of this field equation has been briefly described in Appendix A.

Now it becomes imperative to compute the inverse of the non-random linear differential operator L_0 i.e., L_0^{-1} which is defined as

$$L_0G_{ij}\left(\vec{x},\vec{x}'\right)=\delta\left(\vec{x},\ \vec{x}'\right)\delta_{lj}.$$

We will discuss the matter in the next section.

3. Solution

Towards solving the problem of mean wave propagation it is important to define non-random and random parts of parameters characterizing the randomly fluctuating inhomogeneous medium. Following Karal and Keller [39a] and Chow [11], assuming small random fluctuation, the fluctuating parameters are expressed as the sum of an average non-random part and a slightly randomly fluctuating part as follows:

$$\left(\lambda,\mu,\rho,A,B,m,m^*,\ \gamma,\nu\right)\left(\vec{x}\right)=\left(\lambda_0,\mu_0,\rho_0,A_0,B_0,m_0,m_0^*,\gamma_0,\nu_0\right)+\epsilon\left(\lambda_1,\ \mu_1,\ \rho_1,\ A_1,\ B_1,m_1,m_1^*,\gamma_1,\nu_1\right)\left(\vec{x}\right).$$

The factor ϵ measures the degree of smallness of the fluctuation.

Here $m^*(\vec{x})=m(\vec{x})t_1(\vec{x})$ such that $m_1^*=m_1\left(\vec{x}\right)t_1\left(\vec{x}\right)$. Also following [39a], [11] it is assumed again that in this case $m_0=m_0^*=0$ and that expected values of fluctuating parts vanish such that

$$\left\langle\lambda_1\left(\vec{x}\right)\right\rangle=\left\langle\mu_1\left(\vec{x}\right)\right\rangle=\left\langle\rho_1\left(\vec{x}\right)\right\rangle=\left\langle A_1\left(\vec{x}\right)\right\rangle=\left\langle B_1\left(\vec{x}\right)\right\rangle=\left\langle\gamma_1\left(\vec{x}\right)\right\rangle=\left\langle\nu_1\left(\vec{x}\right)\right\rangle=0,$$

except, $\left\langle m_1\left(\vec{x}\right)\right\rangle=m_2\neq 0$, (constant), $\left\langle m_1^*\left(\vec{x}\right)\right\rangle=m_3\neq 0$, (constant).

These last two assumptions indicate that only the case of weak thermo-elasticity is being considered. The last condition m_3 = constant $\neq 0$, clearly conforms to conditions of generalized thermoelasticity [32] and assumptions of Chow [11].

Using the perturbation operator Eq. (6) and putting $f = 0$, the field Eq. (5) can now be set in the form

$$\left[L_0 + \epsilon L_1\left(\vec{x}\right) + \epsilon^2 L_2\left(\vec{x}\right)\right] V\left(\vec{x}\right) = \mathbf{0} \tag{8}$$

such that

$$L_0 = \begin{pmatrix} M_0 & P_0 & 0 \\ N_0 & Q_0 & 0 \\ 0 & 0 & S_0 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} M_1 & P_1 & K_1 \\ N_1 & Q_1 & 0 \\ R_1 & 0 & S_1 \end{pmatrix}, \qquad L_2 = 0,$$

where

$$M_0 = \rho_0 \varpi^2 + (\lambda_0 + \mu_0)\vec{\nabla}\left(\vec{\nabla}\cdot\right) + \mu_0(\nabla^2)$$
$$P_0 = -i\omega A_0(\vec{\nabla}\times)$$
$$N_0 = B_0\nabla^2(\vec{\nabla}\times)$$
$$Q_0 = 2A_0 i\omega + B_0(\nabla^2)$$
$$S_0 = \gamma_0\left(i\omega + \omega^2 t_0\right) + \nu_0(\nabla^2)$$

and

$$M_1 = \rho_1\varpi^2 + (\lambda_1 + \mu_1)\vec{\nabla}\left(\vec{\nabla}\cdot\right) + \mu_1\left(\nabla^2\right) + \left(\vec{\nabla}\lambda_1\right)\left(\vec{\nabla}\cdot\right) + \left(\vec{\nabla}\mu_1\right)\times\left(\vec{\nabla}\times\right) + 2\left(\vec{\nabla}\mu_1\right)\cdot\vec{\nabla}$$
$$P_1 = -i\omega\left[A_1\left(\vec{\nabla}\times\right) + \left(\vec{\nabla}A_1\times\right)\right]$$
$$K_1 = -\vec{\nabla}(m_1 - i\omega_1 m_1^*)$$
$$N_1 = B_1\nabla^2\left(\vec{\nabla}\times\right) + (\vec{\nabla}\mathrm{B}_1\cdot\ \vec{\nabla})(\vec{\nabla}\ \times)$$
$$Q_1 = 2A_1 i\omega + \left(B_1\nabla^2\right) + (\vec{\nabla}B_1\cdot\vec{\nabla})$$
$$R_1 = \theta_0 m_1\left(i\omega + t_0\omega^2\delta_{lk}\right)(\vec{\nabla}\cdot)$$
$$S_1 = \gamma_1\left(i\omega + \omega^2 t_0\right) + \nu_1\left(\nabla^2\right) + (\vec{\nabla}\nu_1\cdot\vec{\nabla})$$

where $\varpi^2 = \omega^2 - \vec{\Omega}\times\left(\vec{\Omega}\times\right) + 2i\omega\left(\vec{\Omega}\times\right)$.

Next let us assume for the mean field equation (7)

$$\left\langle V\left(\vec{x}\right)\right\rangle = \begin{pmatrix} \vec{A} \\ \vec{B} \\ C \end{pmatrix} e^{i\vec{k}\cdot\vec{x}} \tag{9}$$

where $\vec{k}$ is the wave vector. The components of Green's tensor corresponding to $L_0\langle V\rangle = 0$, were computed earlier by Chattopadhyay and Bhattacharyya [40] in the form:

$$G_{lj} = \begin{pmatrix} G_0 & G_1 & 0 \\ G_2 & G_3 & 0 \\ 0 & 0 & G_4 \end{pmatrix} \tag{10}$$

where

$$G_4(r) = \frac{-1}{4\pi r}e^{i\beta r}, \qquad r = |\vec{x} - \vec{\acute{x}}| \tag{11}$$

$$\beta = \sqrt{\left\{\frac{\omega\gamma_{0(i+t_0\omega)}}{\nu_0}\right\}} = \sqrt{\left\{\frac{\omega\gamma_0}{2\nu_0}\right\}}\left[\left\{\sqrt{1 + t_0^2\omega^2} + t_0\omega\right\}^{\frac{1}{2}} + i\left\{\sqrt{1 + t_0^2\omega^2} - t_0\omega\right\}^{\frac{1}{2}}\right]$$
$$= \beta_1 + i\beta_2$$

and

$$\beta_1 = \sqrt{\left\{\frac{\omega\gamma_0}{2\nu_0}\right\}}\left\{\sqrt{1 + t_0^2\omega^2} + t_0\omega\right\}^{\frac{1}{2}}, \qquad \beta_2 = \sqrt{\left\{\frac{\omega\gamma_0}{2\nu_0}\right\}}\left\{\sqrt{1 + t_0^2\omega^2} - t_0\omega\right\}^{\frac{1}{2}}.$$

Tensors G_0, G_1, G_2, G_3 can be found in [40], the components having been reproduced in Appendix B. Here $\vec{x}$ is the field point and $\vec{x}'$ is the source point where $r = |\vec{x} - \vec{x}'|$, $d\vec{r} = -d\vec{x}'$.

Now writing

$$
\begin{aligned}
e^{i\vec{k}\cdot\vec{x}} f_1 \vec{A} &= \langle M_1G_0M_1 + M_1G_1N_1 + P_1G_2M_1 + P_1G_3N_1 + K_1G_4R_1\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} f_2 \vec{B} &= \langle M_1G_0P_1 + M_1G_1Q_1 + P_1G_2P_1 + P_1G_3Q_1\rangle(\vec{B}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} f_3 C &= \langle M_1G_0K_1 + P_1G_2K_1 + K_1G_4S_1\rangle(Ce^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} g_1 \vec{A} &= \langle N_1G_0M_1 + N_1G_1N_1 + Q_1G_2M_1 + Q_1G_3N_1\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} g_2 \vec{B} &= \langle N_1G_0P_1 + N_1G_1Q_1 + Q_1G_2P_1 + Q_1G_3Q_1\rangle(\vec{B}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} g_3 C &= \langle N_1G_0K_1 + Q_1G_2K_1\rangle(Ce^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} h_1 \vec{A} &= \langle R_1G_0M_1 + R_1G_1N_1 + S_1G_4R_1\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} h_2 \vec{B} &= \langle R_1G_0P_1 + R_1G_1Q_1\rangle(\vec{B}e^{i\vec{k}\cdot\vec{x}})\\
e^{i\vec{k}\cdot\vec{x}} h_3 C &= \langle R_1G_0K_1 + S_1G_4S_1\rangle(Ce^{i\vec{k}\cdot\vec{x}}),
\end{aligned}
\tag{12}
$$

and substituting these values in (7), setting $f = 0$ and neglecting ϵ^3 and ϵ^4 terms and after simplifications we derive the following equation:

$$
\begin{aligned}
&(\lambda_0+\mu_0)\left(\vec{k}\cdot\vec{A}\right)\vec{k} + \left(\mu_0k^2 - \rho_0\varpi^2\right)\vec{A} + \frac{i\omega A_0B_0k^2}{B_0k^2 - 2i\omega A_0}\vec{k}\times\left(\vec{k}\times\vec{A}\right)\\
&+\epsilon^2\frac{i\theta_0m_2\left(\vec{k}\cdot\vec{A}\right)\left(\omega - i\omega^2t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_0\left(i\omega+\omega^2t_0\right)-\nu_0k^2}\vec{k} - \epsilon^2 i\theta_0m_2\left(\vec{k}\cdot\vec{A}\right)(m_2 - i\omega m_3)\left(i\omega+\omega^2t_0\delta_{lk}\right)\\
&\times\int\vec{\nabla}G_4(r)\,e^{-i\vec{k}\cdot\vec{r}}d\vec{r} - \epsilon^2\frac{\omega A_0}{B_0k^2 - 2A_0i\omega}\vec{k}\times\int g_1\vec{A}e^{-i\vec{k}\cdot\vec{r}}\,d\vec{r} - \epsilon^2\frac{i\omega A_0B_0k^2}{\left(B_0k^2 - 2A_0i\omega\right)^2}\vec{k}\\
&\times\int g_2\left(\vec{k}\times\vec{A}\right)e^{-i\vec{k}\cdot\vec{r}}d\vec{r} - \epsilon^2\int\left(f_1\vec{A} - f_2\frac{iB_0k^2\left(\vec{k}\times\vec{A}\right)}{B_0k^2 - 2A_0i\omega}\right)e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = \mathbf{0}.
\end{aligned}
\tag{13}
$$

This is the **general dispersion equation** for the propagation of waves in the random weakly generalized thermal granular elastic medium under the rotating frame of reference. The presence of ϖ^2 indicates dependence of $\vec{\Omega}$ on the propagation of waves. No ϵ order terms appear in the equation, indicating that effects of random granular elastic character as also of thermal field are small to order ϵ^2 only.

The ϵ^2 order terms except one, are integrals involving numerous auto- and cross-correlation functions between various elastic, thermal and granular parameters of the interacting medium. These terms represent effects of randomness of inhomogeneities of the medium. It would however be extremely laborious to undertake an analysis of the general dispersion equation (13). If however, we decide to omit all correlation functions, then the dispersion equation (13) reduces to (for both L–S and G–L thermoelasticity):

$$
\begin{aligned}
&(\lambda_0+\mu_0)\left(\vec{k}\cdot\vec{A}\right)\vec{k} + \left(\mu_0k^2 - \rho_0\varpi^2\right)\vec{A} + \frac{i\omega A_0B_0k^2}{B_0k^2 - 2A_0i\omega}\left[\left(\vec{k}\cdot\vec{A}\right)\vec{k} - k^2\vec{A}\right]\\
&+\epsilon^2\frac{i\theta_0m_2\left(\vec{k}\cdot\vec{A}\right)\left(\omega - i\omega^2t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_0\left(i\omega+\omega^2t_0\right)-\nu_0k^2}(\vec{k}\cdot\vec{A})\vec{k} = \mathbf{0}.
\end{aligned}
\tag{14}
$$

We are left with only one ϵ^2 level term representing effects of randomness due to generalized thermal field being dependent on expected values m_2 and m_3. No integral terms appear in the simplified equation. The effects of non-random granular elastic parameters and rotation of the frame of reference are also discernible from this equation. This equation therefore is significant in that it exhibits random thermal behaviour as also non-random granular elastic properties. We will later take up this equation for further investigation.

Analysis of Eqs. (13) and (14):

The analyses carried out in cases I–IV below, based on the generalized dispersion equation (13), propose to highlight some interesting properties of wave propagation phenomena in the interacting random and non-random media.

CASE I

An analysis of the simpler dispersion equation (14) involving a single ϵ^2 order term assumes importance in the absence of an analysis of the more complicated general dispersion equation (13). This analysis will enable one to discern effects of randomness of different parameters under varying conditions of wave propagation. For compression waves we get from Eq. (14):

$$(\lambda_0+2\mu_0)k^2\vec{A}=\left[\omega^2\vec{A}-\left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega}+\Omega^2\vec{A}+2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right]-\epsilon^2\frac{i\theta_0 m_2\left(\omega-i\omega^2 t_0\delta_{lk}\right)(m_2-i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}}k^2\vec{A}. \tag{15}$$

This means that for compression waves even non-random granular character of the medium is not discernible at all but the rotation of the frame and random generalized thermal field parameters are observed to be quite effective.

Next taking $\vec{\Omega}=(0,0,\Omega)$, $\vec{A}=(A_1,A_2,A_3)$, we get two fourth degree equations in k:

$$\begin{aligned}&\nu_0(\lambda_0+2\mu_0)k^4-\left\{(\lambda_0+2\mu_0)\left(i\omega+\omega^2 t_0\right)\gamma_0+\rho_0\nu_0\omega^2\right.\\&\quad\left.+\epsilon^2 i\theta_0 m_2\left(\omega-i\omega^2 t_0\delta_{lk}\right)(m_2-i\omega m_3)\right\}k^2+\rho_0\gamma_0\omega^2\left(i\omega+\omega^2 t_0\right)=0\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\nu_0(\lambda_0+2\mu_0)k^4-\left\{(\lambda_0+2\mu_0)\left(i\omega+\omega^2 t_0\right)\gamma_0+\rho_0\nu_0(\omega^2+\Omega^2)+\epsilon^2 i\theta_0 m_2\left(\omega-i\omega^2 t_0\delta_{lk}\right)(m_2-i\omega m_3)\right.\\&\quad\left.\pm 2\omega\Omega\rho_0\nu_0\right\}k^2+\rho_0\gamma_0(\omega^2+\Omega^2)\left(i\omega+\omega^2 t_0\right)\pm 2\omega\Omega\rho_0\gamma_0\left(i\omega+\omega^2 t_0\right)=0.\end{aligned} \tag{17}$$

Eq. (16) is independent of angular velocity $\vec{\Omega}$ but dependent on generalized thermal parameters θ_0, t_0 and mean non-zero thermomechanical coupling parameters m_2 and m_3. Eq. (17) is dependent on angular velocity $\vec{\Omega}$ and generalized thermal parameters, θ_0, $t_{0.}$ as also m_2 and m_3. So, two different types of compression waves are found to propagate in the interacting granular elastic medium.

CASE II

Rejecting the term to the order of ϵ^2, one gets from (14), the purely non-random period equation:

$$(\lambda_0+\mu_0)\left(\vec{k}\cdot\vec{A}\right)\vec{k}+\left(\mu_0 k^2-\rho_0\varpi^2\right)\vec{A}+\frac{i\omega A_0 B_0 k^2}{B_0 k^2-2A_0 i\omega}\left[\left(\vec{k}\cdot\vec{A}\right)\vec{k}-k^2\vec{A}\right]=\mathbf{0}. \tag{18}$$

Eq. (14) is decoupled. This turns out to be the dispersion equation for waves dependent on $\vec{\Omega}$ and propagating in the non-random, non-thermal granular elastic medium.

In this case, for compression waves one gets the dispersion equation as

$$\left[(\lambda_0+2\mu_0)k^2-\rho_0(\omega^2+\Omega^2)\right]\vec{A}+\rho_0\left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega}-2i\omega\rho_0\left(\vec{\Omega}\times\vec{A}\right)=\mathbf{0}. \tag{19}$$

Eq. (14) is further decoupled. This means for compression waves period equation becomes altogether independent of granular character of the medium but the elastic wave propagation is strongly dependent on rotation of the frame.

Next setting $\vec{\Omega}=(0,0,\Omega)$, $\vec{A}=(A_1,A_2,A_3)$, one gets from (19) the following two equations:

$$k=\omega\sqrt{\frac{\rho_0}{(\lambda_0+2\mu_0)}} \tag{20}$$

and

$$k=\sqrt{\frac{\rho_0(\omega^2+\Omega^2)\pm 2\omega\Omega\rho_0}{(\lambda_0+2\mu_0)}}. \tag{21}$$

In Eq. (20), k depends on λ_0, μ_0, ρ_0 but is independent of angular velocity. In Eq. (21), k strongly depends on angular velocity. If $\Omega=$ then these equations reduce to familiar equations of elastic wave propagation in the non-random medium.

CASE III

Let us consider Eq. (18) generally:

$$(\lambda_0+\mu_0)\left(\vec{k}\cdot\vec{A}\right)\vec{k}+\left(\mu_0 k^2-\rho_0\varpi^2\right)\vec{A}+\frac{i\omega A_0 B_0 k^2}{B_0 k^2-2A_0 i\omega}\left[\left(\vec{k}\cdot\vec{A}\right)\vec{k}-k^2\vec{A}\right]=\mathbf{0}.$$

This equation is independent of generalized thermo-elastic parameters θ_0, t_0 but dependent on λ_0, μ_0, ρ_0 and angular velocity, $\vec{\Omega}$. The above equation can be re-written as

$$(n_1k^2+n_2)n_2^2+n_2\left[n_1\left(k_1^2+k_2^2\right)+n_2\right]\rho_0\Omega^2-4\omega^2\rho_0^2\Omega^2\left(n_1k_3^2+n_2\right)-4\omega^2\rho_0^3\Omega^4=0 \tag{22}$$

where

$$\begin{aligned}g&=\frac{i\omega A_0B_0k^2}{B_0k^2-2A_0i\omega},\\n_1&=(\lambda_0+\mu_0)+g,\\n_2&=(\mu_0-g)k^2-\rho_0(\omega^2+\Omega^2).\end{aligned}$$

This dispersion equation is clearly dependent on the direction of $\vec{k}$ where $k^2 = k_1^2 + k_2^2 + k_3^2$ and angular velocity $\vec{\Omega}$ but independent of generalized thermo-elastic parameters θ_0, t_0.

Let us assume that $\vec{k} = (k, 0, 0)$. Then neglecting Ω^3 and higher power terms of Ω we get from Eq. (22):

$$n_2 = 0, \tag{23}$$

and

$$\left(n_1k^2 + n_2\right) n_2 + \rho_0\Omega^2 \left(n_1k^2 + n_2\right) - 4\omega^2\rho_0^2\Omega^2 = 0. \tag{24}$$

From (23) we get after simplification

$$(\mu_0B_0 - i\omega A_0B_0)\, k^4 - \{2i\omega\mu_0A_0 + \rho_0B_0(\omega^2 + \Omega^2)\}k^2 + 2iA_0\rho_0(\omega^2 + \Omega^2)\omega = 0.$$

Neglecting $\omega^3,\ \omega^4,\ \Omega^4,\ \Omega^2\omega^2$ order terms and solving for k^2 we get

$$k^2 = \frac{\rho_0B_0\mu_0(\omega^2 + \Omega^2) - 4A_0^2\mu_0\omega^2}{2B_0(\mu_0^2 + \omega^2A_0^2)} + i\frac{2A_0\,\omega\mu_0^2}{B_0(\mu_0^2 + \omega^2A_0^2)} \tag{25}$$

and

$$k^2 = \frac{(\omega^2 + \Omega^2)\rho_0\mu_0}{2(\mu_0^2 + \omega^2A_0^2)} + \mathrm{i}\frac{(\omega^2 + \Omega^2)\rho_0A_0\omega}{2(\mu_0^2 + \omega^2A_0^2)}. \tag{26}$$

Eq. (25) represents the period equation dependent on rotation vector and granularity of the medium. Eq. (26) shows that k depends on parameters of granular character of the medium, viz., $\omega,\ A_0,\ \rho_0,\ \mu_0$ and angular velocity $\vec{\Omega}$ but k is independent of B_0. This equation therefore bears significance in that it identifies the dominance of one granular parameter over the other. From the physical point of view the absence of granular elastic parameters λ and B with the presence of elastic parameter μ presents an interesting contrast in this case.

CASE IV

The integrals in the general dispersion equation (13) consist of a large number of terms involving various auto- and cross-correlation functions between the thermal and granular elastic parameters. Consequently the task of analysing the equation for determining the effect of thermal field in general and for low and high frequencies becomes extremely cumbrous and laborious. Besides these, it may be considered more important to study the effect of random variation of granular character of the medium on the phenomena of elastic wave propagation in the generalized thermal field. With this end in view we first decide, on trial basis, to consider the rather simplified case where

$$\begin{aligned}
&\left\langle m\left(\vec{x}\right) m(\vec{x}')\right\rangle = R_{mm}(r) \neq 0,\\
&\left\langle A\left(\vec{x}\right) A\left(\vec{x}'\right)\right\rangle = R_{AA}(r) \neq 0,\\
&\left\langle B\left(\vec{x}\right) B(\vec{x}')\right\rangle = R_{BB}(r) \neq 0,\\
&\left\langle \rho\left(\vec{x}\right) \rho\left(\vec{x}'\right)\right\rangle = R_{\rho\rho}\left(r\right) \neq 0
\end{aligned}$$

but

$$R_{\lambda\lambda}\left(r\right) = 0 = R_{\mu\mu}\left(r\right),$$

and also all cross correlation functions like $R_{\rho\mu}(r)$, etc., are zeros. It is clear that even under this simplification a large number of terms involving the non-vanishing correlation functions remain to be considered for the proposed analysis. Obviously the analysis will still be lengthy and hence further simplification becomes imperative.

Now we discuss the case where $R_{mm}(r) \neq 0$ but all other $R_{ij}\left(r\right) = 0$. Eq. (13) now simplifies to

$$\begin{aligned}
&(\lambda_0 + \mu_0)\left(\vec{k}\cdot\vec{A}\right)\vec{k} + \mu_0k^2\vec{A} - \rho_0\left[\omega^2\vec{A} - \left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right]\\
&\quad + \frac{i\omega A_0B_0k^2}{B_0k^2 - 2A_0i\omega}\left[\left(\vec{k}\cdot\vec{A}\right)\vec{k} - k^2\vec{A}\right] + \epsilon^2\frac{i\theta_0m_2\left(\vec{k}.\vec{A}\right)\left(\omega - i\omega^2t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2t_0)-\nu_0k^2}}\vec{k}\\
&\quad - \epsilon^2\theta_0m_2\left(\vec{k}\cdot\vec{A}\right)\vec{k}\,(m_2 - i\omega m_3)\left(i\omega + t_0\omega^2\delta_{lk}\right)\int_0^\infty (R_{mm} - m_2^2)G_4\left(r\right)e^{-i\vec{k}\cdot\vec{r}}\,d\vec{r} = \mathbf{0}.
\end{aligned}\tag{27}$$

For **compression waves** Eq. (27) reduces to

$$\begin{aligned}
&(\lambda_0 + 2\mu_0)k^2\vec{A} - \rho_0\left[\omega^2\vec{A} - \left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] + \epsilon^2k^2\vec{A}\frac{i\theta_0m_2\left(\omega - i\omega^2t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2t_0)-\nu_0k^2}}\\
&\quad - \epsilon^2\theta_0m_2k^2\vec{A}\,(m_2 - i\omega m_3)\left(i\omega + t_0\omega^2\delta_{lk}\right)\int_0^\infty (R_{mm} - m_2^2)G_4\left(r\right)e^{-i\vec{k}\cdot\vec{r}}\,d\vec{r} = \mathbf{0}.
\end{aligned}\tag{28}$$

It is observed that for compression waves the term representing effect of granular character of the medium again disappears. It is concluded that propagation of compression waves is independent of granularity of the medium but dependent on rotation vector $\vec{\Omega}$ and generalized thermo-elastic parameters θ_0, t_0. On the other hand the thermal effects are discernible to term to the order of ϵ^2 in this case. Both the non-random and random parts of thermal parameters influence the propagation of compression waves in the medium. **It may be pointed out that the mean and auto-correlation function of the thermo-mechanical coupling parameter greatly influence the mean wave propagation.** The effect, however, is small to the order of ϵ^2 only. Eq. (28) will be studied numerically after performing the integration involved in the equation in a later section.

For propagation of **shear waves** Eq. (27) reduces to

$$\mu_0 k^2\vec{A} - \rho_0\left[\omega^2\vec{A} - \left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] - i\omega A_0B_0k^4\left(\frac{B_0k^2 + 2iA_0\omega}{B_0^2k^4 + 4\omega^2A_0^2}\right)\vec{A} = \mathbf{0}. \tag{29}$$

It is observed that randomness has no effect on wave propagation in this case. Terms to the order ϵ^2 disappear. Moreover the propagation of shear waves is completely independent of thermal character of the medium. Thermal parameters including the thermomechanical coupling parameter do not have any impact on mean wave propagation in this case even to the ϵ^2 order terms. However the granularity of the medium effectively determines the shear wave propagation in the medium.

Further discussion on Eq. (28): **Compression waves**:

Eq. (28) can be re-written as

$$(D + \epsilon^2 D_5)\vec{A} + \rho_0\left[\left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] = \mathbf{0} \tag{30}$$

where

$$D_5 = D_3 - D_4,$$
$$D = (\lambda_0 + 2\mu_0)k^2 - \rho_0\left(\omega^2 + \Omega^2\right),$$
$$D_3 = \frac{i\theta_0 m_2\left(\omega - i\omega^2 t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}}k^2,$$
$$D_4 = \theta_0 m_2 k^2(m_2 - i\omega m_3)\left(i\omega + t_0\omega^2\delta_{lk}\right)\int_0^\infty (R_{mm} - m_2^2)\,G_4(r)\,e^{-i\vec{k}\cdot\vec{r}}\,d\vec{r}.$$

From Eq. (30) we get

$$(D + \epsilon^2 D_5)(A_1, A_2, A_3) + \rho_0\left[\left(0, 0, \Omega^2 A_3\right) + 2i\omega(\Omega A_2, -\Omega A_1, 0)\right] = \mathbf{0}.$$

From this, eliminating, A_1, A_2, A_3, we obtain the following two equations

$$(\lambda_0 + 2\mu_0)k^2 - \rho_0\omega^2 + \epsilon^2(D_3 - D_4) = 0 \tag{31}$$

and

$$(\lambda_0 + 2\mu_0)k^2 = \rho_0\left[\left(\omega^2 + \Omega^2\right) + 2\omega\Omega\right] - \epsilon^2(D_3 - D_4). \tag{32}$$

These two dispersion equations (31) and (32) may be analysed in a separate section.

Discussion on Eq. (29) **: shear waves:**

Eq. (29) can be re-written as

$$D_7\vec{A} + \rho_0\left[\left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] = \mathbf{0} \tag{33}$$

where

$$D_7 = \mu_0 k^2 - \rho_0(\omega^2 + \Omega^2) + D_6,$$
$$D_6 = -i\omega A_0B_0k^4\left(\frac{B_0k^2 + 2iA_0\omega}{B_0^2k^4 + 4\omega^2A_0^2}\right).$$

From Eq. (33) we get

$$D_7(A_1, A_2, A_3) + \rho_0\left[\left(0, 0, \Omega^2 A_3\right) + 2i\omega(\Omega A_2, -\Omega A_1, 0)\right] = \mathbf{0}.$$

From this equation eliminating A_1, A_2, A_3 we get the following two equations

$$\mu_0 k^2 = \rho_0\omega^2 - D_6 \tag{34}$$

and

$$\mu_0 k^2 = \rho_0\left[\left(\omega^2 + \Omega^2\right) + 2\omega\Omega\right] - D_6. \tag{35}$$

The pair of shear wave dispersion equation (34) and (35) is independent of randomness and consequently not of much interest to us; however these equations may be further analysed numerically in future.

4. Discussion

The mean wave propagation under specified circumstances is characterized by two physical situations: the compression wave propagation is dependent on granularity but the shear wave propagation is dependent only on granularity but not on the thermal field. The shear wave propagation is actually independent of randomness to the ϵ^2 order. The physical implication may be found out only by carrying out a detailed numerical computational work firstly based on Eqs. (27) and (28) and then on the general dispersion equation (13). These studies will be carried out later (excepting on Eq. (13), though) in separate sections. In all cases the correlation functions will be chosen in the exponentially decaying form such as (Chernov [1], Karal and Keller [39a] and Karal and Keller [39b], Chow [11]):

$$R_{mm} = m_2^2 e^{-br}, \qquad b > 0.$$

In Eq. (25) it has been assumed that

$$D_3 = \frac{i\theta_0 m_2 \left(\omega - i\omega^2 t_0 \delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}} k^2.$$

Neglecting ω^3, ω^4, ω^5 terms we get

$$D_3 = \omega^2 \theta_0 \left\{ \frac{m_2^2 \gamma_0}{\nu_0^2 k^2} - \frac{m_2}{\nu_0}(t_0 m_2 \delta_{lk} + m_3) - \frac{2\gamma_0 t_0 m_2}{\nu_0 k^2} \right\} - i\frac{\theta_0 m_2^2 \omega}{\nu_0}.$$

From above we note that D_3 is independent of A_0, B_0 but dependent on generalized thermal parameters θ_0, t_0 and mean thermo-mechanical coupling parameters m_2, m_3. Then following Chow [11] it can be shown that D_3 causes dissipation of waves and increase in phase speed.

Next the term D_4 is considered:

$$D_4 = \theta_0 m_2 k^2 (m_2 - i\omega m_3)\left(i\omega + t_0\omega^2 \delta_{lk}\right) \int_0^\infty (R_{mm} - m_2^2) G_4(r) e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}.$$

Setting

$$R_{mm}(r) = m_2^2 e^{-br}, \qquad b > o$$

and using

$$\iiint F(r)\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r} = \frac{4\pi}{k}\int_0^\infty r\,F(r)\sin krdr,$$

one can easily perform integrations to get

$$D_4 = \frac{-ib}{k^2}\theta_0 (m_2 - i\omega m_3) m_2^3 \left(i\omega + t_0\omega^2\delta_{lk}\right)(2i\beta_1 - 2\beta_2 - b)$$

where

$$\beta = \beta_1 + i\beta_2,$$

$$\beta_1 = \sqrt{\left\{ \frac{\omega\gamma_0}{2\nu_0} t_0\omega + \sqrt{(1 + t_0^2\omega^2)} \right\}}^{\frac{1}{2}},$$

$$\beta_2 = \sqrt{\left\{ \frac{\omega\gamma_0}{2\nu_0} - t_0\omega + \sqrt{(1 + t_0^2\omega^2)} \right\}}^{\frac{1}{2}}.$$

From the above expression we observe that D_4 is independent of granular character of the medium but dependent on generalized thermal parameter. The dispersion equation (28) for compression waves becomes

$$(\lambda_0 + 2\mu_0)k^2\vec{A} - \rho_0\left[\omega^2\vec{A} - \left(\vec{\Omega}\cdot\vec{A}\right)\vec{\Omega} + \Omega^2\vec{A} - 2i\omega\left(\vec{\Omega}\times\vec{A}\right)\right] + \epsilon^2 k^2 \vec{A}\left[\frac{i\theta_0 m_2\left(\omega - i\omega^2 t_0\delta_{lk}\right)(m_2 - i\omega m_3)}{\gamma_{0(i\omega+\omega^2 t_0)-\nu_0 k^2}}\right]$$
$$+ \epsilon^2 \frac{ib}{k^2}\theta_0 (m_2 - i\omega m_3) m_2^3 \left(i\omega + t_0\omega^2\delta_{lk}\right)(2i\beta_1 - 2\beta_2 - b)\vec{A} = \mathbf{0}. \tag{36}$$

From the last term it may be concluded that the phase velocity and attenuation depend upon the factor, b^{-1}, the correlation length for the thermomechanical coupling parameter [11].

Setting $\vec{A} = (A_1, A_2, A_3)$, $\vec{\Omega} = (0, 0, \Omega)$ and eliminating A_1, A_2, A_3 we get from (36) the following equation:

$$(\lambda_0 + 2\mu_0)\nu_0 k^6 - k^4\left[\gamma_0(\lambda_0 + 2\mu_0)\left(i\omega + t_0\omega^2\right) + \epsilon^2 i\theta_0 m_2\left(\omega - i\omega^2 t_0\delta_{lk}\right)(m_2 - i\omega m_3) + \rho_0\nu_0\omega^2\right]$$
$$+ k^2\left[\rho_0\omega^2\gamma_0\left(i\omega + t_0\omega^2\right) + ib\theta_0\nu_0 m_2^3\epsilon^2(m_2 - i\omega m_3)\left(i\omega + t_0\omega^2\delta_{lk}\right)(2i\beta_1 - 2\beta_2 - b)\right]$$
$$- ib\theta_0\gamma_0 m_2^3\epsilon^2\left(i\omega + t_0\omega^2\right)(m_2 - i\omega m_3)\left(i\omega + t_0\omega^2\delta_{lk}\right)(2i\beta_1 - 2\beta_2 - b) = 0. \tag{37}$$

Substituting $k = k_c - i\delta_c$ in (37) and neglecting higher power of ω we get

$$\delta_c = \frac{\nu_0 k_c^3 - k_c\gamma_0\left(i\omega + t_0\omega^2\right)}{6i\nu_0 k_c^2 - 4i\gamma_0\left(i\omega + t_0\omega^2\right)} + \epsilon^2\frac{D_8}{D_9} \tag{38}$$

where

$$\begin{aligned} D_8 \;=\; & -5\nu_0\theta_0 m_2\left(\lambda_0 + 2\mu_0\right)\left(\omega m_2 - i\omega^2 m_3 - i\omega^2 m_2 t_0\delta_{lk} + \rho_0\nu_0\omega^2\right)k_c^4 + 4i\theta_0\gamma_0 m_2^2\omega^2 k_c^2\left(\lambda_0 + 2\mu_0 - 1\right)\\ & - 6\theta_0\gamma_0\nu_0 m_2^4 b\omega^2 k_c^2\left(2i\beta_1 - 2\beta_2 - b\right) - 6\theta_0\gamma_0 m_2^4 b\omega^2\left(2i\beta_1 - 2\beta_2 - b\right)\left(\lambda_0 + 2\mu_0\right).\end{aligned}$$

$$D_9 = -36\nu_0^2\left(\lambda_0 + 2\mu_0\right)^2 k_c^5 + 48\gamma_0\nu_0\left(\lambda_0 + 2\mu_0\right)^2\left(i\omega + t_0\omega^2\right)k_c^3 + 16\left(\lambda_0 + 2\mu_0\right)^2\omega^2\gamma_0^2 k_c.$$

From (38) it can be shown that

$$\delta_c = Re\delta_c + iIm\delta_c \tag{39}$$

where

$$Re\delta_c = \frac{D_{10}}{D_{11}}, \qquad Im\delta_c = \frac{D_{12}}{D_{11}},$$

$$\begin{aligned} D_{10} \;=\; & 6\left(\lambda_0 + 2\mu_0\right)m_2 t_0\delta_{lk}\nu_0\omega^2 k_c^9 + 6\left(\lambda_0 + 2\mu_0\right)\left(2\beta_2 + b\right)\left(k_c^7 + 1\right)\nu_0\theta_0\gamma_0 m_2^4 b\omega^2\\ & - 2\left(\lambda_0 + 2\mu_0\right)\theta_0\nu_0\omega\epsilon^2 m_2^2 k_c^9 - 4\left(\lambda_0 + 2\mu_0\right)\mu_0\nu_0\gamma_0\omega\, k_c^9\\ & - 4\left(\lambda_0 + 2\mu_0\right)\nu_0\gamma_0 t_0\lambda_0\omega^2 k_c^9 - 2\left(\lambda_0 + 2\mu_0\right)\gamma_0\nu_0\lambda_0\omega\, k_c^9\end{aligned}$$

$$\begin{aligned} D_{11} \;=\; & 16\gamma_0^2\lambda_0^2\omega^2 k_c^6 + 64\lambda_0\mu_0\gamma_0^2\omega^2 k_c^6 + 32\lambda_0\gamma_0\theta_0\epsilon^2 m_2^2\omega^2 k_c^6 + 64\mu_0\gamma_0^2\omega^2 k_c^6\\ & + 16\mu_0\gamma_0\theta_0\epsilon^2\omega^2 k_c^6 + 36\left(\lambda_0 + 2\mu_0\right)^2\nu_0^2 k_c^{10} - 96\left(\lambda_0 + 2\mu_0\right)\nu_0\mu_0 t_0\lambda_0\omega^2 k_c^8\\ & - 48\left(\lambda_0 + 2\mu_0\right)\nu_0\theta_0 m_2 m_3\omega^2\epsilon^2 k_c^8\end{aligned}$$

$$\begin{aligned} D_{12} \;=\; & 8\left(\lambda_0 + 2\mu_0\right)\mu_0 t_0\nu_0\gamma_0\omega^2 k_c^9 + 10\left(\lambda_0 + 2\mu_0\right)\nu_0\theta_0 m_2 m_3\omega^2\epsilon^2 k_c^9 + 6\left(\lambda_0 + 2\mu_0\right)\nu_0\gamma_0 t_0\lambda_0\omega^2 k_c^9\\ & + 12\left(\lambda_0 + 2\mu_0\right)\nu_0\mu_0 t_0\gamma_0\omega^2 k_c^9 + 6\left(\lambda_0 + 2\mu_0\right)\nu_0^2\rho_0\omega^2 k_c^9 - 12\left(\lambda_0 + 2\mu_0\right)\nu_0\theta_0\gamma_0 b\beta_1 m_2^4\omega^2\epsilon^2 k_c^7\\ & - \left(\lambda_0 + 2\mu_0\right)^2\nu_0^2 k_c^{11} - 8\mu_0\gamma_0\theta_0\omega^2\epsilon^2 k_c^7 - 16\gamma_0^2\mu_0^2\omega^2 k_c^7 - 8\gamma_0^2\mu_0\lambda_0\omega^2 k_c^7 - 4\lambda_0\gamma_0\theta_0 m_2^2\omega^2\epsilon^2 k_c^7\\ & - 8\lambda_0\mu_0\gamma_0^2\omega^2 k_c^7 - 4\gamma_0^2\lambda_0^2\end{aligned}$$

and therefore

$$k = \left(k_c + Im\delta_c\right) - i\,Re\delta_c \tag{40}$$

where

$$Rek = k_c + Im\delta_c = \omega\sqrt{\frac{\rho_0}{\lambda_0 + 2\mu_0}} + \frac{D_{12}}{D_{11}},$$

$$Imk = \frac{D_{10}}{D_{11}},$$

$$e^{ikx} = e^{i(k_c + Im\delta_c)x}e^{Re\delta_c x}. \tag{41}$$

From Eq. (41) we see that apart from change in phase speed there may occur attenuation of k_c-type waves only if $Re\delta_c < 0$. The attenuation factor is of the form $e^{Re\delta_c x}$, $Re\delta_c < 0$.

The change of phase speed positive or negative and attenuation can be clearly estimated by numerical computation only. Eq. (30), through (28), (31), (32), (36), (40) and (41) therefore has been finally analysed theoretically.

5. Numerical computation

The numerical computation of attenuation co-efficient $e^{Re\delta_c x}$ given in Eq. (41) for high frequency waves has been done. The corresponding graph $e^{Re\delta_c x}$ (attenuation coefficient) *versus* ω (frequency) has been drawn. The units of different dimensionless parameters have been chosen to suit the purpose of the present analysis. The following numerical values are used:

$$\rho_0 = 2.190\ \text{gm/cm}^3, \qquad \lambda_0 = 7.59\times 10^{11}\ \text{dyne/cm}^2, \qquad \mu_0 = 1.89\times 10^{11}\ \text{dyne/cm}^2$$

$$\gamma_0 = 0.268\times 10^{11}\ \text{dyne}, \qquad \nu_0 = 4\pi\times 10^7\ \text{Hm}^1, \qquad C^* = 0.23\ \text{cal/C}^0,$$

$$K^* = 0.6\times 10^{-2}\ \text{cal/cm s °C}, \qquad t_0 = \frac{3K^*}{\rho_0 C^*}, \qquad \epsilon = 0.001, \qquad x = 0.1, \qquad \theta_0 = 0.1,$$

$$m_2 = 0.001, \qquad m_3 = 0.005, \qquad b = -0.5,$$

$$\delta_{lk} = 1.0\ (\textit{This value } \delta_{lk} \textit{ conforms to L–S theory}).$$

We get the following attenuation *versus* frequency graph (see Fig. 1).

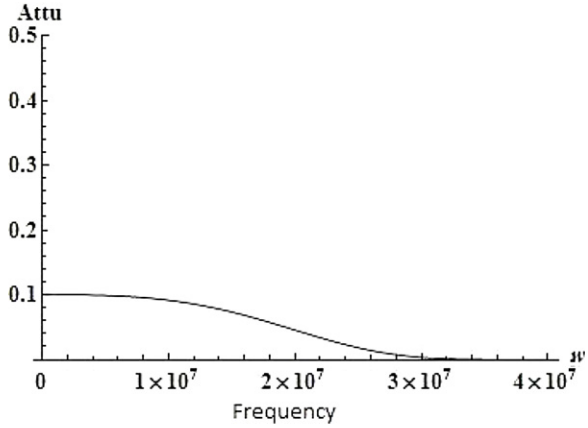

Fig. 1. Attenuation coefficient *versus* frequency graph.

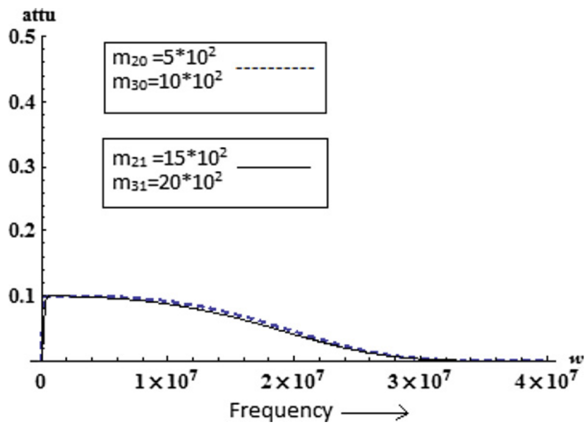

Fig. 2. Attenuation *versus*frequency graph for different values of m_2, m_3.

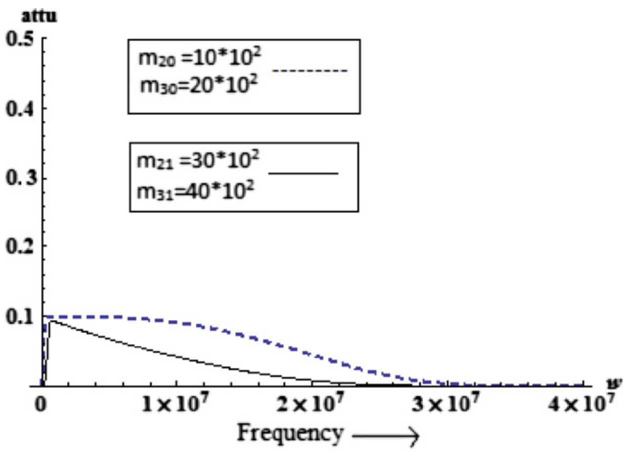

Fig. 3. Attenuation *versus* frequency graph for different values of m_2, m_3.

Clearly waves propagate only in the limited frequency range.

Now changing the numerical values of the parameters m_2, m_3 we draw the attenuation versus frequency graph which is given in Fig. 2.

There is not much appreciable change for change of values (see Fig. 3).

In this case however the change is highly appreciable in the limited frequency domain.

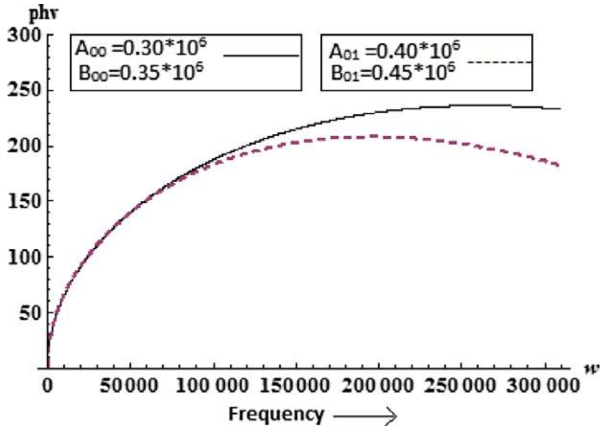

Fig. 4. Phase velocity *versus* frequency graph in non-random media.

Again from (25), neglecting, ω^3, ω^4, Ω^4, $\Omega^2\omega^2$, $\Omega\omega^2$ terms we get

$$k = Rek + iImk \tag{42}$$

where

$$Rek = \frac{1}{2}\sqrt{\frac{\omega A_0(4\mu_0^2 + \rho_0 B_0 \Omega^2)}{2B_0(\mu_0^2 + \omega^2 A_0^2)}}\left\{\left(\frac{\rho_0 B_0 \mu_0(\omega^2 + \Omega^2) - 4\mu_0\omega^2 A_0^2 + A_0\omega\rho_0 B_0 \Omega^2 + 4\omega A_0 \mu_0^2}{A_0\omega\rho_0 B_0\Omega^2 + 4\omega A_0\mu_0^2}\right)^{\frac{1}{2}}\right\},$$

$$Imk = \frac{1}{2}\sqrt{\frac{\omega A_0(4\mu_0^2 + \rho_0 B_0 \Omega^2)}{2B_0(\mu_0^2 + \omega^2 A_0^2)}}\left\{\left(\frac{A_0\omega\rho_0 B_0 \Omega^2 + 4\omega A_0\mu_0^2 - \rho_0 B_0 \mu_0(\omega^2 + \Omega^2) + 4\mu_0\omega^2 A_0^2}{A_0\omega\rho_0 B_0\Omega^2 + 4\omega A_0\mu_0^2}\right)^{\frac{1}{2}}\right\}.$$

Now taking the following numerical values of different parameters in *Rek* given in Eq. (42) we draw phase velocity (*Rek*) *versus* frequency (ω) graph which is given in Fig. 4.

$\rho_0 = 2.190 \text{ gm/cm}^3$
$\mu_0 = 1.89 \times 10^{11} \text{ dyne/cm}^2$
$\Omega = 10^5 \text{ s}^1$
$A_{00} = 0.30 \times 10^6$
$B_{00} = 0.35 \times 10^6$
$A_{01} = 0.40 \times 10^6$
$B_{01} = 0.45 \times 10^6$.

Effect of granular character is discernible from the graph.

Again taking the following numerical values in *Rek* given in Eq. (42) we get the phase velocity (*Rek*) *versus* frequency (ω) graph for different values of Ω which is given in Fig. 5:

$\rho_0 = 2.190 \text{ gm/cm}^3$
$\mu_0 = 1.89 \times 10^{11} \text{ dyne/cm}^2$
$A_0 = 0.30 \times 10^6$
$B_0 = 0.35 \times 10^6$
$\Omega_{00} = 10^5 \text{ s}^1$
$\Omega_{01} = 35^5 \text{ s}^1$.

The graph shows effect of rotation of the frame.

From (26) neglecting ω^3, ω^4, Ω^4, $\Omega^2\omega^2$ terms we get

$$k = \sqrt{\frac{\rho_0\mu_0(\omega^2 + \Omega^2)}{(\mu_0^2 + \omega^2 A_0^2)}}. \tag{43}$$

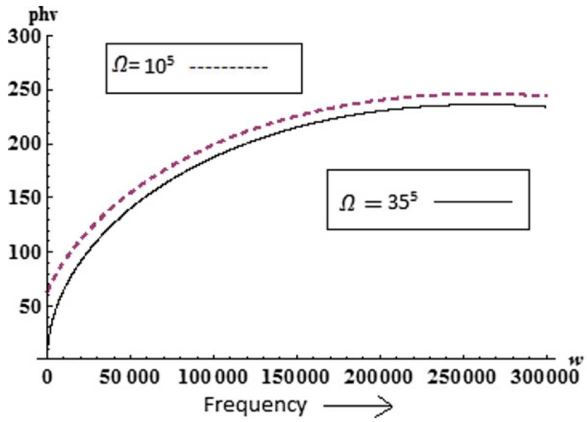

Fig. 5. Phase velocity *versus* frequency graph in non-random media for different values of Ω.

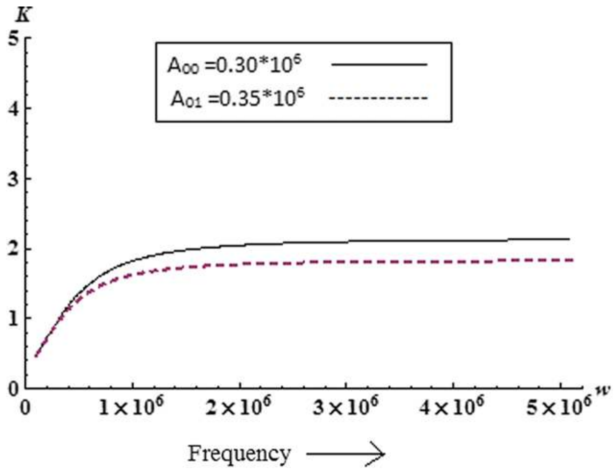

Fig. 6. Wave number *versus* frequency graph in non-random media.

Taking the following numerical values in Eq. (43) we get wave number *versus* frequency graph in non-random media which is given in Fig. 6:

$$\rho_0 = 2.190 \text{ gm/cm}^3$$
$$\mu_0 = 1.89 \times 10^{11} \text{ dyne/cm}^2$$
$$A_{00} = 0.30 \times 10^6$$
$$A_{01} = 0.35 \times 10^6$$
$$\Omega = 10^5 \text{ s}^1.$$

This graph shows effect of the change due to the variation of granular parameter $A(\vec{x})$.

Taking the following numerical values in Eq. (20) we draw wave number *versus* frequency graph which is given in Fig. 7:

$$\rho_0 = 2.190 \text{ gm/cm}^3$$
$$\mu_0 = 1.89 \times 10^{11} \text{ dyne/cm}^2$$
$$\lambda_0 = 7.59 \times 10^{11} \text{ dyne/cm}^2.$$

Again taking the following numerical values in Eq. (21) we get the three graphs (see Figs. 8–10):

$$\rho_0 = 2.190 \text{ gm/cm}^3$$
$$\mu_0 = 1.89 \times 10^{11} \text{ dyne/cm}^2$$
$$\lambda_0 = 7.59 \times 10^{11} \text{ dyne/cm}^2$$
$$\Omega = 10^5 \text{ s}^1.$$

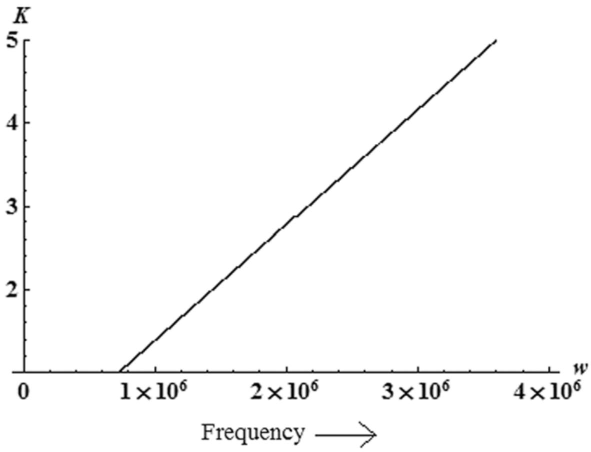

Fig. 7. Wave number *versus* frequency graph.

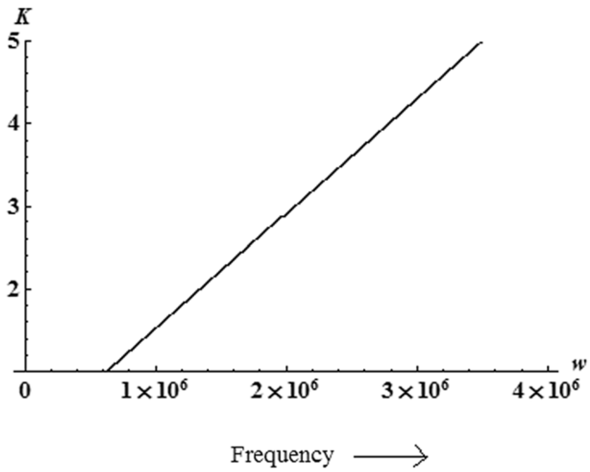

Fig. 8. Wave number *versus* frequency graph with rotation.

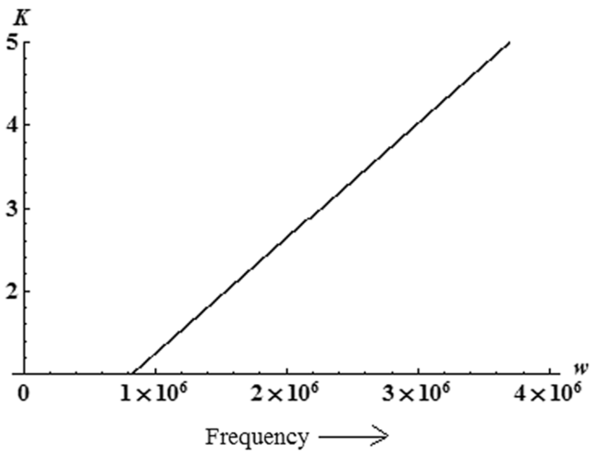

Fig. 9. Wave number *versus* frequency graph with rotation.

6. Conclusion

The general dispersion equation for waves propagating in the rotating random granular generalized thermal elastic medium has been represented. The effects of non-random granular elastic parameters and rotation of the frame of reference

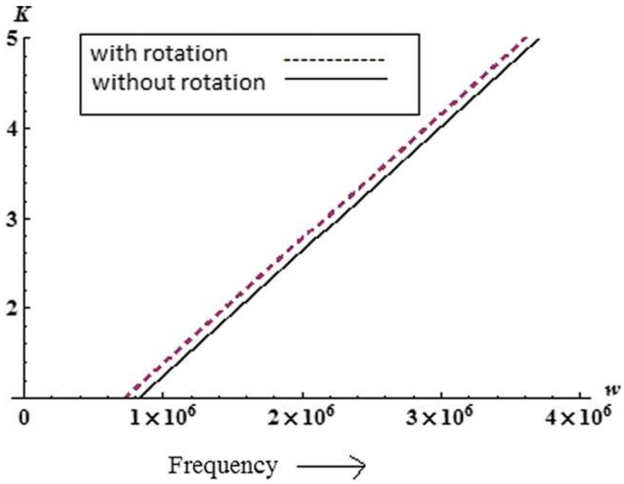

Fig. 10. Wave number *versus* frequency graph with rotation and without rotation.

has been studied from the dispersion equation. The compression and shear wave propagation have been discussed in different cases. We observe that propagation of compression waves is independent of granularity of the medium but dependent on generalized thermal parameters. Also the mean and auto-correlation functions of the thermo-mechanical coupling parameters greatly influence the wave propagation. The propagation of shear waves is completely independent of thermal character of the medium. The change of phase speed and attenuation of waves have been computed. The numerical computation of attenuation coefficient for high frequency waves has been done. Some illustrative graphs have been drawn to show the effect of rotation in generalized theory for both random and non-random media.

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Appendix A

Following Karal and Keller [3a] let $Lu_0 = 0$, where L is a linear operator characterizing a homogeneous medium. Let α designate a different medium characterized by

$$L - \epsilon L_1(\alpha) - \epsilon^2 L_2(\alpha) + O(\epsilon^3), \tag{A.1}$$

where ϵ measures departure from homogeneity. The probability distribution is given by

$$\langle f \rangle = \int_A f(\alpha) p(\alpha) d\alpha. \tag{A.2}$$

Then a wave $u(\alpha)$ satisfies the equation

$$\left[L - \epsilon L_1(\alpha) - \epsilon^2 L_2(\alpha) + O\left(\epsilon^3\right)\right] u(\alpha) = 0. \tag{A.3}$$

Rewriting (A.3) one gets

$$u = u_0 + \epsilon L^{-1}(L_1 + \epsilon L_2) u + O(\epsilon^3). \tag{A.4}$$

Iteration of (A.3) yields

$$u = u_0 + \epsilon L^{-1} L_1 u_0 + \epsilon^2 \left(L^{-1} L_1 L^{-1} L_1 + L^{-1} L_2\right) u_0 + O\left(\epsilon^2\right). \tag{A.5}$$

Taking expectation

$$\langle u \rangle = u_0 + \epsilon L^{-1} \langle L_1 \rangle u_0 + \epsilon^2 L^{-1}(\langle L_1 L^{-1} L_1 \rangle + \langle L_2 \rangle) u_0 + O\left(\epsilon^2\right). \tag{A.6}$$

Hence

$$u_0 = \langle u \rangle - \epsilon L^{-1} \langle L_1 \rangle \langle u \rangle + O(\epsilon^3). \tag{A.7}$$

Substituting (A.7) into (A.6) one gets finally

$$L\langle u \rangle - \epsilon \langle L_1 \rangle \langle u \rangle - \epsilon^2 \left[\langle L_1 L^{-1} L_1 \rangle - \langle L_1 \rangle L^{-1} \langle L_1 \rangle + \langle L_2 \rangle\right] \langle u \rangle = O\left(\epsilon^2\right).$$

Appendix B

Non-vanishing components of G_0 are

$$\begin{aligned} G_{11} &= \left(\frac{1}{2\pi}\right)^3 \Bigg[-\frac{2\pi^2 e^{ira}}{\mu_0 r} - \frac{4\pi^2(\lambda_0+\mu_0)}{(\lambda_0+2\mu_0)\mu_0 r^3}\left\{\frac{(ira-1)e^{ira}}{a^2-b^2}+\frac{(irb-1)e^{irb}}{b^2-a^2}\right\} \\ &\quad - \frac{4\pi^2 A_0\omega}{\mu_0(\mu_0+i\omega A_0)r^3}\left\{\frac{a^2\left(1+ira-a^2r^2\right)e^{ira}}{\left(a^2-k_\alpha^2\right)\left(a^2-k_\beta^2\right)} + \frac{k_\alpha^2\left(1+irak_\alpha-k_\alpha^2r^2\right)e^{ik_\alpha r}}{\left(k_\alpha^2-k_\beta^2\right)\left(k_\alpha^2-a^2\right)} \right. \\ &\quad \left. + \frac{k_\beta^2\left(1+irk_\beta-k_\beta^2r^2\right)e^{ik_\beta r}}{\left(k_\beta^2-a^2\right)\left(k_\beta^2-k_\alpha^2\right)}\right\}\Bigg] \\ &= G_{22}, \\ G_{33} &= \left(\frac{1}{2\pi}\right)^3 \Bigg[-\frac{2\pi^2 e^{ira}}{\mu_0 r} - \frac{4\pi^2(\lambda_0+\mu_0)}{(\lambda_0+2\mu_0)\mu_0 r^3}\left\{\frac{(a^2r^2+2ira-2)e^{ira}}{a^2-b^2}+\frac{(b^2r^2+2irb-2)e^{irb}}{b^2-a^2}\right\} \\ &\quad - \frac{4\pi^2 A_0\omega}{\mu_0(\lambda_0+2\mu_0)r^3}\left\{\frac{a^2\left(-2-2ira+a^2r^2\right)e^{ira}}{\left(a^2-k_\alpha^2\right)\left(a^2-k_\beta^2\right)} + \frac{k_\alpha^2\left(-2-2irk_\alpha+k_\alpha^2r^2\right)e^{ik_\alpha r}}{\left(k_\alpha^2-k_\beta^2\right)\left(k_\alpha^2-a^2\right)} \right. \\ &\quad \left. + \frac{k_\beta^2\left(-2-2irk_\beta+k_\beta^2r^2\right)e^{ik_\beta r}}{\left(k_\beta^2-a^2\right)\left(k_\beta^2-k_\alpha^2\right)}\right\}\Bigg], \end{aligned}$$

where $a^2 = \frac{\rho_0\omega^2}{\mu_0}$, $b^2 = \frac{\rho_0\omega^2}{\lambda_0+2\mu_0}$ and $c^2 = -\frac{2i\omega A_0}{B_0}$.

Non-vanishing components of G_1 are

$$\bar{G}_{12} = \frac{1}{(2\pi)^3}\frac{4\pi^2(\lambda_0+\mu_0)}{(\mu_0+i\omega A_0)B_0r^2}\left\{\frac{(1-irk_\alpha)e^{irk_\alpha}}{k_\alpha^2-k_\beta^2}+\frac{\left(1-irk_\beta\right)e^{irk_\beta}}{k_\beta^2-k_\alpha^2}\right\}, \qquad G_{21} = -G_{12}.$$

Non-vanishing components of G_2 are

$$\bar{\bar{G}}_{12} = \frac{1}{(2\pi^3)}\frac{4\pi^2(\lambda_0+\mu_0)}{(\mu_0+i\omega A_0)B_0r^2}\left\{k_\alpha^2\frac{(1-irk_\alpha)e^{irk_\alpha}}{k_\alpha^2-k_\beta^2}+k_\beta^2\frac{\left(1-irk_\beta\right)e^{irk_\beta}}{k_\beta^2-k_\alpha^2}\right\}, \qquad \bar{\bar{G}}_{21} = -\bar{\bar{G}}_{12}.$$

Non-vanishing components of G_3 are

$$\begin{aligned} \bar{\bar{\bar{G}}}_{11} &= \frac{1}{(2\pi)^3}\Bigg[\frac{4\pi^2e^{-cr}}{B_0r} - \frac{4\pi^2A_0\omega}{(\mu_0+i\omega A_0)B_0}\left\{\frac{\left(c^2r^2+cr+1\right)e^{-cr}}{\left(k_\alpha^2-c^2\right)\left(k_\beta^2-c^2\right)} \right. \\ &\quad \left. + \frac{(k_\alpha^2r^2+k_\alpha r-1)e^{-irk_\alpha}}{(k_\alpha^2+c^2)(k_\alpha^2-k_\beta^2)} + \frac{(k_\beta^2r^2+k_\beta r-1)k^{-irk_\beta}}{(k_\beta^2+c^2)(k_\beta^2-k_\alpha^2)}\right\}\Bigg] = \bar{\bar{\bar{G}}}_{22}, \\ \bar{\bar{\bar{G}}}_{33} &= \frac{1}{(2\pi)^3}\Bigg[\frac{4\pi^2e^{-cr}}{B_0r} - \frac{8\pi^2A_0\omega}{(\mu_0+i\omega A_0)B_0}\left\{\frac{(cr+1)e^{-cr}}{\left(k_\alpha^2-c^2\right)\left(k_\beta^2-c^2\right)} \right. \\ &\quad \left. + \frac{k_\alpha^2(-k_\alpha r+1)k^{-irk_\alpha}}{(k_\alpha^2+c^2)(k_\alpha^2-k_\beta^2)} + \frac{k_\beta^2(-k_\beta r+1)e^{-irk_\beta}}{(k_\beta^2+c^2)(k_\beta^2-k_\alpha^2)}\right\}\Bigg]. \end{aligned}$$

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