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Unraveling the combined actions of a Holling type III predator–prey model incorporating Allee response and memory effects

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Abstract

In this article, we have studied a memory dependent prey–predator model with logistic type prey growth subject to the Allee effect and Holling type-III functional response. To include the memory effect, we have used here a fractional order system. The model contains at most four equilibrium points among which one is trivial, two are axial, and the last one is interior. Here we have studied behavior of these four equilibrium points using eigen analysis method. Our investigation shows that the trivial equilibrium point is stable for strong Allee effect, and unstable for weak Allee effect. Between the two axial equilibrium points, one is stable under certain condition but the other is always unstable. This unstable equilibrium point exists only for strong Allee case. The interior equilibrium point arises when the axial equilibrium point is unstable. Stability condition of this interior equilibrium point depends on the parametric relationship. The schematic diagram divides the a-p plane into five sub-regions in both the cases of memory dependent and memory less system. The number and nature of the equilibrium points in different sub-regions are different. Finally, we have drawn the bifurcation diagrams considering all the equilibrium points in different sub-regions and the results are concluded at the end.

KEYWORDS

Allee effect, bifurcations, fractional-order prey–predator model, global asymptotic stability, Holling type-III functional response, predator harvesting

1 | INTRODUCTION

In classical calculus, order of the differentiation and the integration are always positive integers. On the other hand, in fractional calculus, they are of any arbitrary order including rational number. This implies fractional calculus is the generalization of classical calculus.[1] Integer order systems are memory less whereas fractional order systems possess memory.[2] In case of integer order systems, the rate of change is independent of previous effects; it depends only on the current state. On the other hand, in case of fractional order systems, rate of change depends on the size of population at that instant as well as on previous history of the system. To investigate the nature of the system which depends on previous history, the integer order rate of change does not provide good analytical findings but the fractional order derivative and integration provide more accurate results.[3-5]

Biological systems are memory dependent as their growth depends on current population density of the system as well as on previous history of the environment where they are growing. As for example in any predator–prey system if the amount of prey is initially large and prey population stay in a group then consumption by predator population will be small because of their group defense compared to the case when the prey population are scattered. Prey population uses their previous knowledge for safety of new born baby and gives birth in a safe place. Entire population keeps in mind where food is available as well as where they can get attacked. To include previous effects in biological models, currently authors are using fractional order prey–predator models.[6-8]

So far most of the work has been done by formulating the fractional models only replacing the integer order derivatives by fractional order derivatives.[9] They have not introduced the memory dependence using memory dependent kernel. But in this article we have introduced the memory effect of prey and predators using memory dependent kernels. In most of the previous works on fractional order prey–predator models, authors mainly focused on stability analysis. As most of population systems such as Lotka–Volterra systems have long-range temporary memory, fractional order population models are more advantageous than integer order population models. El-Sayed et al.[10] investigated the stability, existence, uniqueness, and found numerical solution of fractional order logistic equation using characteristic equation method.

Javidi and Nyamoradi[11] investigated a fractional order predator–prey system with harvesting; Tian et al.[12] analyzed the stability and bifurcation of three-dimensional fractional order Lotka–Volterra system.

Correlation of population size with individual population fitness is a biological phenomenon called Allee effect.[13] Allee effect arises due to the mechanisms, inherently tried to survival and reproduction. Inclusion of Allee effect in the prey predator model gives us more realistic findings. However, lot of examples show that population growth naturally exhibits Allee dynamics[14-17] (the Allee effect) where the growth rate per capita increases at low density of species. There are two types of Allee effect: the strong Allee and the weak Allee effect. In case of strong Allee effect, there exists a population threshold when the species density is small compared to the net population. Growth rate is negative below this threshold population.[15] On the other hand, for weak Allee, growth rate is low for low population densities keeping its value positive.

Holling proposed three types of functional response (Holling I-III) experimentally and authors used it to model the ecological systems. For Holling type-I functional response, the response function takes the form $g(x) = bx$. The most frequently used and studied functional response is the Holling type-II and its form is $g(x) = \frac{bx}{1+ax}$. This type response is used when the predator cause maximum mortality rate at low prey densities. Holling type-III functional response gives better description of the dynamics of the ecosystem if the predators are more efficient at higher densities and less efficient at lower prey densities. The functional form[18-21] looks like $g(x) = \frac{bx^2}{a^2+x^2}$. Effectiveness of using Holling type functional responses in prey predator models has already been explained by many researchers.[19-21]

To formulate the fractional order predator prey model with memory effect, we first consider an integer order prey–predator model with Holling type-III functional response along with Allee effect.[22] Olivares et al.[22] have shown complex dynamical behavior of similar biological system in absence of memory i.e. for integer order system. They have shown that trajectories of such system can have three different-limit cycles for the same set of parameter values which are highly sensitive to initial conditions. There are three potential attractors: (1) the origin; (2) an interior equilibrium; and (3) a limit cycle of large amplitude. A separatrix curve exists in presence of strong Allee effect. If we take initial point above the separatrix curve, then that goes to interior or saturation equilibrium point otherwise extinction.

Study of classical or integer order prey–predator model shows that among three Holling type functional responses, HT-III gives best findings.[19-21] So far some fractional order prey–predator models have been analyzed using HT-I[9] and HT-II[23] functional responses. We are using HT-III functional response in our proposed fractional order prey–predator model.

In our proposed model, we have considered prey population $X = X(\tau)$ and predator population $Y = Y(\tau)$ at time τ satisfy the governing equations,

$$\left.\begin{aligned} \frac{dX}{d\tau} &= RX\left(1-\frac{X}{k}\right)(X-m) - \frac{SX^2}{X^2+A^2}Y \\ \frac{dY}{d\tau} &= \frac{PY^2}{Y^2+A^2}Y - cY \end{aligned}\right\} \tag{1}$$

where R, k, S, P, c, A, m are all positive constants. The parameters R and k represent the intrinsic growth rate and environmental carrying capacity of the prey population, respectively, S is the attack rate of predators to prey population, (P/S) is the conversion factor, c is the mortality rate of predator population, and a is the half saturation (environmental protection) constant. Here m is the quantitative characteristic of Allee effects. Allee effect to the model depends on this characteristic quantity m. If $0 < m < k$ then strong Allee effect occurs, whereas for weak Allee effect, the value of m satisfies the condition $-k < m \leq 0$. The above model shows that the prey population grows logistically under Allee effect and the predators are more active for large prey population. As a consequence, Holling type-III functional response has been considered.

Now for simplicity of analysis, we have normalized the above model. As a result number of constants in the model has been decreased. For this normalization, the following transformations are used:

$$x = \frac{X}{k}, y = \frac{SY}{kc}, r = \frac{Rk}{c}, \beta = \frac{m}{k}, a = \frac{A}{k}, p = \frac{P}{c}, t = \tau c.$$

This reduces model (1) to,

$$\left.\begin{array}{l} \frac{dx}{dt} = rx(1-x)(x-\beta) - \frac{x^2 y}{x^2+a^2} \\ \frac{dy}{dt} = \frac{px^2}{x^2+a^2} y - y \end{array}\right\}$$

To get a deeper insight of this model, we shall now convert it to a memory dependent fractional order model via memory dependent kernel and the model can be written as follows.

$$\begin{aligned} \frac{dx}{dt} &= \int \left(k(\mathrm{t}-t') \left(rx(1-x)(x-\beta) - \frac{x^2 y}{x^2+a^2} \right) \right) dt' \\ \frac{dy}{dt} &= \int \left(k(\mathrm{t}-t') \frac{px^2}{x^2+a^2} y - y \right) dt' \end{aligned} \tag{2}$$

Here the kernel $k(\mathrm{t}-t')$ plays the role of a time-dependent kernel. For Markov process, it is equal to a delta function $\delta(\mathrm{t}-t')$ and generates the Equation (1). In fact, any arbitrary function can be replaced by a sum of delta functions, thereby leading to a given type of time correlations. A proper choice, in order to include long-term memory effects, is a power-law function which exhibits a slow decay so that the state of the system at quite early times can contribute to the evolution of the system.[2] This type of kernel guarantees the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model, we consider $k(\mathrm{t}-t') = \frac{1}{\Gamma(q-1)}(\mathrm{t}-t')^{q-2}$, where $0 < q \leq 1$ and $\Gamma(x)$ denote the gamma function. Substituting this in the right-hand side of (2), and then using the definition of fractional derivative,[2] we can replace the integral in the Equation (2) to form the fractional differential equations with the Caputo-type fractional derivative in the following form

$$\begin{aligned} \frac{dx}{dt} &= {}_{t_0}D_t^{-(q-1)} \left(\left(rx(1-x)(x-\beta) - \frac{x^2 y}{x^2+a^2} \right) \right) \\ \frac{dy}{dt} &= {}_{t_0}D_t^{-(q-1)} \left(\frac{px^2}{x^2+a^2} y - y \right) \end{aligned} \tag{3}$$

Now, applying fractional Caputo derivative of order $(q-1)$ on both sides of Equation (3), and using the fact the fractional derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the above model:

$$\left.\begin{array}{l} {}_{t_0}^{C}D_t^q(x) = rx(1-x)(x-\beta) - \frac{x^2 y}{x^2+a^2} \\ {}_{t_0}^{C}D_t^q(y) = \frac{px^2}{x^2+a^2} y - y, \;\; 0 < q < 1 \end{array}\right\} \tag{4}$$

Here ${}_{t_0}^{C}D_t^q$ denotes the Caputo derivative of order q, defined for an arbitrary function $y(t)$ as follows:

$${}_{t_0}^{C}D_t^q(y(t)) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{y'(\tau)d\tau}{(t-\tau)^q}, \tag{5}$$

Introducing a convolution integral with a power-law memory kernel in a fractional order prey–predator model, it is useful to describe memory effects in prey–predator dynamics. The decaying rate of the memory kernel (a time-correlation function) depends on the parameter q. For slow decaying (long memory) time correlation function, we need low values of q. The strength (through the "length") of the memory is controlled by this parameter. As $q \to 1$, the influence of memory decreases and then the system tend toward a memory less system. It is to be noted that for simplicity, we have assumed the same memory contributions (same value of q) for both prey and predator. We have studied the dynamics of this memory dependent prey predator model by eigenfunction analysis method.

Section 2 contains some basic results. Section 3 is devoted to show boundedness of the solution of the system. Section 4 deals with equilibrium of the fractional model and their stability. Section 5 describes the global asymptotic stability situation very well. Numerical simulation is given in Section 6. Section 7 concludes the whole work done in this article.

2 | SOME BASIC RESULTS

For linear Cauchy type fractional differential equation, the following result holds.

Lemma 1. *The Cauchy problem*[9,24]

$$ {}_a^cD_t^q x(t) = \lambda x(t) + f(t), x(a) = b (b \in \mathbb{R}) $$

with $0 < q < 1$ *and* $\lambda \in \mathbb{R}$ *has the solution of the form*

$$ x(t) = bE_q[\lambda(t-a)^q] + \int_a^t (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s)ds \tag{6} $$

while the solution to the problem ${}_a^cD_t^q x(t) = \lambda x(t), x(a) = b (b \in \mathbb{R})$ *is given by* $x(t) = bE_q[\lambda(t-a)^q]$.

To prove uniform boundedness of the system (4), we need the following lemma, which generalizes lemma 2 of Reference 25 from the integer-order system to fractional-order system.

Lemma 2. *Let* $u(t)$ *be a continuous function on* $[t_0, +\infty]$ *satisfying*

$$ \begin{cases} {}_a^cD_t^q u(t) \le -\lambda u(t) + \mu \\ u(t_0) = u_{t_0} \end{cases} \tag{7} $$

where $0 < q < 1$, λ, $\mu \in \mathbb{R}$ *and* $\lambda \neq 0$, and $t_0 \ge 0$ *is the initial time, then u(t) satisfies the inequality*

$$ u(t) \le \left(u_{t_0} - \frac{\mu}{\lambda} \right) E_q[-\lambda(t-t_0)^q] + \frac{\mu}{\lambda}. $$

Proof. Let $U(t) = u(t) - \frac{\mu}{\lambda}$ then the system (7) reduces to

$$ \begin{cases} {}_a^cD_t^q U(t) \le -\lambda U(t) \\ U(t_0) = u_{t_0} - \frac{\mu}{\lambda} \end{cases} \tag{8} $$

from which we can say that there exists a nonnegative function $m(t)$ satisfying

$$ \begin{cases} {}_a^cD_t^q U(t) = -\lambda U(t) - m(t), \\ U(t_0) = u_{t_0} - \frac{\mu}{\lambda} \end{cases} \tag{9} $$

The solution of the system (9) can be written as

$$ U(t) = U(t_0)E_q[-\lambda(t-t_0)^q] - \int_{t_0}^t (t-s)^{q-1} E_q[-\lambda(t-a)^q] m(s)ds, t \ge t_0 \tag{10} $$

Let us compare this system with (9)

$$\begin{cases} {}_{t_0}^{c}D_t^q V(t) = -\lambda V(t), \\ V(t_0) = U(t_0) \end{cases} \tag{11}$$

where $0 < q < 1$, $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, and $t_0 \geq 0$ is the initial time. According to Lemma 1, solution of the system (11) can be written as

$$V(t) = U(t_0)E_q[-\lambda(t-a)^q], t \geq t_0 \tag{12}$$

With the help of (10) and (12)

$$U(t) - V(t) = -\int_{t_0}^{t} (t-s)^{q-1}E_{q,q}[-\lambda(t-a)^q]m(s)ds, t \geq t_0 \tag{13}$$

Since $E_{q,q}(x) > 0$ for $0 < q < 1$ and $x \in R$, $m(t)$ is a nonnegative function, it follows from (12) and (13) that $u(t) \leq \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_q[-\lambda(t-t_0)^q] + \frac{\mu}{\lambda}, t \geq t_0$. ∎

3 | EXISTENCE, UNIQUENESS, AND BOUNDEDNESS OF THE SOLUTION

In this section, we shall establish the existence, uniqueness and boundedness of the solution of the proposed system (4). In Theorems 1 and 2, we have established the uniqueness and boundedness of the solutions.

Theorem 1. *The solution of the system* (4) *is unique for any non-negative initial conditions.*

Proof. The solutions of fractional order system (4) exist and uniqueness of the in the domain $\psi \times (0, \text{T})$ where $\psi = \{(x, y) \in R^2 : Max(|x|, |y| < \eta)\}$, η-is a positive constant.

Let us construct the function $G(X) = (G_1(X), G_2(X)), X = \begin{pmatrix} x \\ y \end{pmatrix}$, where $G_1(X)$ and $G_2(X)$ are given by $G_1(X) = rx(1-x)(x-\beta) - \frac{x^2y}{x^2+a^2}$ and $G_2(X)=\frac{px^2}{x^2+a^2}y - y$.

Now for any $X, \overline{X} \in \psi$ it follows that

$$\|G(X) - G(\overline{X})\| \leq |G_1(X) - G_1(\overline{X})| + |G_2(X) - G_2(\overline{X})|$$

$$= \left| rx(1-x)(x-\beta) - \frac{x^2y}{x^2+a^2} - r\overline{x}(1-\overline{x})(\overline{x}-\beta) + \frac{\overline{x}^2\overline{y}}{\overline{x}^2+a^2} \right| + \left| \frac{px^2}{x^2+a^2}y - y - \frac{p\overline{x}^2}{\overline{x}^2+a^2}\overline{y} + \overline{y} \right|$$

$$= r(2\eta + 3\eta^2 + \beta + 2\eta\beta)|x-\overline{x}| + 3|y-\overline{y}| + 4\eta|x-\overline{x}|, \text{ since } X, \overline{X} \in \psi$$

$$= r(2\eta + 3\eta^2 + \beta + 2\eta\beta)|x-\overline{x}| + 4\eta|x-\overline{x}| + 3|y-\overline{y}|$$

$$\leq H\|X - \overline{X}\|, \text{where } H = Max(3, r(2\eta + 3\eta^2 + \beta + 2\eta\beta) + 4\eta)$$

$$\therefore \|G(X) - G(\overline{X})\| \leq H\|X - \overline{X}\|.$$

Thus $G(X)$ satisfies the Lipschitz condition and consequently, the existence and uniqueness of fractional order system (4) is proved. To establish the boundedness, we establish the following theorem. ∎

Theorem 2. *All the solutions of the system which start* $\mathbb{R}_+^2$, *are uniformly bounded and enter the region* $\Omega = \left\{(x, y) \in \mathbb{R}_+^2 : 0 < x + \frac{1}{p}y < \frac{M}{v}\right\}$, *where u and v are defined in the text.*

Proof. Let $x(0)>0$, $y(0)>0$ be any initial condition of the system and $(x(t),y(t))$ be the solution of the system (4) at any time $t>0$ and consider the function $F=x+\frac{1}{p}y$.

The fractional time derivative of F is

$$
\begin{aligned}
D^qF &= D^qx+\frac{1}{p}D^qy\\
\Rightarrow \frac{d^qF}{dt^q} &= \left[rx(1-x)(x-\beta)-\frac{x^2}{x^2+a^2}y\right]+\frac{1}{p}\left[\frac{px^2}{x^2+a^2}y-y\right]\\
&= rx(1-x)(x-\beta)-\frac{y}{p}.
\end{aligned}
$$

Now, for $v>0$ we have

$$
\begin{aligned}
\frac{d^qF}{dt^q}+vF &= rx(1-x)(x-\beta)-\frac{y}{p}+vx+\frac{v}{p}y\\
&\leq r\left[-x^3+(1+\beta)x^2+\frac{1}{r}(\beta r+v)x\right],\ \frac{d^qF}{dt^q}+vF<M\\
\text{where } M &= \max_{x\in R^+}\left(r\left[-x^3+(1+\beta)x^2+\frac{1}{r}(\beta r+v)x\right]\right)>0
\end{aligned}
$$

Following Lemma 1, we get $\mathrm{F}(x(t),y(t))<\frac{M}{v}+E_q(-vt^q)\mathrm{F}(x(0),y(0))\to\frac{M}{v}$ as $t\to\infty$. Thus, all the solutions of the system (4) that start in R_+^2 are confined to the region

$$
\Omega=\left\{(x,y)\in\mathbb{R}_+^2\,:\,0<x+\frac{1}{p}y<\frac{M}{v}\right\}.
$$

This completes the proof. ■

4 | EQUILIBRIA OF THE FRACTIONAL MODEL AND THEIR STABILITY

The system (4) has four equilibrium points, those are: trivial equilibrium point $E_0=(0,0)$, axial equilibrium points $E_1=(1,0)$, $E_2=(\beta,0)$, and the interior equilibrium point

$$
E_3(x^*,y^*)=\left(a\sqrt{\frac{1}{p-1}},\,apr\left[(1+\beta)a\sqrt{\frac{1}{p-1}}-\beta-\frac{a^2}{(p-1)}\right]\sqrt{\frac{1}{p-1}}\right)
$$

The trivial and predator free equilibrium points always exist and the interior equilibrium point exist only when $1+a^2<p<1+\frac{a^2}{\beta^2}$. It should be noted that the axial equilibrium point $E_2=(\beta,0)$ does not exist for weak Allee case.

Let us rewrite the model (4) in the following form:

$$
\left.\begin{aligned}
D_t^q(x(t)) &= f(x,y)=rx(1-x)(x-\beta)-\tfrac{x^2y}{x^2+a^2}\\
D_t^q(y(t)) &= g(x,y)=\tfrac{px^2}{x^2+a^2}y-y
\end{aligned}\right\}
$$

The characteristic equation of the system (4) corresponding to any fixed point (x^*,y^*) is given by

$$
\begin{vmatrix}
-r[3x^{*^2}-2(1+\beta)x^*+\beta]-\frac{2a^2x^*}{(x^{*^2}+a^2)^2}y^*-\lambda & \frac{2a^2px^*}{(x^{*^2}+a^2)^2}y^*\\
-\frac{x^{*^2}}{(x^{*^2}+a^2)} & \left|\frac{px^*2}{x^{*^2}+a^2}-1\right|-\lambda
\end{vmatrix}=0
$$

After simplifications it becomes $\lambda^2-\tau\lambda+\Delta=0$ (14)

where, $\tau=-r[3x{*}^2-2(1+\beta)x^*+\beta]-\frac{2a^2x^*}{(x^{*^2}+a^2)^2}y^*$ and $\Delta=\frac{2pa^2x{*}^3y^*}{(x{*}^2+a^2)^3}\geq 0$.

So, the roots of the quadratic equation (14) are $\lambda=\frac{\tau+\sqrt{\tau^2-4\Delta}}{2},\frac{\tau-\sqrt{\tau^2-4\Delta}}{2}$.

Let us consider "a" as the control parameter and define $h = q\frac{\pi}{2} - \min_{i=1,2} |\arg(\lambda_i)|$,[26] the fractional model (4) will stable or unstable according to the following conditions satisfied by the roots of the Equation (14):

1. If $h < 0$ then fixed point is locally asymptotically stable.
2. If $h > 0$ then fixed point is unstable.
3. If $h = 0$ occurs at $a = a^*$ then the system will experience hopf bifurcation when "a" crosses the critical value $a = a^*$ provided the transversality condition[26] is satisfied

$$\left.\frac{dh(\mathrm{a})}{da}\right|_{a=a*} \neq 0 \tag{15}$$

The critical value of a can be found from the condition $h = 0$.

Now we will discuss the stability behavior of the model in two separate sections. First section will deal with the strong Allee effects, whereas the second section will describe the weak Allee part.

Case I: Stability analysis related with strong Allee effects.

Theorem 3. *The trivial equilibrium point* $(0, 0)$ *is always stable.*

Proof. For the trivial equilibrium point $(0, 0)$ roots of the Equation (14) are$\lambda_1 = -r\beta$ and $\lambda_2 = -1$.

For strong Allee effect $0 < \beta < 1$, so

$$h = q\frac{\pi}{2} - \min |\arg(\lambda_1), \arg(\lambda_2)| = q\frac{\pi}{2} - \min |\pi, \pi| < 0.$$

Hence the equilibrium point is always stable. ∎

Theorem 4. *The predator free equilibrium point* $(1, 0)$ *is stable only for*$(\mathrm{p}-1) - a^2 < 0$ *and saddle if* $(\mathrm{p}-1) - a^2 > 0$.

Proof. For the fixed point (1, 0), roots of the characteristic equation are $\lambda_1 = \frac{p}{1+a^2} - 1$ and $\lambda_2 = -r(1-\beta)$. Hence $\arg(\lambda_1) = -\pi$ if $(\mathrm{p}-1) - a^2 < 0$ or $\arg(\lambda_1) = 0$ if $(\mathrm{p}-1) - a^2 > 0$ and $\arg(\lambda_2) = -\pi$.

So, $h(a) = q\frac{\pi}{2} - \min |\arg(\lambda_1), \arg(\lambda_2)| < 0$ only when $(\mathrm{p}-1) - a^2 < 0$.

Hence the axial equilibrium point (1, 0) is stable when $(\mathrm{p}-1) - a^2 < 0$ but the point changes its nature to a saddle when $(\mathrm{p}-1) - a^2 > 0$. ∎

Theorem 5. *The equilibrium point* $(\beta, 0)$ *is always unstable and become saddle when* $(\mathrm{p}-1)\beta^2 - a^2 < 0$.

Proof. For the fixed point $(\beta, 0)$, roots of the characteristic equation are $\lambda_1 = \frac{p\beta^2}{\beta^2+a^2} - 1$ and $\lambda_2 = r\beta(1-\beta)$. Therefore we have $\arg(\lambda_2) = 0$. So, $h(a) = q\frac{\pi}{2} - \min |\arg(\lambda_1), \arg(\lambda_2)| \geq 0$.

Hence, the equilibrium point $(\beta, 0)$ is always unstable but if $(\mathrm{p}-1)\beta^2 - a^2 < 0$ then $\arg(\lambda_1) = -\pi$, so the point $(\beta, 0)$ behave as a saddle point. In the next theorem, we will establish the stability of the interior equilibrium point. ∎

Theorem 6. *Equilibrium point* $E_2 = (x^*, y^*)$ *of model* (4) *(if exist) is locally asymptotically stable if one of the following conditions holds*:

1. $\tau \leq 0$,
2. $\tau > 0$ *and* $\tau^2 - 4\Delta < 0$, *and* $\tan^{-1}\left(\frac{\sqrt{4\Delta-\tau^2}}{\tau}\right) > \frac{q\pi}{2}$.

Proof. Roots of the characteristic equation (6) evaluated at equilibrium point $E_2 = (x^*, y^*)$ are

$$\lambda_{1,2} = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

1. If $\tau \leq 0$ then the eigenvalues become negative real for which $h = q\frac{\pi}{2} - \pi < 0$ or imaginary with negative real parts which gives $h = q\frac{\pi}{2} - \pi + \tan^{-1}\left(\left|\frac{\sqrt{4\Delta-\tau^2}}{\tau}\right|\right) < 0$. So the fixed point (x^*, y^*) is stable.
2. Assume that $\tau > 0$, $\tau^2 - 4\Delta < 0$ and $\tan^{-1}\left(\frac{\sqrt{\tau^2-4\Delta}}{\tau}\right) > \frac{q\pi}{2}$. Then the corresponding eigenvalues become complex conjugate. From the assumptions, we get $|\tau^2 - 4\Delta| > \tau \tan\left(\frac{q\pi}{2}\right)$ consequently $h < 0$ and hence the interior equilibrium point is stable.

■

In the next part, we shall describe the nature of equilibrium points for the weak Allee effect.

Case II: Stability analysis related with weak Allee effects.

Theorem 7. *The trivial equilibrium point* $(0, 0)$ *is always a saddle point*.

Proof. For the trivial equilibrium point $(0, 0)$ roots of the Equation (14) are $\lambda_1 = -r\beta$ and $\lambda_2 = -c$.

For weak Allee effect $-1 < \beta < 0$, then.

$$h = q\frac{\pi}{2} - \min|\arg(\lambda_1), \arg(\lambda_2)| = q\frac{\pi}{2} - \min|0, \pi| \geq 0 \text{ but } \arg(\lambda_2) = -\pi.$$

Hence the equilibrium point is always an unstable saddle.

The nature of solution of the predator free equilibrium point $(1, 0)$ is similar as in the case of strong Allee effect. ■

Theorem 8. *Equilibrium point* $E_2 = (x^*, y^*)$ *of model* (4) *(if exist) is locally asymptotically stable if one of the following conditions hold*:

1. $\tau \leq 0$,
2. $\tau > 0$ and $\tau^2 - 4\Delta < 0$, and $\tan^{-1}\left(\frac{\sqrt{4\Delta-\tau^2}}{\tau}\right) > \frac{q\pi}{2}$.

Proof. Roots of the characteristic Equation (16) evaluated at equilibrium point $E_2 = (x^*, y^*)$ are

$$\lambda_{1,2} = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}.$$

1. If $\tau \leq 0$ then the eigenvalues become negative real for which $h = q\frac{\pi}{2} - \pi < 0$ or imaginary with negative real parts which gives $h = q\frac{\pi}{2} - \pi + \tan^{-1}\left(\left|\frac{\sqrt{4\Delta-\tau^2}}{\tau}\right|\right) < 0$. So the fixed point (x^*, y^*) is stable.
2. Assume that $\tau > 0$ and $\tau^2 - 4\Delta < 0$, and $\tan^{-1}\left(\frac{\sqrt{\tau^2-4\Delta}}{\tau}\right) > \frac{q\pi}{2}$. Then the corresponding eigenvalues become complex conjugate. From the assumptions, we get $|\tau^2 - 4\Delta| > \tau \tan\left(\frac{q\pi}{2}\right)$ consequently $h < 0$ and hence the interior equilibrium point is stable. An interesting result has been found here.

■

5 | GLOBAL ASYMPTOTIC STABILITY

The global asymptotic stability of any system implies the system will converge to the particular equilibrium point whatever the initial point is chosen.

Theorem 10. *The axial equilibrium point* $E_1 = (1, 0)$ *will be globally asymptotically stable in the domain* $A = \{(x, y) : x > \beta \text{ and } x^2 - px + a^2 \geq 0, x, y > 0\}$.

Proof. We observe that the system (4) always has three equilibrium point $E_0 = (0, 0)$, $E_1 = (1, 0)$, and $E_2 = (\beta, 0)$ for all parameter values. On the other hand, there exists a coexistence equilibrium point $E_2 = (x^*, y^*)$ if $1 + a^2 < p < 1 + \frac{a^2}{\beta^2}$. Here we have studied the global asymptotic stability of the equilibrium point $E_1 = (1, 0)$, $E_1 = (\beta, 0)$, and $E_2 = (x^*, y^*)$ which has lot of importance in the study of mathematical biology. To investigate the global stability of the point $E_1 = (1, 0)$, we construct a Lyapunov function

$$V(x, y) = (x - 1 - \ln x) + \frac{1}{p} y \tag{16}$$

Clearly $V(x, y)$ is a positive definite function. Now performing q-th order Caputo type fractional derivative in both side of expression of $V(x, y)$ we have,

$${}_{t_0}^{C}D_t^q V(x, y) = {}_{t_0}^{C}D_t^q (x - 1 - \ln x) + \frac{1}{p} {}_{t_0}^{C}D_t^q y \tag{17}$$

We have a relation in Lemma 4 in Reference 9,

$${}_{t_0}^{C}D_t^q \left(x - x^* - x^* \ln \left(\frac{x}{x^*} \right) \right) \leq \left(1 - \frac{x^*}{x} \right) {}_{t_0}^{C}D_t^q x(t), x \in \mathrm{R}_+, \forall q \in (0, 1). \tag{18}$$

So, putting (18) in (17), we get,

$${}_{t_0}^{C}D_t^q V(x, y) \leq - \left\{ r(x - 1)^2 (x - \beta) - \left(\frac{x}{a^2 + x^2} - \frac{1}{p} \right) y \right\} \tag{19}$$

We have $x, y > 0$, now if we take $x^2 - px + a^2 \geq 0$ and construct a domain set $A = \{(x, y) : x > \beta \text{ and } x^2 - px + a^2 \geq 0, x, y > 0\}$. Clearly ${}_{t_0}^{C}D_t^q V(x, y) \leq 0$ for all (x, y) belongs to the set A. Then ${}_{t_0}^{C}D_t^q V(x, y) = 0$ implies that $x = 1$ and using this condition in second equation of (4) then solving with taking $t \to \infty$ we get $y = 0$, that is, ${}_{t_0}^{C}D_t^q V(x, y) = 0$ and $t \to \infty$ implies $(x, y) = (1, 0)$. So the only invariant set in which ${}_{t_0}^{C}D_t^q V(x, y) = 0$ is the singleton set $\{E_1\}$. By the lemma-4.6 in Reference 25, it follows that the equilibrium point $E_1 = (1, 0)$ is globally stable in region A. ∎

Now, to investigate global asymptotically stability of the equilibrium points $E_3(x^*, y^*)$, we consider the Lyapunov function

$$V(x, y) = \left(x - x^* - x^* \ln \left(\frac{x}{x^*} \right) \right) + \frac{1}{p} \left(y - y^* - y^* \ln \left(\frac{y}{y^*} \right) \right) \tag{20}$$

Clearly $V(x, y)$ is a positive definite function. Now, again performing q-th order Caputo type fractional derivative in both side of expression of $V(x, y)$ we have,

$${}_{t_0}^{C}D_t^q V(x, y) = {}_{t_0}^{C}D_t^q \left(x - x^* - x^* \ln \left(\frac{x}{x^*} \right) \right) + \frac{1}{p} {}_{t_0}^{C}D_t^q \left(y - y^* - y^* \ln \left(\frac{y}{y^*} \right) \right) \tag{21}$$

So, putting (18) in (21) we get, ${}_{t_0}^{C}D_t^q V(x, y) \leq - \left\{ r(x - x^*)(x - 1)(x - \beta) + \frac{x^* x y}{x^2 + a^2} + \frac{y}{p} \right\}$

$${}_{t_0}^{C}D_t^q V(x, y) \leq -\{ r(x - x^*)(x - 1)(x - \beta) \} \tag{22}$$

Now we construct the domain $B = \{(x, y) : (x - x^*)(x - \beta) < 0\}$. Then, clearly ${}_{t_0}^{C}D_t^q V(x, y) \leq 0$ for all (x, y) belongs to the set B as $x, y > 0$ and ${}_{t_0}^{C}D_t^q V(x, y) = 0$ implies that $x = x^*$. Then the equilibrium condition gives $y = y^*$. That is ${}_{t_0}^{C}D_t^q V(x, y) = 0$ implies $(x, y) = (x^*, y^*)$. So the only invariant set in which ${}_{t_0}^{C}D_t^q V(x, y) = 0$ is the singleton set $\{E_2\}$. By the lemma 4.6 in Reference 25, it follows that the equilibrium point $E_2 = (x^*, y^*)$ is globally stable in region $B = \{(x, y) : (x - x^*)(x - \beta) < 0\}$. For weak Allee effect as $\beta < 0$ the domain looks like $B = \{(x, y) : x < x^*\}$.

Note that the same model in absence of Allee effect is described in Appendix A.

In absence of Allee effect in the model one equilibrium point disappears. So that the new system contains three equilibrium points, those are $(0, 0), (1, 0), \left(\frac{a}{\sqrt{p-1}}, \frac{apr}{\sqrt{p-1}} \left(1 - \frac{a}{\sqrt{p-1}} \right) \right)$. Among them the equilibrium point $(0, 0)$ is always unstable but it was stable in model (4). The second equilibrium point $(1, 0)$ is stable when $p < 1 + a^2$ but the condition

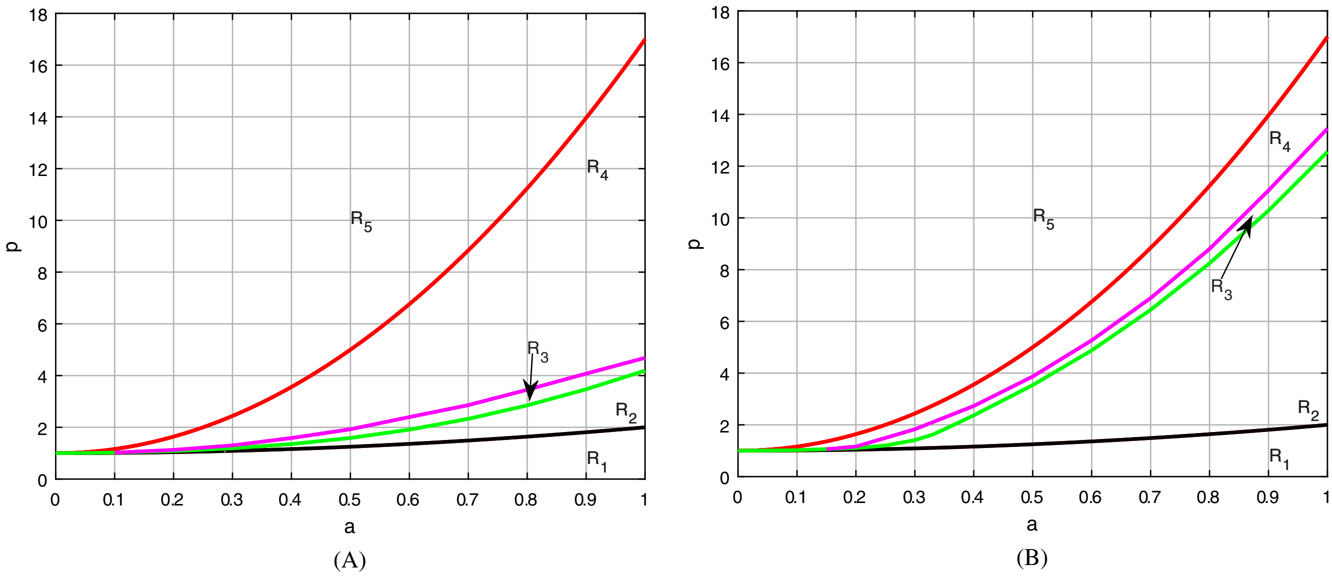

FIGURE 1 The schematic bifurcation diagram of the system (4) in the a-p plane with$r = 0.2$ and $\beta = 0.25$.The black, green, magenta, and the red lines represent the transcritical bifurcation line, Hopf line, and Homoclinic line for the interior equilibrium point and criteria for existence of the interior equilibrium point respectively: (A) $q = 1.0$ (B) $q = 0.9$

of stability was different for the same point in original model which is stated in Theorem 10. In both cases, the interior equilibrium point arises when (1,0) becomes unstable but the condition is different and the condition of stability in both cases is different.

6 | NUMERICAL SIMULATIONS

In the present section, we shall provide numerical simulations of the theoretical results to give a realistic picture of the above describe theoretical results. To generate the numerical results for the fractional order system (4), we have used the algorithm described in References 27,28 (scheme is given in Appendix B). To study the number and nature of equilibrium points, we have drawn the bifurcation diagram in $a - p$ plane (see Figure 1(A) for $q = 1.0$ (B) for $q = 0.9$) taking $r = 0.2$ and $\beta = 0.25$. In Figure 1, the black, green, magenta, and red lines represent the transcritical bifurcation line, Hopf line, and Homoclinic line for the interior equilibrium point and criteria for existence of the interior equilibrium point, respectively. Interior equilibrium point exists only in the region bounded by black and red lines. In both the cases, the $a - p$ plane is divided into five sub-regions, namely, R_1, R_2, R_3, R_4, and R_5. It is clear from the figures that the stability region for the interior equilibrium point increases with the increase of memory in the system, that is, with the decreasing of fractional order rate of change (q).

For values of the parameters in region R_1, the system contains three equilibrium points, namely, E_0, E_1, and E_2 among them the equilibrium point E_0 and E_1 are locally asymptotically stable and other equilibrium point E_2 is unstable (see Figure 2). Biologically this region is insignificant because depending on initial size of the populations, the system in this region goes to extinction or only the predator population goes to extinction. Now we are crossing the transcritical line and enter in the region R_2. Then the interior equilibrium point arises and in this case the equilibrium points are E_0, E_1, E_2, and E_3. Among all the equilibrium points only E_0 and E_3 are locally asymptotically stable and the points E_1 and E_2 are saddle (see Figure 3). Thus the system experiences bi-stability. If the initial size of the prey populations is low then the system goes to extinction and it goes to the co-existence equilibrium point if the initial abundance of the prey population is high. Thus from biological point of view, this region is highly important because if the initial number of prey population is high then the both the populations survive in the system.

It is clear from Figures 2(A,B) and 3(A,B) that memory parameter plays important role for growth and stability of the population. For higher memory, the population growth is low and population takes more time to reach equilibrium points.

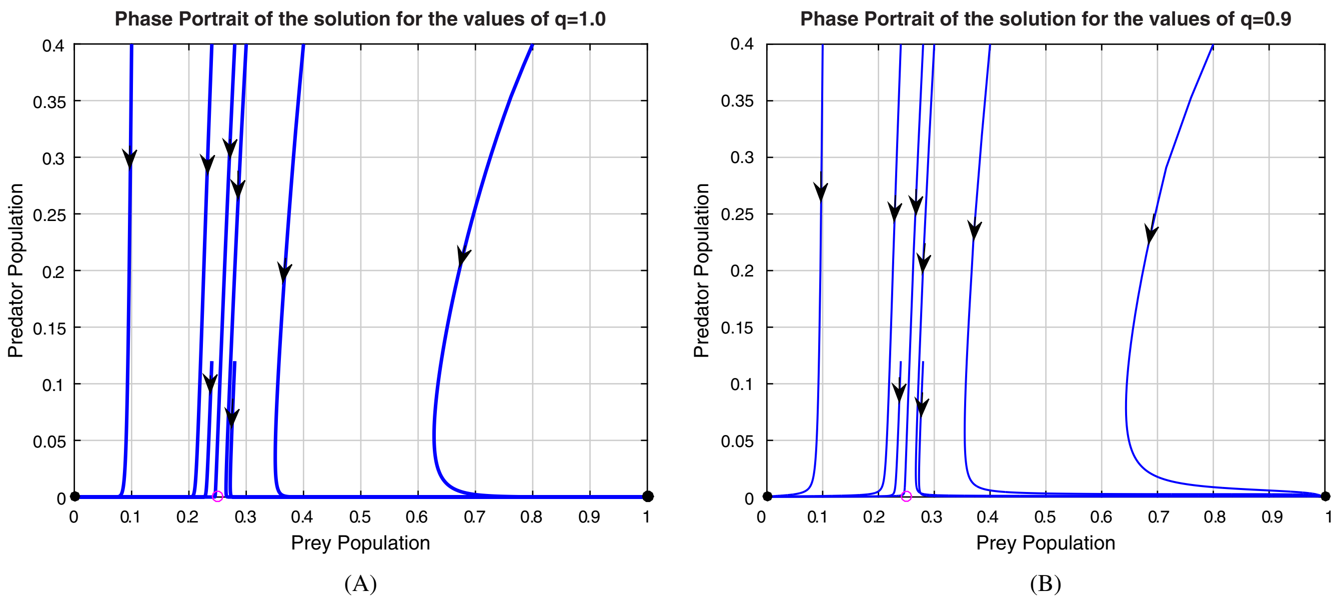

FIGURE 2 Phase portrait for the system (4) for values of the in region R_1 for (A) $q = 1.0$ and (B) $q = 0.9$

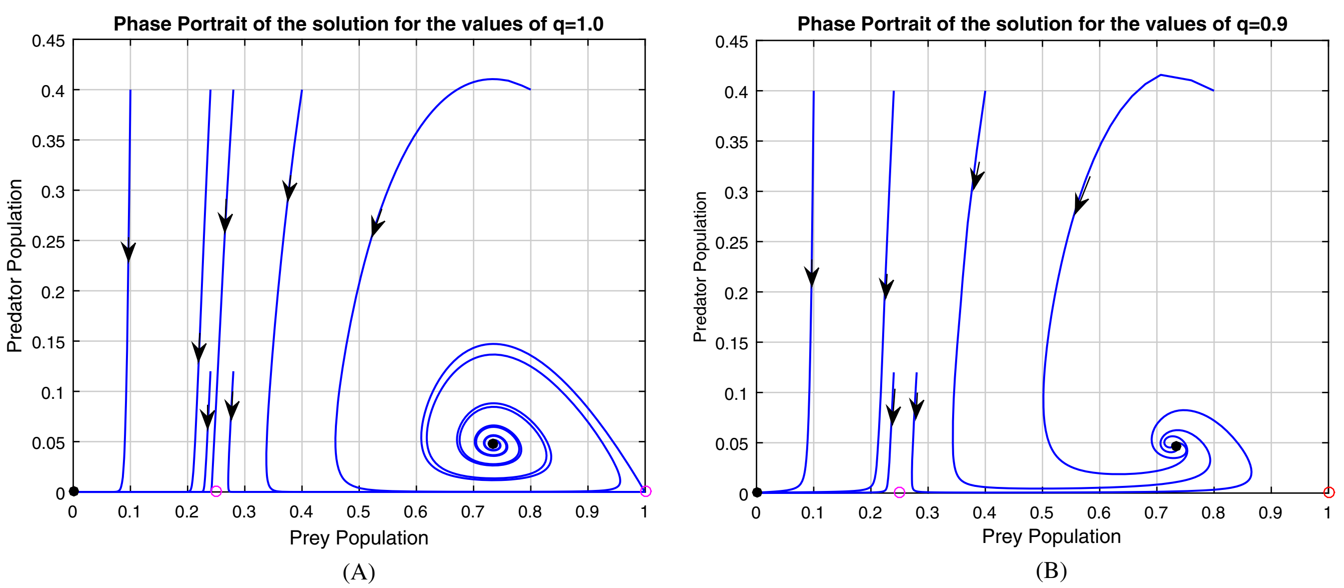

FIGURE 3 Phase portrait for the system (4) for values in region R_2 for (A) $q = 1.0$ and (B) $q = 0.9$

Now, we are crossing the Hopf line (green line) and enter in the region R_3 then the interior equilibrium point becomes unstable and hence the system experiences Hopf bifurcation when we move from region R_2 to R_3. For any values of the parameters in this region, the solution in the neighborhood of the interior equilibrium point reaches to a stable limit cycle. In Figure 4, we have presented the phase portrait considering all the equilibrium points. It is clear from Figure 4(A), a stable limit cycle arises (red circle). This figure is drawn considering $a = 0.9$, $p = 4.5$ in region R_3. But for these values of the parameters the memory dependent system is stable, which is clear from the bifurcation diagram 1(B). So to draw the phase portrait for the memory dependent system we consider the values of a and p in region R_3 as in Figure 1(B). For these values of the parameter here a stable limit cycle arises (see Figure 4(B)) but in Figure 1(B), these values are in the domain where the solution about the interior equilibrium point is unstable spiral and converges to the axial equilibrium point (0,0). Thus due to memory effect the stability region of the interior equilibrium point increases. That means memory effect has stabilization character. It is clear from Figure 4(B), the growth of the population is lower in memory-dependent

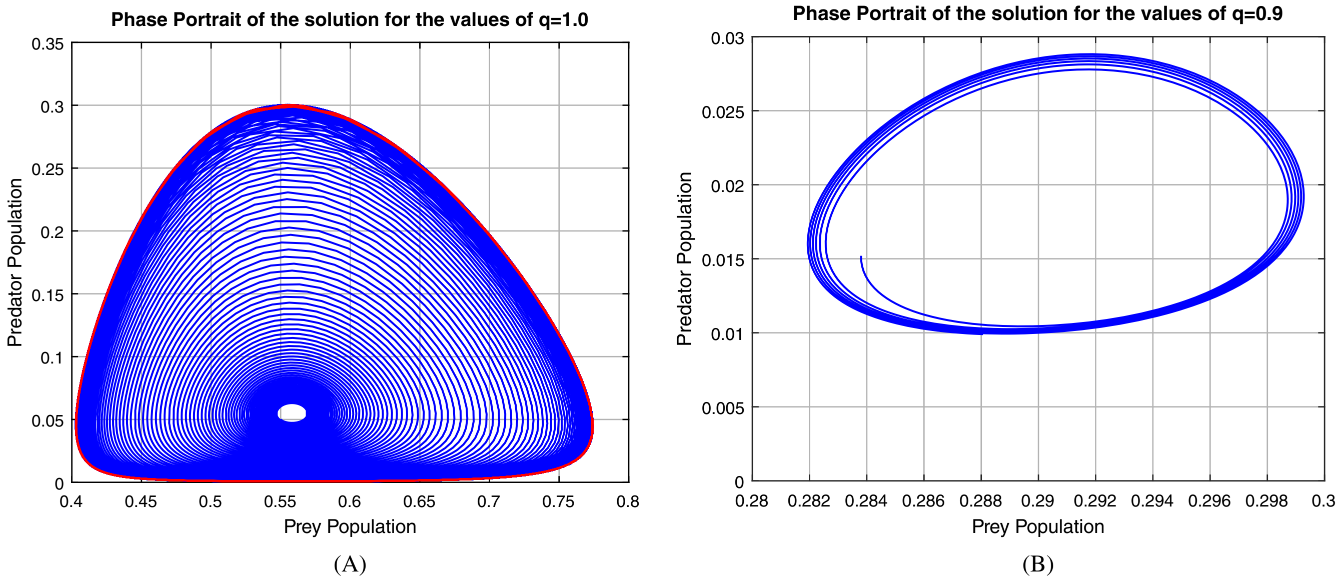

FIGURE 4 Phaseportrait for the system (4) for values of in region R_3 for (A) $q = 1.0$ and (B) $q = 0.9$

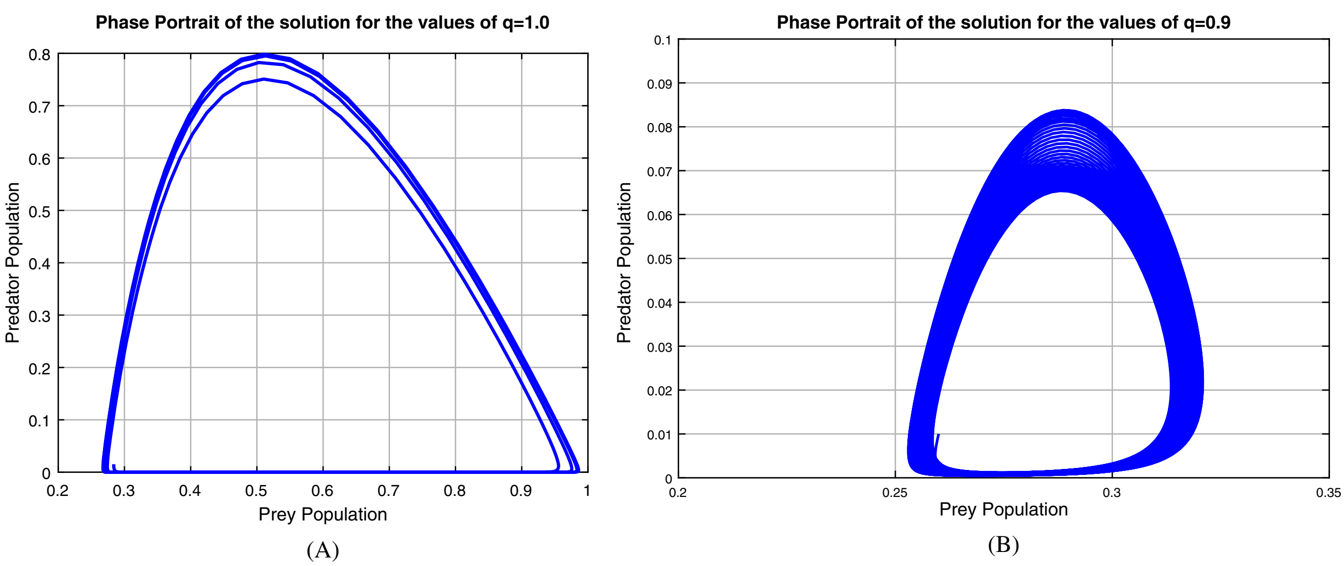

FIGURE 5 Phase portrait for the system (4) for values of on the Homoclinic line for (A) $q = 1.0$ and (B) $q = 0.9$

system compare to memory less system. The radius of the limit cycle increases with decreases of memory effect. This region is biologically important because here the abundance of the populations oscillates when the initial density of the prey population is high and goes to extinction when the initial density of the population is low.

Now, consider the values of the parameter on the homoclinic line. Then the limit cycle around the unstable interior equilibrium point connects the saddle point E_1 or E_2 or both. In this situation, the solution oscillates in connecting the saddles points (see Figure 5) for memory less system. But the system with memory shows different behavior. Here growth of the populations is very low compare to the memory less system. In this case stable limit cycle arises but radius of the limit cycle is very small compare to the memory less system and hence it does not connect the saddle points.

Now, we cross the homoclinic line (magenta line) and enter in the region R_4, the interior equilibrium point unstable spiral (here no stable limit cycle arise). Here the solution about the interior equilibrium point does not form a stable limit cycle. In presence of memory effect, the growth of the population becomes slower compare to the memory less system.

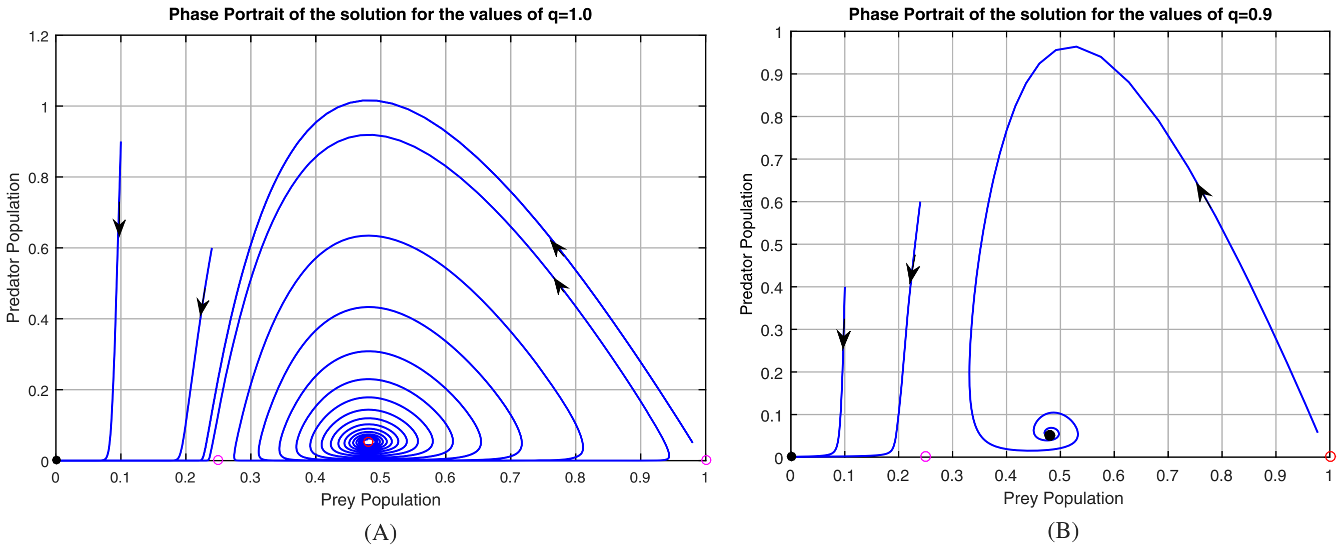

FIGURE 6 Phase portrait for the system (4) for values of in region R_4 for (A) $q = 1.0$ and (B) $q = 0.9$

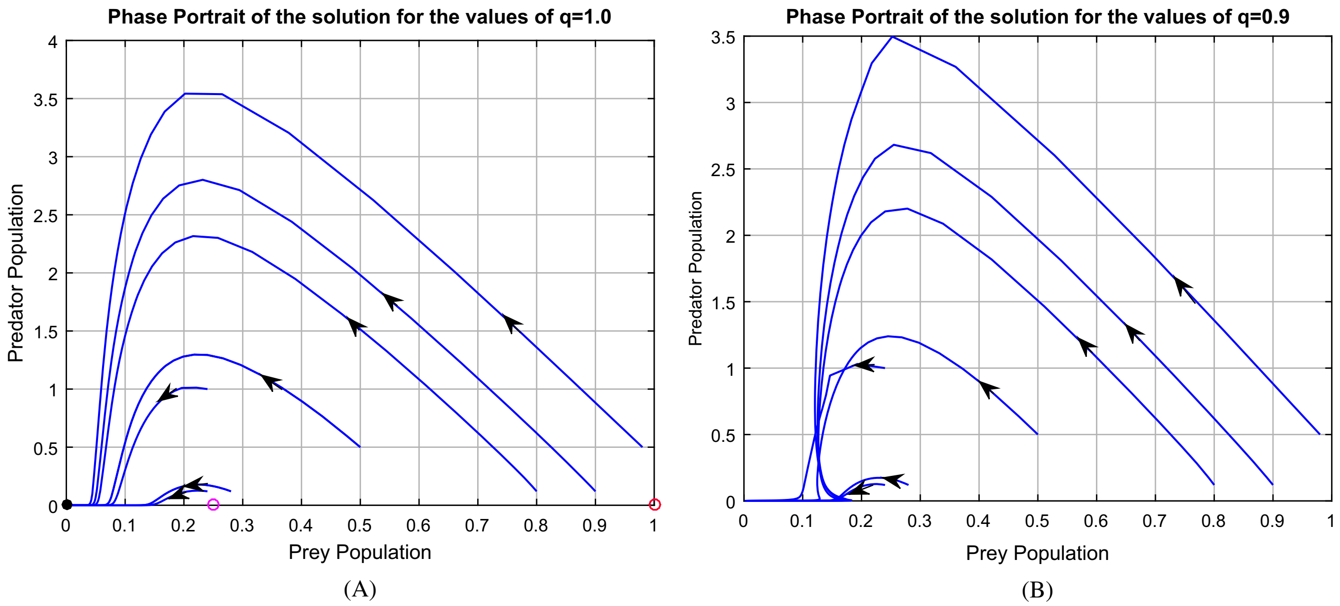

FIGURE 7 Stability situation of all equilibrium point in region R_4 for (A) $q = 1.0$ and (B) $q = 0.9$

In Figure 6, we have presented the phase portrait considering all the equilibrium points for (A) $q = 1.0$ and (B) $q = 0.9$. In Figure 6, we have presented the phase portrait for $a = 0.9$, $p = 12$ which is in region R_4 for both the system. This region is biologically important because here the abundance of the populations oscillates when the initial density of the prey population is high and goes to extinction when the initial density of the population is low.

In region R_5, the interior equilibrium point does not exist. Here only the trivial and axial equilibrium points exist among them the trivial equilibrium point (E_0) is stable and other two points are unstable (see Figure 7). The number and nature of equilibrium points in different regions are summarized in Table 1.

In Figure 8, we have drawn phase portrait of the model in absence of Allee effects from the model (4). We see that there is a basic change in nature of attractor between model (4) and the reduced model. In model (4), (0,0) is an attractor and it attract the population when population size is less than a threshold value otherwise axial equilibrium point (1,0) or interior equilibrium point become attractor. The stable manifold of the equilibrium point $(\beta, 0)$ divides the phase plane divides into two parts in one side (0, 0) is stable and other side (1,0) or the interior equilibrium point is stable (see Figures 2

TABLE 1 The feasible equilibrium points in various sub-regions of Figure 1 and their nature

Region	Points	Nature
R_1	E_0, E_1, E_2	Stable node, stable node, saddle node
R_2	E_0, E_1, E_2, E_3	Stable node, saddle node, saddle node, stable
R_3	E_0, E_1, E_2, E_3	Stable node, saddle node, saddle node, unstable spiral(Limit cycle arise)
R_4	E_0, E_1, E_2, E_3	Stable node, saddle node, saddle node, unstable spiral
R_5	E_0, E_1, E_2	Stable node, saddle node, unstable

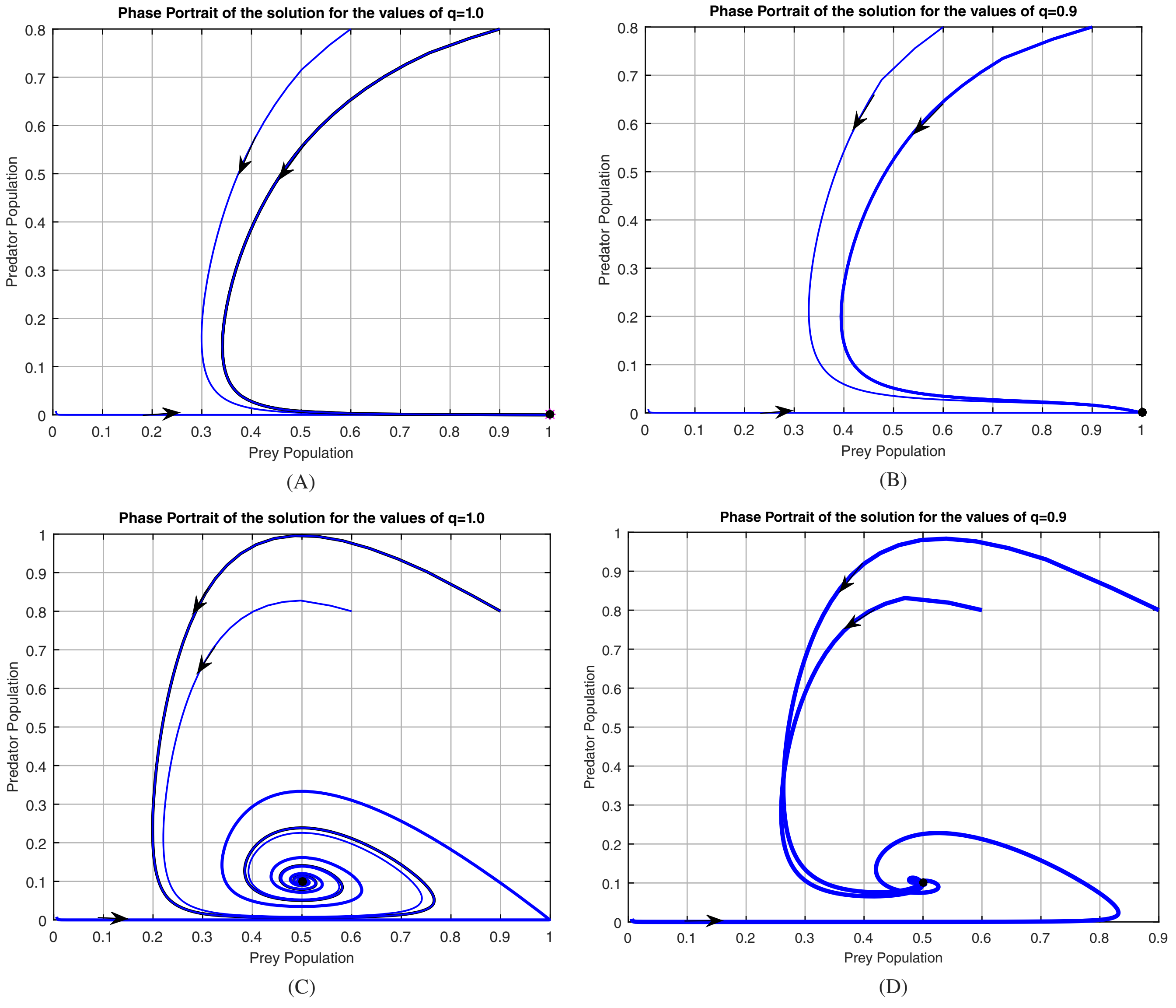

FIGURE 8 Phase portrait of the corresponding model of (4) in absence of Allee effects for (A) $r = 0.2$, $p = 1$, $q = 0.9$, (B) $r = 0.2$, $p = 1$, $q = 1.0$, (C) $r = 0.2$, $p = 2$, $q = 1.0$, and (D) $r = 0.2$, $p = 2$, $q = 0.9$

and 3) but from Figure 8, it is clear that no such separatrix exists. That is introducing Allee effects we get an extra feature of the model.

7 | CONCLUSIONS

In this article, we have formulated a fractional order prey–predator model using time dependent memory kernel. The model is formulated considering Holling type-III functional response. From the above analysis, it is clear that stability of the fixed points depend on order of fractional derivative as well as the parameters included in the model. The graphical presentation of the solutions shows that with the change of order of fractional derivative range of the solution changes. Also we show that there is a huge change in stability region for considering memory in a prey–predator model. Thus the same system becomes stable under same condition if we introduce memory in the considered system. That means in presence of memory, population can survive easily and will stay for longer time periods compare to memory less system.

Thus one can conclude that the dynamics of memory dependent prey predator models are better compare to that of integer order systems. Also the solution takes more time to reach equilibrium point. Thus to interpret the long lasting system with the effect of previous memory, the use of fractional order derivative is justified.

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APPENDIX A. STUDY OF THE MODEL (4) ELIMINATING ALLEE EFFECT FROM THE MODEL

Eliminating the Allee effect from the model (4) we get,

$$\left.\begin{aligned} {}_{t_0}^{C}D_t^q(x) &= rx(1-x) - \tfrac{x^2 y}{x^2+a^2} \\ {}_{t_0}^{C}D_t^q(y) &= \tfrac{px^2}{x^2+a^2}y - y, \quad 0<q<1 \end{aligned}\right\} \tag{A1}$$

Fixed points of this system are $(0,0), (1,0), \left(\frac{a}{\sqrt{p-1}}, \frac{apr}{\sqrt{p-1}}\left(1-\frac{a}{\sqrt{p-1}}\right)\right)$. First two equilibrium point always exist but third one exists only when $p > 1 + a^2$. Among these three equilibrium points, the trivial equilibrium point (0, 0) is always unstable. The axial equilibrium point (1, 0) is stable when $p < 1 + a^2$. The interior equilibrium point exists when axial equilibrium point becomes unstable. The stability of the interior equilibrium point depends on some conditions. The characteristic equation of the system corresponding to the interior equilibrium point (x*,y*) is $\lambda^2 - \tau\lambda + \Delta = 0$ where $\tau = r(1 - 2x*^2) - \frac{2a^2xy*}{(x*^2+a^2)^2}$ and $\Delta = \frac{2a^2px*^3y^*}{(x*^2+a^2)^3} > 0$.

Correspondingly the eigenvalues are $\lambda_1, \lambda_2 = \frac{\tau+\sqrt{\tau^2-4\Delta}}{2}, \frac{\tau-\sqrt{\tau^2-4\Delta}}{2}$.

Equilibrium point (x^*, y^*) of model (if exist) is locally asymptotically stable if one of the following conditions holds: (i) $\tau \le 0$, (ii) $\tau > 0$ and $\tau^2 - 4\Delta < 0$, and$\tan^{-1}\left(\frac{\sqrt{4\Delta-\tau^2}}{\tau}\right) > \frac{q\pi}{2}$.

APPENDIX B

The Caputo derivatives of a function $f(t)$ of order n with $0 < n < 1$ is expressed as,[27]

$$D^n f(t) = \frac{1}{\Gamma(2-n)} \times \left[\frac{f'(t)}{t^{n-1}}\left(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-1+n)}{\Gamma(n-1)(\mathrm{p})!}\right) - \left(\frac{n-1}{t^n}f(t) + \sum_{p=2}^{\infty}\frac{\Gamma(p-1+n)}{\Gamma(n-1)(\mathrm{p}-1)!}\right)\left(\frac{f(t)}{t^n}+\frac{v_p(f)(t)}{t^{p-1+n}}\right)\right], \tag{B1}$$

where $v_p(f)(t) = -(\mathrm{p}-1)\int_0^t \tau^{p-2} f(\tau)d\tau, p = 2, 3, 4, . \ldots$

We approximate $D^n f(t)$ by using the first M terms in the sum appearing in Equation (B1) as follows:

$$D^n f(t) \simeq \frac{1}{\Gamma(2-n)} \times \left[\frac{f'(t)}{t^{n-1}}\left(1+\sum_{p=1}^{M}\frac{\Gamma(p-1+n)}{\Gamma(n-1)(\mathrm{p})!}\right) - \left(\frac{n-1}{t^n}f(t) + \sum_{p=2}^{M}\frac{\Gamma(p-1+n)}{\Gamma(n-1)(\mathrm{p}-1)!}\right)\left(\frac{f(t)}{t^n}+\frac{v_p(f)(t)}{t^{p-1+n}}\right)\right], \tag{B2}$$

We can rewrite Equation (B2) as follows:

$$D^n f(t) \simeq \Omega(n,t,M)f'(t) + \Phi(n,t,M)f(t) + \sum_{p=2}^{M} A(n,t,p)\frac{v_p(f)(t)}{t^{p-1+n}}$$

$$\text{where } \Omega(n,t,M) = \frac{1+\sum_{p=1}^{M}\frac{\Gamma(p-1+n)}{\Gamma(n-1)(\mathrm{p})!}}{\Gamma(2-n)}, R(n,t) = \frac{1-n}{t^n\Gamma(2-n)},$$

$$\text{and } \Phi(n,t,M) = R(n,t) + \sum_{p=2}^{M}\frac{A(n,t,p)}{t^n}, A(n,t,p) = \frac{\Gamma(p-1+n)}{\Gamma(2-n)\Gamma(n-1)(\mathrm{p}-1)!}$$

We set $v_p(x)(t) = w_p(t)$, $v_p(\mathrm{y})(t) = u_p(t)$, p = 2, 3, 4, . … and rewrite system (4) as:

$$\Omega(n,t,M)x'(t) + \Phi(n,t,M)x(t) + \sum_{p=2}^{M} A(n,t,p)\frac{w_p(t)}{t^{p-1+n}} = rx(1-x)(x-\beta) - \frac{x^2y}{x^2+a^2}$$

$$\text{where } w_p(t) = -(\mathrm{p}-1)\int_0^t \tau^{p-2}x(\tau)d\tau, p = 2,3,4,. \ldots, M$$

$$\text{and } \Omega(n,t,M)\mathrm{y}'(t)+\Phi(n,t,M)\mathrm{y}(t)+\sum_{p=2}^{M}A(n,t,p)\frac{u_p(t)}{t^{p-1+n}}=\frac{px^2}{x^2+a^2}y-y,$$

$$\text{where } u_p(t)=-(\mathrm{p}-1)\int_0^t\tau^{p-2}\mathrm{y}(\tau)d\tau, p=2,3,4,.\ \dots\ ,M$$

Now we can rewrite the above equation as follows:

$$x'(t)=\frac{1}{\Omega(n,t,M)}\left[\left(rx(1-x)(x-\beta)-\frac{x^2y}{x^2+a^2}\right)-\Phi(n,t,M)x(t)-\sum_{p=2}^{M}A(n,t,p)\frac{w_p(t)}{t^{p-1+n}}\right]$$

$$\text{where } w_p'(t)=-(\mathrm{p}-1)\mathrm{t}^{p=2}\mathrm{x(t)}, \mathrm{p}=2,3,4,..\ \dots\ ,M$$

$$\text{and } \mathrm{y}'(t)+=\frac{1}{\Omega(n,t,M)}\left[\left(\frac{px^2}{x^2+a^2}y-y\right)-\Phi(n,t,M)\mathrm{y}(t)-\sum_{p=2}^{M}A(n,t,p)\frac{u_p(t)}{t^{p-1+n}}\right]$$

$$\text{where } u_p'(t)=-(\mathrm{p}-1)\mathrm{t}^{p=2}\mathrm{y(t)}, \mathrm{p}=2,3,4,..\ \dots\ ,M$$

With the following initial conditions:

$$x(\delta)=x_0, \mathrm{w}_p(\delta)=0, \mathrm{p}=2,3,4,.\ \dots\ ,\mathrm{M},$$
$$\mathrm{y}(\delta)=y_0, u_p(\delta)=0, \mathrm{p}=2,3,4,.\ \dots\ ,\mathrm{M}.$$