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Subclasses of spirallike multivalent functions

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ABSTRACT

We studied convolution properties of spirallike, starlike and convex functions, and some special cases of the main results are also pointed out. Further, we also obtained inclusion and convolution properties for some new subclasses on p-valent functions defined by using the Dziok–Srivastava operator.

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$.

If f and g are analytic functions in U, we say that f is *subordinate* to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, for all $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U}).$$

We will define the following two subclasses of a multivalent function:

Definition 1.1. Let $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi}{2}$, let $p \in \mathbb{N}$, and let ϕ be a univalent function in the unit disc U with $\phi(0) = 1$, such that

$$\operatorname{Re} \phi(z) > 1 - \frac{1}{p}, \quad z \in \mathrm{U}. \tag{1.2}$$

We define the classes $\mathcal{S}_p^{\lambda}(\phi)$ and $\mathcal{C}_p^{\lambda}(\phi)$ by

$$\mathcal{S}_p^{\lambda}(\phi) = \left\{ f \in \mathcal{A}_p : \mathrm{e}^{i\lambda} \frac{zf'(z)}{f(z)} \prec p\left(\phi(z) \cos\lambda + i \sin\lambda\right) \right\}$$

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and

$$\mathcal{C}_p^\lambda(\phi)=\left\{f\in\mathcal{A}_p:\mathrm{e}^{i\lambda}\left[1+\frac{zf''(z)}{f'(z)}\right]\prec p\left(\phi(z)\cos\lambda+i\sin\lambda\right)\right\}.$$

Remarks 1.1. 1. It is easy to see that

$$f\in\mathcal{C}_p^\lambda(\phi)\Leftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}_p^\lambda(\phi).\tag{1.3}$$

2. For $\phi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, we set $\mathcal{S}_p^\lambda[A,B]:=\mathcal{S}_p^\lambda(\phi)$ and $\mathcal{C}_p^\lambda[A,B]:=\mathcal{C}_p^\lambda(\phi)$.
3. For $\lambda=0$ we write

$$\mathcal{S}_p^0(\phi)=:\mathcal{S}_p^*(\phi)\quad\text{and}\quad\mathcal{C}_p^0(\phi)=:\mathcal{C}_p(\phi),$$
$$\mathcal{S}_p^0[A,B]=:\mathcal{S}_p^*[A,B]\quad\text{and}\quad\mathcal{C}_p^0[A,B]=:\mathcal{C}_p[A,B].$$

Moreover, for the special case $p=1, A=1$ and $B=-1$, the class $\mathcal{C}_1[1,-1]$ represents the class of convex (univalent) and normalized functions (with the conditions $f(0)=f'(0)-1=0$) in U.
4. For $0\le\alpha<1$, the classes $\mathcal{S}_p^\lambda[1-2\alpha,-1]$ and $\mathcal{C}_p^\lambda[1-2\alpha,-1]$ reduce to the classes $\mathcal{S}_p^\lambda(\alpha)$ and $\mathcal{C}_p^\lambda(\alpha)$ of multivalent λ-spirallike functions of order α, and multivalent λ-spirallike convex functions of order α in U respectively.

Lemma 1.1. 1. *If $f\in\mathcal{S}_p^\lambda(\phi)$, then the function $F(z)=\frac{f(z)}{z^{p-1}}$ is a λ-spirallike (univalent) function in* U.
2. *Consequently, for all $\gamma\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, the multivalued function $\left[\frac{f(z)}{z^p}\right]^\gamma$ has an analytic branch in* U, *with $\left.\left[\frac{f(z)}{z^p}\right]^\gamma\right|_{z=0}=1$.*

Proof. Since $f\in\mathcal{A}_p$ we have $F\in\mathcal{A}_1$. Differentiating the definition formula of F we get $\frac{zF'(z)}{F(z)}=\frac{zf'(z)}{f(z)}-p+1$, and using the fact that $f\in\mathcal{S}_p^\lambda(\phi)$ it follows

$$\mathrm{e}^{i\lambda}\frac{zF'(z)}{F(z)}\prec[p\phi(z)+1-p]\cos\lambda+i\sin\lambda=:H(z).\tag{1.4}$$

Using the assumption (1.2) we deduce

$$\operatorname{Re}H(z)=\cos\lambda\cdot\operatorname{Re}[p\phi(z)+1-p]>0,\quad z\in\mathrm{U},$$

whenever $\lambda\in\mathbb{R}$ with $|\lambda|<\frac{\pi}{2}$. Since the function H is univalent, the subordination (1.4) yields that $\operatorname{Re}\left[\mathrm{e}^{i\lambda}\frac{zF'(z)}{F(z)}\right]>0$, $z\in$ U, and this last inequality shows that F is a λ-spirallike function, hence it is univalent in U. It follows $\frac{F(z)}{z}\ne 0$ for all $z\in$ U, i.e. $\frac{f(z)}{z^p}\ne 0$, $z\in$ U, that implies the second part of our lemma. □

Using the above lemma we will introduce the following two subclasses of $\mathcal{A}_p$:

Definition 1.2. Let $\gamma\in\mathbb{C}^*$ be an arbitrary number.

1. A function $g\in\mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^{\lambda,\gamma}(\phi)$, if there exists a function $f\in\mathcal{S}_p^\lambda(\phi)$ such that $g(z)=z^p\left[\frac{f(z)}{z^p}\right]^\gamma$, where $\left.\left[\frac{f(z)}{z^p}\right]^\gamma\right|_{z=0}=1$.
2. A function $g\in\mathcal{A}_p$ is said to be in the class $\mathcal{C}_p^{\lambda,\gamma}(\phi)$, if $G\in\mathcal{S}_p^{\lambda,\gamma}(\phi)$, where $G(z)=\frac{zg'(z)}{p}$.

Remarks 1.2. 1. Note that for the special case $\gamma=1$ the classes $\mathcal{S}_p^{\lambda,\gamma}(\phi)$ and $\mathcal{C}_p^{\lambda,\gamma}(\phi)$ coincide with $\mathcal{S}_p^\lambda(\phi)$ and $\mathcal{C}_p^\lambda(\phi)$ respectively.
2. Also, for $p=1$ and $\phi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ we have $\mathcal{S}_1^{\lambda,\gamma}[A,B]=:\mathcal{S}^{\lambda,\gamma}[A,B]$, which was studied by Ahuja [1].

For the functions $f\in\mathcal{A}_p$, given by (1.1) and $g\in\mathcal{A}_p$ of the form

$$g(z)=z^p+\sum_{k=p+1}^{\infty}b_kz^k,$$

the Hadamard (or convolution) product of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kb_kz^k.$$

For $a_i\in\mathbb{C}$ $(i=1,2,\ldots,q)$ and $b_j\in\mathbb{C}\setminus\mathbb{Z}_-^0$ $(j=1,2,\ldots,s)$, where $\mathbb{Z}_-^0=\{0,-1,-2,-3,\ldots\}$, we define *the generalized hypergeometric function* ${}_qF_s(a_1,\ldots,a_q;b_1,\ldots,b_s;z)$ (see, for example, [2]) by the following infinite series

$$ {}_qF_s(a_1,\dots,a_q;b_1,\dots,b_s;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_q)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!},\quad z\in\mathrm{U}, \tag{1.5}$$

where $q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ with $q\le s+1$, and $(\lambda)_r$ is *the Pochhammer symbol* defined in terms of the Gamma function by

$$(\lambda)_r=\frac{\Gamma(\lambda+r)}{\Gamma(\lambda)}=\begin{cases}1, & \text{for } r=0,\ \lambda\in\mathbb{C}\setminus\{0\},\\ \lambda(\lambda+1)\cdots(\lambda+r-1), & \text{for } r\in\mathbb{N},\ \lambda\in\mathbb{C}.\end{cases}$$

Corresponding to a function $h_p(a_1,\dots,a_q;b_1,\dots,b_s;z)$ given by

$$h_p(a_1,\dots,a_q;b_1,\dots,b_s;z)=z^p\cdot{}_qF_s(a_1,\dots,a_i,\dots,a_q;b_1,\dots,b_s;z).$$

Dziok and Srivastava (see [3,4]) considered the linear operator

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s):\mathcal{A}_p\to\mathcal{A}_p$$

defined by the following Hadamard product:

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s)f(z)=h_p(a_1,\dots,a_q;b_1,\dots,b_s;z)*f(z).$$

Thus, for a function f of the form (1.1) we have

$$H_p(a_1,\dots,a_q;b_1,\dots,b_s)f(z)=z^p+\sum_{k=p+1}^{\infty}\Gamma_{k-p}[a_1;b_1]a_kz^k,$$

where

$$\Gamma_{k-p}[a_1;b_1]:=\frac{1}{(k-p)!}\frac{(a_1)_{k-p}\cdots(a_q)_{k-p}}{(b_1)_{k-p}\cdots(b_s)_{k-p}},$$

and for convenience, we write

$$H_{p;q,s}(a_1):=H_p(a_1,\dots,a_q;b_1,\dots,b_s).$$

For the various properties and special cases of this operator we may refer to [2,5,3,4,6,7], and many others.

Definition 1.3. Let $\lambda\in\mathbb{R}$ with $|\lambda|<\frac{\pi}{2}$, let $p\in\mathbb{N}$, and let ϕ be an univalent function in the unit disc U with $\phi(0)=1$, that satisfies (1.2). For $q,s\in\mathbb{N}_0$, with $q\le s+1$, we define the following two subclasses of analytic functions:

$$\begin{aligned}\mathcal{S}^{\lambda,\gamma}_{p;q,s}[a_1;\phi] &:= \mathcal{S}^{\lambda,\gamma}_{p}[a_1,\dots,a_q;b_1,\dots,b_s;\phi]\\ &= \left\{f\in\mathcal{A}_p: H_{p;q,s}(a_1)f\in\mathcal{S}^{\lambda,\gamma}_p(\phi)\right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}^{\lambda,\gamma}_{p;q,s}[a_1;\phi] &:= \mathcal{C}^{\lambda,\gamma}_{p}[a_1,\dots,a_q;b_1,\dots,b_s;\phi]\\ &= \left\{f\in\mathcal{A}_p: H_{p;q,s}(a_1)f\in\mathcal{C}^{\lambda,\gamma}_p(\phi)\right\}.\end{aligned}$$

In particular, for $\gamma=1$ we denote $\mathcal{S}^{\lambda}_{p;q,s}[a_1;\phi]:=\mathcal{S}^{\lambda,1}_{p;q,s}[a_1;\phi]$ and $\mathcal{C}^{\lambda}_{p;q,s}[a_1;\phi]:=\mathcal{C}^{\lambda,1}_{p;q,s}[a_1;\phi]$.

In the present paper we studied several convolution and inclusion properties of the subclasses introduced in this section, that extend some known results obtained by Ahuja [1], and Aouf and Soudy [5].

2. Convolution properties

Theorem 2.1. *Let $f\in\mathcal{A}_p$, let $\gamma\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ be an arbitrary number, and let $g(z)=z^p\left[\frac{f(z)}{z^p}\right]^{\gamma}$, where the power function is considered to the main branch, i.e. $\left[\frac{f(z)}{z^p}\right]^{\gamma}\Big|_{z=0}=1$. Then, $g\in\mathcal{S}^{\lambda}_p(\phi)$ if and only if*

$$\frac{1}{z^p}\left[f(z)*\frac{z^p-Cz^{p+1}}{(1-z)^2}\right]\neq 0,\quad z\in\mathrm{U}, \tag{2.1}$$

and for all

$$C=C_\gamma(x)=\frac{[1-\phi(x)]\,p\cos\lambda-\gamma e^{i\lambda}}{[1-\phi(x)]\,p\cos\lambda},\quad |x|=1, \tag{2.2}$$

and also for $C=1$, if $\gamma\in\mathbb{C}^*\setminus\{1\}$.

Proof. For any $f \in \mathcal{A}_p$, we may easily deduce that

$$f(z) = f(z) * \frac{z^p}{1-z} \quad \text{and} \quad zf'(z) = f(z) * \left[\frac{z^{p+1}}{(1-z)^2} + \frac{pz^p}{1-z}\right]. \tag{2.3}$$

1. Suppose that $g \in \mathcal{S}_p^{\lambda}(\phi)$, where $g(z) = z^p \left[\frac{f(z)}{z^p}\right]^{\gamma}$. Since

$$\gamma e^{i\lambda} \frac{zf'(z)}{f(z)} + e^{i\lambda} p(1-\gamma) = e^{i\lambda} \frac{zg'(z)}{g(z)},$$

it follows that $g \in \mathcal{S}_p^{\lambda}(\phi)$ if and only if

$$\gamma e^{i\lambda} \frac{zf'(z)}{f(z)} + e^{i\lambda} p(1-\gamma) \prec p\left(\phi(z)\cos\lambda + i\sin\lambda\right). \tag{2.4}$$

Using the fact that the function from the right-hand side of the above subordination is univalent, from (2.4) it follows that

$$\gamma e^{i\lambda} \frac{zf'(z)}{f(z)} + e^{i\lambda} p(1-\gamma) - p\phi(x)\cos\lambda - ip\sin\lambda \neq 0, \quad \forall |x| = 1,\ \forall\, z \in \mathrm{U}. \tag{2.5}$$

From the convolution relations (2.3), a simple computation shows that (2.5) is equivalent to (2.1), where C is given by (2.2).

If $\gamma \in \mathbb{C}^* \setminus \{1\}$, since $g(z) = z^p \left[\frac{f(z)}{z^p}\right]^{\gamma}$ is analytic in U it follows that $\frac{f(z)}{z^p} \neq 0$ for all $z \in \mathrm{U}$, and according to (2.3) this is equivalent to

$$\frac{1}{z^p}\left[f(z) * \frac{z^p}{1-z}\right] \neq 0, \quad z \in \mathrm{U}, \tag{2.6}$$

that represents (2.1) for $C = 1$.

2. Reversely, if $\gamma \in \mathbb{C}^* \setminus \{1\}$, since the assumption (2.1) holds for $C = 1$, it follows that (2.6) holds, and according to (2.3) this shows that $\frac{f(z)}{z^p} \neq 0$ for all $z \in \mathrm{U}$. Thus, the power function $\left[\frac{f(z)}{z^p}\right]^{\gamma}$ has uniform branches in the unit disc U, hence the function g is well defined.

Since it was shown in the first part of the proof that the assumption (2.1) is equivalent to (2.5), we obtain that

$$\gamma e^{i\lambda} \frac{zf'(z)}{f(z)} + e^{i\lambda} p(1-\gamma) \neq p\left(\phi(x)\cos\lambda + i\sin\lambda\right), \quad \forall |x| = 1,\ \forall\, z \in \mathrm{U}. \tag{2.7}$$

Denoting

$$\varphi(z) = \gamma e^{i\lambda} \frac{zf'(z)}{f(z)} + e^{i\lambda} p(1-\gamma), \qquad \psi(z) = p\left(\phi(z)\cos\lambda + i\sin\lambda\right),$$

the relation (2.7) shows that $\varphi(\mathrm{U}) \cap \psi(\partial\mathrm{U}) = \emptyset$, and thus the simply-connected domain $\varphi(\mathrm{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathrm{U})$. From here, and using the fact that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which represents in fact the subordination (2.4), i.e. $g \in \mathcal{S}_p^{\lambda}(\phi)$. □

Theorem 2.2. *If $g \in \mathcal{A}_p$, then $g \in \mathcal{S}_p^{\lambda,\gamma}(\phi)$ if and only if*

$$\frac{1}{z^p}\left[g(z) * \frac{z^p - Dz^{p+1}}{(1-z)^2}\right] \neq 0, \quad z \in \mathrm{U}, \tag{2.8}$$

for all

$$D = D_{\gamma}(x) = \frac{[1-\phi(x)]\,p\gamma\cos\lambda - e^{i\lambda}}{[1-\phi(x)]\,p\gamma\cos\lambda}, \quad |x| = 1. \tag{2.9}$$

Proof. From Definition 1.2, $g \in \mathcal{S}_p^{\lambda,\gamma}(\phi)$ if there exists a function $f \in \mathcal{S}_p^{\lambda}(\phi)$ such that $g(z) = z^p \left[\frac{f(z)}{z^p}\right]^{\gamma}$, where $\left[\frac{f(z)}{z^p}\right]^{\gamma}\Big|_{z=0} = 1$.

Since

$$\gamma \frac{zf'(z)}{f(z)} + p(1-\gamma) = \frac{zg'(z)}{g(z)},$$

and using the fact that $g \in \mathcal{S}_p^{\lambda,\gamma}(\phi)$ if and only if $f \in \mathcal{S}_p^{\lambda}(\phi)$, it follows that $g \in \mathcal{S}_p^{\lambda,\gamma}(\phi)$ is equivalent to

$$e^{i\lambda}\frac{zg'(z)}{g(z)} - e^{i\lambda}p(1-\gamma) = \gamma e^{i\lambda}\frac{zf'(z)}{f(z)} \prec p\gamma\left(\phi(z)\cos\lambda + i\sin\lambda\right).$$

Using the same arguments as in the proof of Theorem 2.1, we find that the above subordination holds if and only if the relation (2.8) is satisfied whenever D has the form (2.9). □

Remark 2.1. Setting $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ and writing $-\bar{x}$ as x in Theorems 2.1 and 2.2, we obtain the convolution results obtained by Ahuja [1].

For $\gamma = 1$ we have $g(z) = f(z)$, where g is defined like in Theorem 2.1, and $C_1(x) \equiv D_1(x)$, hence both of the results of Theorems 2.1 and 2.2 coincide with the following one:

Corollary 2.1. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{S}_p^{\lambda}(\phi)$ if and only if (2.1) holds for all*

$$C = C_1(x) = \frac{[1-\phi(x)]\,p\cos\lambda - e^{i\lambda}}{[1-\phi(x)]\,p\cos\lambda}, \quad |x| = 1. \tag{2.10}$$

Theorem 2.3. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{C}_p^{\lambda}(\phi)$ if and only if*

$$\frac{1}{z^p}\left[f(z) * \frac{pz^p - [p + C(p+1) - 2]\,z^{p+1} + C(p-1)z^{p+2}}{(1-z)^3}\right] \ne 0, \quad z \in \mathrm{U}, \tag{2.11}$$

for all C, where $C = C_1(x)$ is given by (2.10).

Proof. If we let $G(z) = \frac{z^p - Cz^{p+1}}{(1-z)^2}$, then

$$zG'(z) = \left[\frac{pz^p - [p + C(p+1) - 2]\,z^{p+1} + C(p-1)z^{p+2}}{(1-z)^3}\right].$$

Using the duality relation (1.3), we have

$$f \in \mathcal{C}_p^{\lambda}(\phi) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^{\lambda}(\phi),$$

and according to Corollary 2.1 we deduce that $f \in \mathcal{C}_p^{\lambda}(\phi)$ if and only if

$$\frac{1}{z^p}\left[\frac{zf'(z)}{p} * G(z)\right] \ne 0, \quad z \in \mathrm{U}, \tag{2.12}$$

for all C given by (2.2). Since the identity $zf'(z) * g(z) = f(z) * zg'(z)$ holds for all $f, g \in \mathcal{A}_p$, in particular we have $zf'(z) * G(z) = f(z) * zG'(z)$, hence the condition (2.12) reduces to

$$\frac{1}{z^p}\left[f(z) * zG'(z)\right] \ne 0, \quad z \in \mathrm{U}.$$

This last relation is equivalent to (2.11), which gives the required result. □

Theorem 2.4. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{C}_p^{\lambda,\gamma}(\phi)$ if and only if*

$$\frac{1}{z^p}\left[f(z) * \frac{pz^p - [p + D(p+1) - 2]\,z^{p+1} + D(p-1)z^{p+2}}{(1-z)^3}\right] \ne 0, \quad z \in \mathrm{U}, \tag{2.13}$$

for all D, where $D = D_\gamma(x)$ is given by (2.9).

Proof. Letting $\phi(z) = \frac{z^p - Dz^{p+1}}{(1-z)^2}$, we have

$$zH'(z) = \left[\frac{pz^p - [p + D(p+1) - 2]\,z^{p+1} + D(p-1)z^{p+2}}{(1-z)^3}\right].$$

From Definition 1.2 we have that $f \in \mathcal{C}_p^{\lambda,\gamma}(\phi)$ if and only if $\frac{zf'(z)}{p} \in \mathcal{S}_p^{\lambda,\gamma}(\phi)$, and according to Theorem 2.2 we deduce that $f \in \mathcal{C}_p^{\lambda,\gamma}(\phi)$ if and only if

$$\frac{1}{z^p}\left[\frac{zf'(z)}{p} * \phi(z)\right] \ne 0, \quad z \in \mathrm{U},$$

for all D given by (2.9). Now, applying exactly the same method used in the proof of Theorem 2.3 we easily obtain the desired result. □

Remark 2.2. For $\gamma = 1$, since $C_1(x) \equiv D_1(x)$, the results of Theorems 2.3 and 2.4 coincide.

Theorem 2.5. *If $f \in \mathcal{A}_p$ is of the form (1.1), then $f \in \mathcal{S}^{\lambda,\gamma}_{p;q,s}[a_1;\phi]$ if and only if*

$$1 + \sum_{k=p+1}^{\infty} \left[\frac{[1-\phi(x)]\,p\gamma\cos\lambda + (k-p)e^{i\lambda}}{[1-\phi(x)]\,p\gamma\cos\lambda} \right] \Gamma_{k-p}[a_1;b_1]a_k z^{k-p} \neq 0, \quad z \in \mathrm{U}, \tag{2.14}$$

for all $|x| = 1$.

Proof. From Theorem 2.2 we find that $f \in \mathcal{S}^{\lambda,\gamma}_{p;q,s}[a_1;\phi]$ if and only if

$$\frac{1}{z^p}\left[H_{p;q,s}(a_1)f(z) * \frac{z^p - Dz^{p+1}}{(1-z)^2} \right] \neq 0, \quad z \in \mathrm{U}, \tag{2.15}$$

for all $|x| = 1$, where D is given by (2.9). Since

$$\frac{z^p}{(1-z)^2} = z^p + \sum_{k=p+1}^{\infty}(k-p+1)z^k, \qquad \frac{z^{p+1}}{(1-z)^2} = \sum_{k=p+1}^{\infty}(k-p)z^k, \tag{2.16}$$

after some simple computations in Eq. (2.15), with the help of (2.16) we get

$$\frac{1}{z^p}\left[z^p + \sum_{k=p+1}^{\infty}[1+(1-D)(k-p)]\Gamma_{k-p}[a_1;b_1]a_k z^k \right] \neq 0, \quad z \in \mathrm{U},$$

for all $|x| = 1$, which simplifies to (2.14), and the proof of the theorem is complete. □

Theorem 2.6. *If $f \in \mathcal{A}_p$ is of the form (1.1), then $f \in \mathcal{C}^{\lambda,\gamma}_{p;q,s}[a_1;\phi]$ if and only if*

$$1 + \sum_{k=p+1}^{\infty} \frac{k}{p}\left[\frac{[1-\phi(x)]\,p\gamma\cos\lambda + (k-p)e^{i\lambda}}{[1-\phi(x)]\,p\gamma\cos\lambda} \right] \Gamma_{k-p}[a_1;b_1]a_k z^{k-p} \neq 0, \quad z \in \mathrm{U}, \tag{2.17}$$

for all $|x| = 1$.

Proof. From Theorem 2.4 we have that $f \in \mathcal{C}^{\lambda,\gamma}_{p;q,s}[a_1;\phi]$ if and only if

$$\frac{1}{z^p}\left[H_{p;q,s}(a_1)f(z) * \frac{pz^p - [p + D(p+1) - 2]\,z^{p+1} + D(p-1)z^{p+2}}{(1-z)^3} \right] \neq 0, \quad z \in \mathrm{U}, \tag{2.18}$$

for all $|x| = 1$, where D is given by (2.9). Now, it can be easily shown that

$$\frac{z^p}{(1-z)^3} = z^p + \sum_{k=p+1}^{\infty}\frac{(k-p+1)(k-p+2)}{2}z^k, \quad z \in \mathrm{U},$$
$$\frac{z^{p+1}}{(1-z)^3} = \sum_{k=p+1}^{\infty}\frac{(k-p)(k-p+1)}{2}z^k, \quad z \in \mathrm{U},$$
$$\frac{z^{p+2}}{(1-z)^3} = \sum_{k=p+1}^{\infty}\frac{(k-p-1)(k-p)}{2}z^k \quad z \in \mathrm{U}.$$

Using these identities in (2.18), after some simple computations we get

$$\frac{1}{z^p}\left[pz^p + \sum_{k=p+1}^{\infty}k[1+(1-D)(k-p)]\Gamma_{k-p}[a_1;b_1]a_k z^k \right] \neq 0, \quad z \in \mathrm{U},$$

for all $|x| = 1$, which simplifies to (2.17). □

3. Inclusion properties

In this section we will discuss some inclusion relations for the classes $\mathcal{S}^{\lambda}_{p;q,s}[a_1;\phi]$ and $\mathcal{C}^{\lambda}_{p;q,s}[a_1;\phi]$. To prove these results we shall require the following lemma:

Lemma 3.1 ([8]). *Let ϕ be convex (univalent) in* U, *with* $\operatorname{Re}[\beta\phi(z)+\gamma] > 0$ *for all* $z \in$ U. *If* q *is analytic in* U, *with* $q(0) = \phi(0)$, *then*

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} \prec \phi(z) \Rightarrow q(z) \prec \phi(z).$$

Theorem 3.1. *Suppose that the function* ϕ *is convex (univalent) in* U *and satisfies the inequality*

$$p \operatorname{Re}\phi(z) > (p - \operatorname{Re} a_1)\sec^2\lambda - p\,[\operatorname{Im}\phi(z) + \tan\lambda]\tan\lambda, \quad z \in \mathrm{U}. \tag{3.1}$$

If $f \in \mathcal{S}^{\lambda}_{p;q,s}[a_1+1;\phi]$ *such that* $H_{p;q,s}(a_1)f(z) \neq 0$ *for all* $z \in \mathrm{U}\setminus\{0\}$, *then* $f \in \mathcal{S}^{\lambda}_{p;q,s}[a_1;\phi]$.

Proof. Suppose that $f \in \mathcal{S}^{\lambda}_{p;q,s}[a_1+1;\phi]$, and let us define

$$q(z) = \frac{1}{\cos\lambda}\left[e^{i\lambda}\frac{z\left(H_{p;q,s}(a_1)f(z)\right)'}{pH_{p;q,s}(a_1)f(z)} - i\sin\lambda\right]. \tag{3.2}$$

Then q is analytic in U, and using the well-known relation

$$z\left(H_{p;q,s}(a_1)f(z)\right)' = a_1H_{p;q,s}(a_1+1)f(z) - (a_1-p)H_{p;q,s}(a_1)f(z),$$

from (3.2) we obtain

$$q(z)\cos\lambda + \frac{a_1-p}{p}e^{i\lambda} + i\sin\lambda = e^{i\lambda}\frac{a_1H_{p;q,s}(a_1+1)f(z)}{pH_{p;q,s}(a_1)f(z)}. \tag{3.3}$$

Differentiating logarithmically (3.3) and then using (3.2), we deduce that

$$q(z) + \frac{zq'(z)}{p\,[q(z)\cos\lambda + i\sin\lambda]\,e^{-i\lambda} + a_1 - p} \prec \phi(z). \tag{3.4}$$

Since the inequality $\operatorname{Re}\left[p\,[\phi(z)\cos\lambda + i\sin\lambda]\,e^{-i\lambda} + a_1 - p\right] > 0$, $z \in$ U, is equivalent to (3.1), according to Lemma 3.1 the subordination (3.4) implies $q(z) \prec \phi(z)$, which proves that $f \in \mathcal{S}^{\lambda}_{p;q,s}[a_1;\phi]$. □

From the duality formula (1.3) we have

$$H_{p;q,s}(a_1)f \in \mathcal{C}^{\lambda}_{p}(\phi) \Leftrightarrow \frac{z\left(H_{p;q,s}(a_1)f(z)\right)'}{p} \in \mathcal{S}^{\lambda}_{p}(\phi),$$

and using the fact that

$$H_{p;q,s}(a_1)\left(\frac{zf'(z)}{p}\right) = \frac{z\left(H_{p;q,s}(a_1)f(z)\right)'}{p},$$

in the same way we may prove the following inclusion:

Theorem 3.2. *Suppose that the function* ϕ *satisfies the inequality* (3.1). *If* $f \in \mathcal{C}^{\lambda}_{p;q,s}[a_1+1;\phi]$ *such that* $H_{p;q,s}(a_1)\left(\frac{zf'(z)}{p}\right) \neq 0$ *for all* $z \in \mathrm{U}\setminus\{0\}$, *then* $f \in \mathcal{C}^{\lambda}_{p;q,s}[a_1;\phi]$.

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