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Submersions on open symplectic manifolds

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ABSTRACT

Let M be an open manifold with a symplectic form Ω, and N a manifold with $\dim N < \dim M$. We prove that submersions $f:(M,\Omega) \to N$ with symplectic fibres satisfy the h-principle. Such submersions define Dirac manifold structures on the given manifold. As an application to this result we show that $\mathbb{C}P^n \setminus \mathbb{C}P^{k-1}$ admits a submersion into $\mathbb{R}^{2(2k-n)}$ with symplectic fibres for $n/2 < k \leqslant n$.

1. Introduction

A complete classification of submersions on open manifolds was given by Phillips in [6]. In the language of h-principle the result states that submersions on open manifolds satisfy the parametric h-principle (see Section 2). In this paper, we prove a symplectic analogue of this result [3, 3.4.2].

Let M be a symplectic manifold with the symplectic form Ω and let N be any smooth manifold with $\dim N \leqslant \dim M$. A submersion $f: M \to N$ will be called a *symplectic submersion* if the level sets of f are symplectic submanifolds of M. Similarly, a bundle epimorphism $F: TM \to TN$ will be referred as a *symplectic epimorphism* if the kernel of F, denoted by $\ker F$, is a symplectic subbundle of TM. Note that the dimension of N in this case is necessarily even. Further in the equidimensional case the symplectic submersions are nothing but the ordinary submersions.

The space of (smooth) symplectic submersion with the C^∞-compact-open topology will be denoted by $\mathcal{S}_\Omega(M,N)$, while the space of symplectic epimorphisms with the C^0-compact-open topology will be denoted by $\mathcal{E}_\Omega(TM,TN)$. The main result of this paper may be stated as follows.

Theorem 1.1. *Let M be an open manifold with a symplectic form Ω and N be any manifold with $\dim N \leqslant \dim M$. Then every F_0 in $\mathcal{E}_\Omega(TM,TN)$ admits a homotopy F_t of symplectic epimorphisms such that F_1 is the derivative of a symplectic submersion f_1. In other words, the differential d induces a surjective map $\pi_0(\mathcal{S}_\Omega(M,N)) \to \pi_0(\mathcal{E}_\Omega(TM,TN))$.*

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In the language of Gromov, the above result says that 'symplectic submersions satisfy the ordinary h-principle'. Thus, on open manifolds the only obstruction to the existence of a symplectic submersion is purely a topological one.

A submersion $f:M\to N$ induces a regular foliation $\mathcal{F}$ on M for which the leaves are given by the connected components of the level sets of f. If f is a symplectic submersion then the foliation $\mathcal{F}$ is a regular Poisson foliation, where the Poisson structure is induced by the global symplectic form Ω. Thus such an f induces a Dirac manifold structure on M. Recall that a Dirac manifold is a triple $(M,\mathcal{F},\Omega)$, where Ω is a pre-symplectic form on M and $\mathcal{F}$ is a regular foliation such that $T\mathcal{F}$ is a symplectic subbundle of (TM,Ω). It is a general fact that every regular Poisson manifold can be naturally embedded in a Dirac manifold [7].

We would like to mention here that, in [1], Bertelson classifies all regular Poisson structures on a manifold M which have a given foliation $\mathcal{F}$ as their canonical symplectic foliation. In fact, she proves the h-principle for all Poisson structures on an 'open foliated manifold' $(M,\mathcal{F})$ [1] with a regular foliation $\mathcal{F}$ as their canonical symplectic foliation. In contrast, we look for regular (Poisson) foliations $\mathcal{F}$ on M for which the Poisson structures are induced by the given symplectic form Ω.

Let F be as in Theorem 1.1. Then $K=\ker F$ is a symplectic subbundle of TM. Let L denote the symplectic complement of K. Then $F|_L:L\to TN$ is a fibrewise isomorphism. Conversely if L is a symplectic subbundle of TM, then any bundle map $G:L\to TN$ which is a fibrewise isomorphism, easily extends to a symplectic epimorphism by defining F to be identically zero on the symplectic complement of L. In view of this observation and the above theorem the existence of a symplectic submersion $f:M\to N$ is reduced to the existence of a symplectic subbundle L of TM with a vector bundle isomorphism $G:L\to TN$.

If N is an Euclidean space then the existence of an F as in Theorem 1.1 implies that there are n linearly independent vector fields on M which span a symplectic subbundle of TM. Let $\mathcal{F}_n(M)$ denote the bundle of n-frames associated with TM, for $n\leqslant m$. A section of this bundle is called an *n-frame field* on M. Thus, an n-frame field is nothing but a set of n linearly independent vector fields on M. An n-frame field will be referred as a *symplectic n-frame field* if these vector fields span a symplectic subbundle of TM. (Warning: This does not mean that these vector fields define a symplectic basis at each point of M.) If $f:M\to\mathbb{R}^n$ is a submersion with coordinate functions $f_1,f_2,\dots,f_n$, then we can define an n-frame field on M by taking the symplectic gradients of f_i, $i=1,2,\dots,n$. Explicitly, the n-frame field associated with f is given by $X_{f_1},\dots,X_{f_n}$, where $\Omega(\ ,X_{f_i})=df_i$ for $i=1,2,\dots,n$. We shall denote this frame by X_f.

Corollary 1.2. *Let M be an open symplectic manifold. Then every symplectic n-frame field on M is homotopic* (*in the space of symplectic n-frames*) *to an n-frame field X_f for some submersion $f:M\to\mathbb{R}^n$.*

The organisation of the paper goes as follows: In Section 2 we recall the basic language of the theory of h-principle and Gromov's theorem on h-principle for open invariant relations on open manifolds. We prove an analogue of this result (see [3, 3.4.2]) in Section 3 for open relations on open symplectic manifolds which are invariant under the group of symplectic diffeomorphisms. Theorem 1.1 is obtained in Section 4 as an application of this theorem. In Section 5, we prove Corollary 1.2 and discuss an application of this result.

2. Preliminaries

In this section we recall the basic language of the theory of h-principle and some results that we need in developing this paper. We refer the reader to [3] for a detailed discussion.

Let M and N be smooth manifolds and let $J^r(M,N)$ denote the r-jet bundle associated with smooth maps from M to N. There is a canonical projection $p^{(r)}:J^r(M,N)\to M$ which maps the r-jet of germ of a map at x onto x. This projection map is a fibration. A continuous map $\sigma:M\to J^r(M,N)$ is called a *section* of the jet bundle if $p^{(r)}\circ\sigma=\mathrm{id}_M$. We denote the space of sections of $p^{(r)}$ by Γ and endow it with the C^0-compact-open topology.

A subset $\mathcal{R}$ of $J^r(M,N)$ is called a *partial differential relation* of order r. Let $\Gamma(\mathcal{R})$ denote the space of sections of $J^r(M,N)\to M$ whose images lie in $\mathcal{R}$. A C^r-function $f:M\to N$ is said to be a *solution* of $\mathcal{R}$ if the r-jet map j^r_f maps M into $\mathcal{R}$; in other words, $j^r_f\in\Gamma(\mathcal{R})$. A section of $p^{(r)}$ is called *holonomic* if it is the r-jet map of a solution of $\mathcal{R}$.

We denote the space of solutions of $\mathcal{R}$ by $\mathrm{Sol}\,\mathcal{R}$, and endow it with the C^∞-compact-open topology.

Definition 2.1. A relation $\mathcal{R}$ is said to satisfy the *h-principle* if a section of $\mathcal{R}$ can be homotoped within $\Gamma(\mathcal{R})$ to a holonomic section.

$\mathcal{R}$ satisfies the *parametric h-principle* if the r-jet map $j^r:\mathrm{Sol}\,\mathcal{R}\to\Gamma(\mathcal{R})$ is a weak homotopy equivalence.

Definition 2.2. A relation $\mathcal{R}$ is said to be an *open* relation if it is an open subset of the r-jet space $J^r(M,N)$.

Let Diff(M) denote the pseudogroup of local diffeomorphisms of M. Diff(M) has a natural action on the r-jet space $J^r(M,N)$ as follows: If σ is a local diffeomorphism near $x\in M$ and $\alpha\in J^r(M,N)$ lies over $\sigma(x)$ then the action is defined by $\sigma.\alpha=:j^r_{f\sigma}(x)$, where f is any representative of the r-jet α.

Let $\mathcal{D}$ denote a pseudosubgroup of Diff(M). A relation $\mathcal{R}$ is said to be $\mathcal{D}$-invariant if it is invariant under the action of $\mathcal{D}$; that is $\sigma.\alpha\in\mathcal{R}$ for $\sigma\in\mathcal{D}$ and $\alpha\in\mathcal{R}$.

Theorem 2.3. *([3, 2.2.2], [2, 7.2.4]) Let M be an open manifold. If $\mathcal{R}$ is open and* $\mathrm{Diff}(M)$ *invariant then it satisfies the parametric h-principle.*

Further, if B is a closed subset of M such that $\mathrm{Int}\, M \setminus \mathrm{Op}\, B$ *has no compact connected component then any section of $\mathcal{R}$ which is holonomic on* $\mathrm{Op}\, B$ *can be homotoped to a solution of $\mathcal{R}$ in such a way that the homotopy remains constant on* $\mathrm{Op}\, B$.

The pseudogroup $\mathrm{Diff}(M)$ plays a very crucial role in the proof of the above theorem. It not only keeps the relation invariant but also 'sharply moves' any submanifold of M of positive codimension. We explain this concept in the following definition.

Definition 2.4. Let d be a fixed metric on M and let M_0 be a submanifold of M of positive codimension. $\mathcal{D}$ is said to *sharply move* M_0 if given any hypersurface S in a small open set U of M and any $\varepsilon > 0$, there exists an isotopy δ_t, $0 \leqslant t \leqslant 1$, in $\mathcal{D}$ such that (i) δ_0 is the identity map of U, (ii) δ_t fixes all points outside an ε-neighbourhood of S and (iii) $d(\delta_1(x), M) \geqslant r$ for all $x \in S$ and for a given $r > 0$.

Given a relation $\mathcal{R} \subset J^r(M, N)$, let $\Phi(M_0)$ denote the space of germs of C^∞ solutions of $\mathcal{R}$ near M_0 and $\Psi(M_0)$ denote the space of germs of C^0 sections of $\mathcal{R}$ near M_0 with 'quasitopological structures' (we refer to [3, 1.4.1] for a detailed discussion on this). We now state a result of Gromov from [3, 2.2.3, 2.2.4].

Theorem 2.5. *Let $\mathcal{R} \subset J^r(M, N)$ be an open relation which is invariant under a pseudosubgroup $\mathcal{D}$ of* $\mathrm{Diff}(M)$. *If $\mathcal{D}$ sharply moves a submanifold M_0 of M of positive codimension then $j^{(r)} : \Phi(M_0) \to \Psi(M_0)$ is a weak homotopy equivalence. In other words the solutions of $\mathcal{R}$ near M_0 satisfy the parametric h-principle.*

Theorem 2.3 can be easily deduced from the above theorem. Indeed, on an open manifold M there exists a Morse function with critical points of indices strictly less than the dimension of M. This defines a simplicial complex K embedded in M (called the core of M), with $\dim K < \dim M$, such that M is isotopic to an arbitrarily small neighbourhood of K. Hence, given any open neighbourhood U of K there exists an embedding of M into U which is homotopic to the identity map of M. On the other hand, by an application of Theorem 2.5 we get a solution of $\mathcal{R}$ defined on some open neighbourhood U of K. Composing this solution with the embedding of M into U we get a global solution of $\mathcal{R}$.

3. *h*-principle for open relations on open symplectic manifolds invariant under symplectomorphism group

In this section we shall state and prove an analogue of Theorem 2.3 for open relations on open symplectic manifolds which are invariant under the symplectic diffeomorphisms. Recall that a diffeomorphism δ of a symplectic manifold (M, Ω) is called a *symplectic diffeomorphism* or a *symplectomorphism* if $\delta^*\Omega = \Omega$.

Theorem 3.1. *Let M be an open manifold with a symplectic form Ω. Let $\mathcal{R}$ be an open relation in $J^r(M, N)$ which is invariant under the action of the pseudogroup of local symplectic diffeomorphisms of (M, Ω). Then $\mathcal{R}$ satisfies the h-principle.*

We first recall the following lemma from [2, 12.2.2].

Lemma 3.2. *Let $(\tilde{M}, \tilde{\omega})$ be a symplectic manifold without boundary and let M be an equidimensional submanifold of $\tilde{M}$ with boundary. Suppose, ω_t, $0 \leqslant t \leqslant 1$, is a family of symplectic forms on M with boundary which belong to the same cohomology class in $H^2(M)$. If $\tilde{\omega}|_M = \omega_0$, then there exists a regular homotopy $f_t : M \to \tilde{M}$ such that $\omega_t = f_t^*\tilde{\omega}$, where f_0 is the inclusion map.*

Further, suppose that M_0 is a compact subset of M lying in its interior and $\omega_t = \omega_0$ on a neighbourhood of M_0. If the forms ω_t belong to the same relative cohomology class in $H^2(M, M_0)$ then we can choose f_t so that $f_t(x) = x$ for all x in some neighbourhood of M_0.

Corollary 3.3. *Let M be an open symplectic manifold and let K be the core of M. Then there exists a homotopy of symplectic immersions $f_t : (M, \Omega) \to (M, \Omega)$, $0 \leqslant t \leqslant 1$, such that $f_0 = \mathrm{id}_M$ and $f_1(M) \subset U$ for a given open neighbourhood of U in M.*

Proof. Let K be the core of M. We can write M as an increasing union of equidimensional submanifolds M_i with boundary such that

(1) M_0 is a neighbourhood of K in M, and
(2) $M_{i+1} = M_i \cup (\partial M_i \times [0, 1])$; in particular, $\mathrm{closure}\, M_i \subset \mathrm{interior}\, M_{i+1}$ and M_i is a deformation retract of M_{i+1}.

(If M is compact with boundary then this sequence terminates after two elements.)

Our aim is to construct, for each $i = 0, 1, 2, \ldots$, a homotopy of immersions, $f_t^i : M \to M$, having the following properties:

(1) $f_0^i = \mathrm{id}_M$;
(2) $f_1^i(M) \subset U$;
(3) $f_t^i = f_t^{i-1}$ on M_{i-1};
(4) $(f_t^i)^*\Omega = \Omega$ on M_i.

Given the existence of the above sequence of homotopy, we may define $f_t(x) = \lim_{i\to\infty} f_t^i(x)$. Clearly, f_0 is the identity map of M and f_1 maps M into the closure of U. Moreover, f_t is a homotopy of symplectic immersions.

We shall now construct by iteration the sequence f_t^i, $0 \leqslant t \leqslant 1$. Supposing that f_t^i has already been constructed, let $\omega_t = (f_t^i)^*\Omega$. Then $\omega_0 = \Omega$, each ω_t is symplectic and $\omega_t = \Omega$ on M_i. By an application of the parametric version of Lemma 3.2, there exists a regular homotopy $g_{t,s} : M_{i+2} \to M$, $(t,s) \in [0,1] \times [0,1]$, such that

(1) $g_{t,0}$ and $g_{0,s}$ are the inclusion map of M_{i+2} in M.
(2) $(g_{t,s})^*\omega_t = \omega_{(1-s)t}$; in particular, $g_{t,1}^*\omega_t = \omega_0 = \Omega$.
(3) Since the relative cohomology group $H^2(M_{i+2}, M_i)$ vanishes (there exists a homotopy α_t of 1-forms vanishing on M_i and satisfying $\omega_t - \omega_0 = d\alpha_t$ for all $t \in [0,1]$ and hence) $g_{t,s}$ can be chosen to be a constant homotopy on M_i.

Define $f_t^{i+1} = f_t^i \circ g_{t,1} : M_{i+2} \to M$ so that $(f_t^{i+1})^*\Omega = \Omega$ on M_{i+2} and $f_t^{i+1} = f_t^i$ on M_i. Finally, $f_t^{i+1}|_{M_{i+1}}$ can be lifted to a homotopy of immersions on M which we shall denote by the same symbol. This has all the desired properties.

To start the process, we observe that, since M is an open manifold and K is its core, there exists an isotopy $\phi_t : M \to M$ of embeddings such that $\phi_0 = \mathrm{id}$ and $\phi_1(M) \subset U$. Let $\omega_t = \phi_t^*\Omega$. Then $\omega_0 = \Omega$ and ω_t is a family of symplectic forms on M in the cohomology class of Ω. The construction of f_t^0 now follows from the above by taking $M_{-1} = \emptyset$. □

Example 3.4. Here is an explicit description of a symplectic immersion of $\mathbb{R}^2$ into the unit disc $\mathbb{D}^2$ of $\mathbb{R}^2$. Let $f : \mathbb{R} \to (0, \frac{1}{2})$ be a diffeomorphism onto the open interval $(0, \frac{1}{2})$. Define two maps $\phi : \mathbb{R}^2 \to (0, \frac{1}{2}) \times \mathbb{R}$ and $\psi : (0, \frac{1}{2}) \times \mathbb{R} \to \mathbb{D}^2$ as follows:

$$\phi(x,y) = \big(f(x), y/f'(x)\big), \qquad \psi(x,y) = (\sqrt{2x}\cos y, \sqrt{2x}\sin y).$$

It may be easily verified that $\psi \circ \phi : \mathbb{R}^2 \to \mathbb{D}^2$ is a symplectic immersion.

Proof of Theorem 3.1. Since M is an open manifold there is a smooth function f on M which has no local maxima. The Morse complex K of such an f has codimension greater than or equal to 1.

Let $\mathcal{D}$ denote the pseudogroup of strictly exact symplectic diffeomorphisms of (M, Ω) (see [3, 3.4.1]). Gromov proves in [3, 3.4.1] that $\mathcal{D}$ sharply moves any submanifold of M of positive codimension. Since $\dim K < \dim M$ and $\mathcal{R}$ is an open relation invariant under $\mathcal{D}$, it follows from Theorem 2.5 that there exists a homotopy F_t of sections of $\mathcal{R}$ which is defined over some open neighbourhood U of K and is such that F_1 is holonomic. Let $f_t : U \to N$, $0 \leqslant t \leqslant 1$, denote the underlying map of F_t.

By Corollary 3.3 there exists a homotopy of symplectic immersions $g_t : (M, \Omega) \to (M, \Omega)$ such that $g_0 = \mathrm{id}_M$ and $g_1(M) \subset U$. Since $\mathcal{R}$ is invariant under the action of local symplectic diffeomorphisms, composing the homotopy F_t with dg_1 we obtain a homotopy of sections in $\mathcal{R}$ between $(F_0 \circ dg_1, f_0 \circ g_1)$ and $(F_1 \circ dg_1, f_1 \circ g_1)$. On the other hand $(F_0 \circ dg_t, f_0 \circ g_t)$ defines a homotopy of sections in $\mathcal{R}$ between (F_0, f_0) and $(F_0 \circ dg_1, f_0 \circ g_1)$. This completes the proof of the theorem since $F_1 \circ dg_1$ is holonomic. □

Remark 3.5. The homotopy joining f_0 and $f_1 \circ g_1$ in the above lemma cannot be constrained to lie in a given C^0 neighbourhood of f_0 unlike in the Symplectic Immersion Theorem for $\dim M < \dim N$.

We would like to mention here that symplectic immersions between equidimensional manifolds satisfy the ordinary h-principle [3, 3.4.2(A)]. This follows from the proof of the symplectic immersion theorem [3, 3.4.2] together with Corollary 3.3.

4. Proof of Theorem 1.1

We shall prove Theorem 1.1 using Lemmas 4.1 and 4.2.

Let (M, Ω) be a symplectic manifold and N any smooth manifold. We consider smooth submersions $f : M \to N$ such that the level sets of f are symplectic submanifolds of M. Such maps correspond to the solution space of the relation $\mathcal{R}$ defined by the following conditions: $\mathcal{R}$ consists of all 1-jets (x, y, α) such that

$$\alpha : T_xM \to T_yN \text{ is an epimorphism} \quad \text{and} \quad \Omega|_{\ker\alpha} \text{ is non-degenerate.}$$

Lemma 4.1. *$\mathcal{R}$ is an open relation.*

Proof. Let (V, Ω) be a symplectic vector space and W a vector space of dimension $\dim W \leqslant \dim V$. Let $L(V, W)$ denote the space of all linear maps from V to W. It will be enough to show that all epimorphisms $\alpha : V \to W$ such that $\Omega|_{\ker\alpha}$ is non-degenerate, form an open subset of $L(V, W)$.

Let $\mathrm{Epi}(V, W)$ denote the open subset of $L(V, W)$ consisting of all epimorphisms and let $\mathrm{Gr}_k(V)$ denote the Grassmannian of k planes in V, where $k = \dim M - \dim N$. Observe that the map $\Phi : \mathrm{Epi}(V, W) \to Gr_k(V)$ which takes β onto $\ker\beta$ is continuous.

To see this, we fix an $\alpha \in \mathrm{Epi}(V, W)$. Let $K_\alpha = \ker\alpha$ and let T_α be a complementary subspace so that $V = T_\alpha \oplus K_\alpha$. Let π_α denote the projection map $\pi_\alpha : V \to K_\alpha$ relative to this decomposition so that $\ker\pi_\alpha = T_\alpha$. Recall that the sets

$$U(\alpha) = \{K \in \mathrm{Gr}_k(V):\ \pi_\alpha|_K : K \to K_\alpha \text{ is an isomorphism}\}$$

are subbasic open sets for the topology of the Grassmannian manifold.

We now note that

$$\begin{aligned} \beta \in \Phi^{-1}\big(U(\alpha)\big) \quad &\Leftrightarrow \quad \Phi(\beta) = \ker\beta \in U(\alpha) \\ &\Leftrightarrow \quad \pi_\alpha|_{\ker\beta} : \ker\beta \to K_\alpha \text{ is an isomorphism} \\ &\Leftrightarrow \quad \ker\beta \cap T_\alpha = \{0\} \\ &\Leftrightarrow \quad \beta|_{T_\alpha} :\ T_\alpha \to W \text{ is an isomorphism.} \end{aligned}$$

Hence $\Phi^{-1}(U(\alpha))$ is open. This proves the continuity of Φ.

Finally, the map $\Psi : U(\alpha) \to \Lambda^2(K_\alpha)$ defined by $K \mapsto [i_K(\pi_\alpha|_K)^{-1}]^*(\Omega)$ is continuous; indeed, it can be expressed as the composition of two continuous maps, namely, $\psi_1 : U(\alpha) \to L(K_\alpha, V)$ and $\psi_2 : L(K_\alpha, V) \to \Lambda^2(K_\alpha)$ defined by $\psi_1(K) = i_K(\pi_\alpha|_K)^{-1}$ and $\psi_2(\beta) = \beta^*\Omega$, where i_K denotes the inclusion map of K in V. The continuity of ψ_1 follows easily if we recall the identification of $U(\alpha)$ with the space $L(K_\alpha, T_\alpha)$.

Finally we note that $\Psi(K_\alpha) = \Omega|_{K_\alpha}$ is non-degenerate. Consequently, there is an open set V_α containing K_α such that $[i_K \circ (\pi_\alpha|_K)^{-1}]^*\Omega$ is non-degenerate for all $K \in V_\alpha$. This implies that $i_K^*\Omega$ is non-degenerate for all $K \in V_\alpha$. □

Lemma 4.2. *The relation $\mathcal{R}$ is invariant under the action of the symplectomorphisms of (M, Ω).*

Proof. Let $\alpha : V \to W$ be a surjective linear map such that $\Omega|_{\ker\alpha}$ is symplectic. Let $\ell : V \to V$ be a linear isomorphism such that $\ell^*\Omega = \Omega$. Then $\ker(\alpha \circ \ell) = \ell^{-1}(\ker\alpha)$. Since $\Omega|_{\ker\alpha}$ is non-degenerate and $\ell^*\Omega = \Omega$ we obtain $\Omega|_{\ker(\alpha\circ\ell)}$ is non-degenerate. This completes the proof of openness of the set of all epimorphisms $\alpha : V \to W$ such that $\Omega|_{\ker\alpha}$ is non-degenerate. Hence $\mathcal{R}$ is invariant under the action of symplectic diffeomorphisms. □

Proof of Theorem 1.1. Theorem follows from the above lemmas in view of Theorem 3.1. □

5. Applications

We begin this section with the proof of Corollary 1.2. Recall that if f is a submersion with coordinate functions $f_1, f_2, \ldots, f_n$ then the Hamiltonian vector fields (or symplectic gradients) $X_{f_1}, X_{f_2}, \ldots, X_{f_n}$ are linearly independent. Hence, they define an n-frame field on M. Moreover, if $\ker df$ is a symplectic subbundle of (TM, Ω), then the subspace spanned by these vector fields is a symplectic subbundle of TM. To see this, observe that $\ker df = \bigcap_{i=1}^n \ker df_i$; consequently, $\ker df$ is the symplectic complement of the subspace spanned by the vector fields X_{f_i}, $i = 1, 2, \ldots, n$. Now, since $\ker df$ is a symplectic subbundle of (TM, Ω), so is the subspace spanned by the Hamiltonian vector fields.

Now, suppose that $F : TM \to T\mathbb{R}^n$ is an epimorphism for which $\ker F$ is a symplectic subbundle of (TM, Ω). Then the symplectic complement of $\ker F$, denoted by $(\ker F)^\Omega$, is a symplectic subbundle of TM. Moreover, $(\ker F)^\Omega$ is isomorphic to the trivial vector bundle $f^*(T\mathbb{R}^n)$, where f is the underlying map of F. Hence, there is a canonical choice of n-frame field on M defined via this isomorphism. On the other hand, suppose $X_1, X_2, \ldots, X_n$ be a symplectic n-frame field on M. Let L denote the subbundle spanned by these vector fields and let K be the symplectic complement of L. Since the chosen frame field is symplectic, K is a symplectic subbundle. We now define an epimorphism $F : TM \to T\mathbb{R}^n$ (over a constant map) whose kernel is K and which takes $X_1, X_2, \ldots, X_n$ onto the coordinate vector fields of $\mathbb{R}^n$.

The corollary now follows from Theorem 1.1 together with the above observation.

Remark 5.1. Submersions $f : M \to \mathbb{R}^n$ obtained by Corollary 1.2 define transversely orientable, symplectic foliations on M.

Remark 5.2. Given any symplectic form on a manifold M there is a compatible almost complex structure J which turns the tangent bundle into a complex vector bundle. In fact, the maximal compact subgroup of $Sp(2n)$ can be identified with the unitary group $U(n)$ [5]. Therefore, it is enough to get a complex n-frame field on (M, J) in order to conclude the existence of a symplectic submersion $f : M \to \mathbb{R}^{2n}$.

Example 5.3. Let M_k denote the manifold $\mathbb{C}P^n \setminus \mathbb{C}P^{k-1}$, $0 < k \leqslant n$. Then M_k deforms onto $\mathbb{C}P^{n-k}$ and hence $TM_k|_{\mathbb{C}P^{n-k}}$ has a p-frame field if and only if TM_k has a p-frame field. Since TM_k is a complex vector bundle of fibre dimension n, it follows from the general theory of bundles that $TM_k|_{\mathbb{C}P^{n-k}}$ has a complex p-frame field, where p satisfies the relation $n - p \geqslant 2(n-k) - 1$ [4]; that is $0 < p \leqslant 2k - n$. Hence, for $n/2 < k \leqslant n$, there is a symplectic submersion of M_k into $\mathbb{R}^{2(2k-n)}$. In particular, if n is even then there exists a symplectic submersion $\mathbb{C}P^n \setminus \mathbb{C}P^{n/2} \to \mathbb{R}^4$. The case $k = n$ says that there is a submersion $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1} \to \mathbb{R}^{2n}$, in other words M_n is trivialisable which is true indeed.

References

[1] M. Bertelson, A h-principle for open relations invariant under foliated isotopies, J. Symplectic Geom. 1 (2) (2002) 369–425.
[2] Y. Eliashberg, N. Mishachev, Introduction to the h-Principle, Grad. Stud. Math., vol. 48, Amer. Math. Soc., Providence, RI, 2002.
[3] M. Gromov, Partial Differential Relations, Ergeb. Math. Grenzgeb. (3), Folge Band 9, Springer-Verlag, 1986.
[4] D. Husemoller, Fibre Bundles, Grad. Texts in Math., vol. 20, Springer-Verlag, New York, 1994.
[5] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Oxford Math. Monogr., The Clarendon Press, Oxford, 1995.
[6] A. Phillips, Submersions of open manifolds, Topology 6 (1967) 171–206.
[7] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Progr. Math., vol. 118, Birkhäuser-Verlag, 1994.