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Some new characterizations of finite frames and F-frames

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Let $\mathcal{R}(L)$ stand for the ring of all real valued continuous functions on a completely regular frame L. We first show that if L is generated by the cozeros of functions in $\mathcal{R}_\Psi(L)$, the ring of all functions in $\mathcal{R}(L)$ which have pseudocompact support, then L is finite if and only if $\mathcal{R}_\Psi(L)$ is a Noetherian ring. We next show for a supportively normal frame L, which in addition is also generated by the cozeros of functions in $\mathcal{R}_\Phi(L)$, the ring of all functions in $\mathcal{R}(L)$ having realcompact support, that L is finite if and only if $\mathcal{R}_\Phi(L)$ is Noetherian. We further check that a frame (resp., a completely regular frame) L is finite if and only if $\mathcal{R}(L)$ is a Noetherian/Artinian/semisimple/hereditary ring if and only if each maximal ideal of $\mathcal{R}(L)$ is principal. This last result expands one of the basic theorems of the present authors already achieved. In the next stage we show that L is an F-frame if and only if each ideal of $\mathcal{R}(L)$ is flat if and only if for each $f \in \mathcal{R}(L)$, pos(f) and neg(f) are completely separated. If $\mathcal{P}$ is an ideal of closed quotients of L, then a normal $\mathcal{P}$-continuous frame turns out to be an F-frame when and only when for each $f \in \mathcal{R}_\mathcal{P}(L)$, the functions in $\mathcal{R}(L)$ having their support on $\mathcal{P}$, pos(f) and neg(f) are completely separated. Finally we realise that L is a P-frame if and only if each prime ideal of $\mathcal{R}(L)$ is a z-ideal if and only if every $\mathcal{R}(L)$-module is flat and also that L is basically disconnected if and only if $\mathcal{R}(L)$ is a semihereditary ring. It is worth mentioning that most of the above results, barring a few have their classical counterparts already available in the literature.

1. Introduction

Throughout this paper by a frame L we shall always mean a completely regular frame and a ring will always stand for a commutative one. $\mathcal{R}(L)$ stands for the ring of all real valued continuous functions on L. The main aim of this article is to characterize finite frames L and also F-frames through $\mathcal{R}(L)$ and some

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of their subrings. The support of an f in $\mathcal{R}(L)$ is the closed quotient $\uparrow(\mathrm{coz}(f))^*$. For any ideal $\mathcal{P}$ of closed quotients of L, let $\mathcal{R}_{\mathcal{P}}(L)$ designate the aggregate of all those functions in $\mathcal{R}(L)$, whose supports lie in $\mathcal{P}$. Taking $\mathcal{P} = \Psi =$ the ideal of all closed pseudocompact quotients of L, it was established by Dube [16] that $\mathcal{R}_\Psi(L)$ is an ideal of the ring $\mathcal{R}(L)$. We first show in Section 2 that if L is generated by the cozeros of functions lying in $\mathcal{R}_\Psi(L)$, then L is finite if and only if $\mathcal{R}_\Psi(L)$ is a Noetherian ring (Theorem 2.1). The classical counterpart of this result on choosing $L = \mathfrak{O}X =$ the frame of all open subsets of a Tychonoff space X reads as follows: if a space X is locally pseudocompact, meaning that each point of X has a pseudocompact neighbourhood, then X is finite if and only if the ring $C_\Psi(X)$ of all functions in $C(X)$ with pseudocompact support is Noetherian. This last result has been established recently in [3] by using an entirely different technique. L is called *realcompact*, if whenever I is a maximal ideal of the cozero part of L, i.e., $\mathrm{Coz}\, L$ with the property that the join of any countable subset of I is not the top element $(= 1)$ of L, then $\bigvee I < 1$ [23]. The set of functions in $\mathcal{R}(L)$ with realcompact support is in general not an ideal. However if supports of L are C-quotients, in which case L is called *supportively normal*, then $\mathcal{R}_\Phi(L)$ is very much an ideal of $\mathcal{R}(L)$, here $\mathcal{R}_\Phi(L)$ stands for the set of all functions in $\mathcal{R}(L)$, which have realcompact support [17]. Incidentally we have established in the same Section 2 of this paper that, if in addition such an L happens to be generated by the cozeros of functions lying in $\mathcal{R}_\Phi(L)$, then L is finite if and only if $\mathcal{R}_\Phi(L)$ is a Noetherian ring (Theorem 2.2). The authors are not aware of whether the classical counterpart of the last result is already there in the literature. Detailed information about these two rings $\mathcal{R}_\Psi(L)$ and $\mathcal{R}_\Phi(L)$, introduced by Dube can be found in his papers [16] and [17]. Finally in this section we show by using Axiom of Choice (AC) that, L is finite if and only if $\mathcal{R}(L)$ is a semisimple/hereditary ring if and only if $\mathcal{R}(L)$ is a Noetherian/Artinian ring if and only if each maximal ideal of $\mathcal{R}(L)$ is principal (Theorem 2.5). It may be mentioned that the classical counterpart of a portion of this theorem, which specifically says that a space X is finite if and only if $C(X)$ is a hereditary ring is already established by Brookshear (see [9, Theorem 2]). The ring $\mathcal{R}(L)$ considered as a module over itself, is called *semisimple* if it is a direct sum of simple modules. On the other hand $\mathcal{R}(L)$ is termed as *hereditary* if every ideal of it is projective [27]. We would like to mention in this context that, the last Theorem 2.5 in this section is an expansion of Theorem 3.11 in our paper [1] by adding some interesting equivalent characterizations of finite frames highlighting a number of module theoretic structural specialities of $\mathcal{R}(L)$ and its ideals, to the list of equivalences in it. Almost a similar kind of comment can be made in the context of the last theorem viz. Theorem 4.6 of this paper, which says that L is basically disconnected[2] if and only if $\mathcal{R}(L)$ is semihereditary[3]/coherent and which is surely an augmented version of Theorem 5.1 of our paper [1] quoted in the last sentence.

Section 3 of this paper is entirely devoted to finding out several new characterizations of F-frames and one characterization of quasi-F-frames. We would like to recall that, L is an *F-frame* if the open quotient of each cozero element of it is a C^*-quotient. F-frames were introduced by Ball and Walters-Wayland, who had offered a number of equivalent descriptions of these frames [5]. Incidentally some other characterizations of F-frames are given by Dube [12]. L is called a *quasi-F-frame*, an entity also introduced by Ball and Walters-Wayland [5], if the open quotient of each dense cozero element of it is a C^*-quotient. A number of equivalent descriptions of these frames is given by Dube and Matlabyane [14]. We first show that L is an F-frame if and only if each finitely generated ideal of $\mathcal{R}(L)$ is flat equivalently each ideal of $\mathcal{R}(L)$ is flat (Theorem 3.1 and Corollary 3.2). An ideal I of $\mathcal{R}(L)$ is called *flat* if the tensor product $I \otimes_{\mathcal{R}(L)} -$ is an exact functor, i.e., if

$$0 \longrightarrow N_1 \xrightarrow{\ i\ } N \xrightarrow{\ p\ } N_2 \longrightarrow 0$$

is an exact sequence of $\mathcal{R}(L)$-modules, then

[2] A frame L is called *basically disconnected* if $(\mathrm{coz}(f))^* \vee (\mathrm{coz}(f))^{**} = 1$ for each $f \in \mathcal{R}(L)$.

[3] $\mathcal{R}(L)$ is called *semihereditary* if every finitely generated ideal of it is projective.

$$0 \longrightarrow M \otimes_{\mathcal{R}(L)} N_1 \xrightarrow{1_M \otimes i} M \otimes_{\mathcal{R}(L)} N \xrightarrow{1_M \otimes p} M \otimes_{\mathcal{R}(L)} N_2 \longrightarrow 0$$

is an exact sequence of abelian groups. Incidentally a result of ours viz. Corollary 3.2 turns out to be the pointfree extension of the classical result, which says that X is an F-space if and only if each ideal of $C(X)$ is flat, and this was established by Neville (see [25, Corollary 1.6]). A well known characterization of F-spaces says that, X is an F-space if and only if each ideal of the lattice ordered ring $C(X)$ is convex if and only if the positive and negative parts of each function in $C(X)$ are completely separated (see [19, Theorem 14.25]). We like to mention that in this paper, we have established the pointfree versions of these results too (Theorem 3.5 and Theorem 3.4). For the definition of convexity of ideals see [19, Chapter 5]. We have observed that by imposing an additional condition of normality together with a continuity like condition on L, there emerges a seemingly new description of F-frames. Indeed if the frame L is normal and $\mathcal{P}$*-continuous* in the sense that $\{\operatorname{coz}(f) : f \in \mathcal{R}_{\mathcal{P}}(L)\}$ generates L, then L is an F-frame if and only if for each $f \in \mathcal{R}_{\mathcal{P}}(L)$, $\operatorname{pos}(f)$ and $\operatorname{neg}(f)$ are completely separated (Theorem 3.7). This we feel is the exact pointfree extension of the classical result in $C(X)$, recently achieved by Ghosh in [18, Theorem 2.1], see also [2]. Finally in this section, we have proved that a frame L is a quasi-F-frame if and only if $\mathcal{R}(L)$ is a *Prüfer ring*, meaning that every finitely generated regular ideal I of it is *invertible*, i.e., for which $II^{-1} = \mathcal{R}(L)$, where $I^{-1} = \{a \in q(\mathcal{R}(L)) : aI \subseteq \mathcal{R}(L)\}$ and $q(\mathcal{R}(L))$ is the ring of fractions of $\mathcal{R}(L)$.

In the final Section 4 of this article, we have established a few results on P-frames,[4] z-ideals, flatness of $\mathcal{R}(L)$-modules and also on basically disconnected frames each of which is of independent interest. Mention may be made of Theorem 4.1 and Theorem 4.3, which combined together tell that L is a P-frame if and only if every prime ideal of $\mathcal{R}(L)$ is a z-ideal if and only if every $\mathcal{R}(L)$-module is flat. We further like to add in this context that the classical counterparts of these two facts are quite well known (see [19, 14B(4)] and [25, Theorem 1.8]).

For the general theory of frames we refer the reader to ([20] and [26]), and for their pointfree function rings we refer the reader to ([6] and [5]). For the algebraic terminologies we refer the reader to the books ([27,21] and [10]). Finally we would like to mention that, in this article we have freely used the Boolean Ultrafilter Theorem (BUT), the Axiom of Countable Choice (ACC) and the Axiom of Choice (AC).

2. Characterizations of finite frames

Theorem 2.1. *Let L be a frame with the property that $\{\operatorname{coz}(f) : f \in \mathcal{R}_{\Psi}(L)\}$ generates L. Then L is finite if and only if $\mathcal{R}_{\Psi}(L)$ is a Noetherian ring.*

Proof (BUT, ACC). Dube shows that, $\mathcal{R}_{\Psi}(L)$ is an ideal of $\mathcal{R}(L)$ and is equal to $\mathcal{R}(L)$ if and only if L is pseudocompact [16].

First let L be finite. Then $\mathcal{R}_{\Psi}(L) = \mathcal{R}(L)$ as L is pseudocompact and hence $\mathcal{R}_{\Psi}(L)$ is a Noetherian ring as $\mathcal{R}(L)$ is Noetherian [1, Theorem 3.11].

Conversely suppose that L be infinite. Then if L is pseudocompact then again $\mathcal{R}_{\Psi}(L) = \mathcal{R}(L)$ and hence $\mathcal{R}_{\Psi}(L)$ is not a Noetherian ring as $\mathcal{R}(L)$ is not Noetherian [1, Theorem 3.11].

Again if L is not pseudocompact then L is not compact. Then there is a subset A of L such that $\bigvee A = 1$ but for any finite subset S of A, $\bigvee S \neq 1$. Let

$$B = \left\{\bigvee S : S \text{ is a finite subset of } A\right\}.$$

Then $A \subseteq B$ and hence B is infinite with $\bigvee B = 1$, but $1 \notin B$.

[4] A frame L is called a *P-frame* if $\operatorname{coz}(f) \vee (\operatorname{coz}(f))^* = 1$ for each $f \in \mathcal{R}(L)$.

Choose $b_1 \in B$. Then it will never happen that for each $b \in B$, $b \leq b_1$. This means that there is a $b_0 \in B$ such that $b_0 \leq b_1$ is not true. Set $b_2 = b_1 \vee b_0$, then $b_2 \in B$ and $b_1 \lneqq b_2$. A use of Principle of Mathematical Induction then yields a strictly increasing sequence $\{b_n : n \in \omega_0\}$ of elements of B. For each $n \in \omega_0$, let $I_n = \{f \in \mathcal{R}_\Psi(L) : \mathrm{coz}(f) \leq b_n\}$. Then I_n is an ideal of $\mathcal{R}_\Psi(L)$ and $I_n \subseteq I_{n+1}$. Since L is generated by $\{\mathrm{coz}(f) : f \in \mathcal{R}_\Psi(L)\}$, we can write

$$b_{n+1} = \bigvee\{\mathrm{coz}(f_\beta) : \beta \in \Gamma\},$$

for some (set-indexed) family $(f_\beta)_{\beta\in\Gamma}$ in $\mathcal{R}_\Psi(L)$. From this we can assert that there is at least one f_β for which $\mathrm{coz}(f_\beta) \nleq b_n$, because otherwise it would yield $b_{n+1} \leq b_n$, a contradiction. Therefore $f_\beta \in I_{n+1}$ but $f_\beta \notin I_n$. This proves $I_n \subsetneqq I_{n+1}$ for each $n \in \omega_0$. Hence $\mathcal{R}_\Psi(L)$ is not a Noetherian ring. □

Theorem 2.2. *Let L be a supportively normal frame with the property that $\{\mathrm{coz}(f) : f \in \mathcal{R}_\Phi(L)\}$ generates L. Then L is finite if and only if $\mathcal{R}_\Phi(L)$ is a Noetherian ring.*

Proof (<u>BUT</u>, <u>ACC</u>). Dube shows for a supportively normal frame L that, $\mathcal{R}_\Phi(L)$ is an ideal of $\mathcal{R}(L)$ and equals $\mathcal{R}(L)$ if and only if L is realcompact [17].

A simple adaptation of the proof of Theorem 2.1, can be applied to complete the proof. Indeed if L is infinite and not realcompact then L is not compact and hence by the same arguments of the last part of the above theorem together with the fact that, $\{\mathrm{coz}(f) : f \in \mathcal{R}_\Phi(L)\}$ generates L, $\mathcal{R}_\Phi(L)$ is not a Noetherian ring. □

Lemma 2.3. *For any completely regular frame L, $\mathcal{R}(L)$ is a J-semisimple*[5] *ring.*

Proof. Let $\mathcal{M}(\{\mathbf{0}\})$ be the collection of all maximal ideals of $\mathcal{R}(L)$ containing $\{\mathbf{0}\}$. Then $\mathcal{M}(\{\mathbf{0}\}) = \mathcal{M}$, where $\mathcal{M}$ is the collection of all maximal ideals of $\mathcal{R}(L)$. Now Lemma 3.7 of [11] tells us that, $\bigcap \mathcal{M} = \{f \in \mathcal{R}(L) : r_L(\mathrm{coz}(f)) \leq r_L(\mathrm{coz}(\mathbf{0}))\} = \{f \in \mathcal{R}(L) : r_L(\mathrm{coz}(f)) = \{0\}\} = \{f \in \mathcal{R}(L) : \mathrm{coz}(f) = 0\} = \{\mathbf{0}\}$, here $r_L : L \to \beta L$ is the right adjoint of the join map $j_L : \beta L \to L$ and βL is the Stone–Čech compactification of L (see [5] for details). □

Remark 2.4. This result has also been established by Dube and Ighedo [15] using the <u>BUT</u>, however the above proof doesn't require this assumption.

Theorem 2.5. *The following statements are equivalent for a frame L:*

(1) *L is finite.*
(2) *$\mathcal{R}(L)$ is a Noetherian ring.*
(3) *$\mathcal{R}(L)$ is an Artinian ring.*
(4) *$\mathcal{R}(L)$ is a semisimple ring.*
(5) *$\mathcal{R}(L)$ is a hereditary ring.*
(6) *Every maximal ideal of $\mathcal{R}(L)$ is principal.*

Proof (<u>AC</u>). (1) ⇒ (6): Let L be finite. Then it is a P-frame (see [1, Theorem 3.11]) and hence an F-frame. So every finitely generated ideal of $\mathcal{R}(L)$ is principal (see [12, Proposition 3.2]). Also L is finite implies $\mathcal{R}(L)$ is a Noetherian ring (see [1, Theorem 3.11]) and hence every ideal of it is finitely generated (see [27, Corollary 3.16]). Therefore each ideal and hence each maximal ideal of $\mathcal{R}(L)$ is principal.

[5] $\mathcal{R}(L)$ is called *J-semisimple* if the Jacobson radical, i.e., the intersection of all the maximal ideals of it is the zero ideal.

(6) ⇒ (1): Let every maximal ideal of $\mathcal{R}(L)$ be principal. Then they become fixed and hence L becomes a compact frame (see [13, Lemma 4.7]). Since L is regular, it is a spatial frame and hence it is finite (see [19, 4B(2)]).

Since every semisimple ring is hereditary (see [27, Example 4.12(i)]) (4) ⇒ (5) follows and (3) ⇒ (4) follows from Theorem 4.14 of [21] and Lemma 2.3 and furthermore the equivalence of the three statements (with the support of BUT) (1), (2) and (3) is established in [1], it suffices to prove the relation (5) ⇒ (2) only. So let $\mathcal{R}(L)$ be a hereditary ring which means that each ideal of $\mathcal{R}(L)$ is projective. In particular therefore from Corollary 2 of [22], a prime ideal P of $\mathcal{R}(L)$ has a projective basis $B = \{\phi_\alpha, f_\alpha\}_{\alpha\in\Lambda}$ with $\{f_\alpha\}_{\alpha\in\Lambda}$ a star finite set. We claim that the family $\{f_\alpha\}_{\alpha\in\Lambda}$ is finite and therefore every prime ideal of $\mathcal{R}(L)$ becomes finitely generated and hence $\mathcal{R}(L)$ turns out to be a Noetherian ring (see [10, Theorem 8.22]). We argue by contradiction; let $\{f_\alpha\}_{\alpha\in\Lambda}$ be an infinite set. Then a simple induction together with the star finiteness of the last family yields a countably infinite subfamily $\{f_i\}_{i\in\mathbb{N}}$ such that $f_i f_j = \mathbf{0}$, whenever $i \neq j$, $i, j \in \mathbb{N}$. Now by using the complete regularity of L, we can pick for each $i \in \mathbb{N}$, an $h_i \in \mathcal{R}(L)$ such that $\mathbf{0} \leq h_i \leq \frac{1}{2^i}$ and $0 \neq \mathrm{coz}(h_i) \leq \mathrm{coz}(f_i)$. Since $\mathcal{R}(L)$ is uniformly complete (see [5, Theorem 4.1.5]), i.e., every Cauchy sequence in $\mathcal{R}(L)$ converges uniformly to a limit in $\mathcal{R}(L)$, it follows that $k_1 = \sum_{i=1}^{\infty} h_{2i}$ and $k_2 = \sum_{i=1}^{\infty} h_{2i-1}$ both belong to $\mathcal{R}(L)$. Also from Section 6 of [7], $\mathrm{coz}(k_1) = \bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i})$ and $\mathrm{coz}(k_2) = \bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i-1})$. Then $\mathrm{coz}(k_1k_2) = \mathrm{coz}(k_1) \wedge \mathrm{coz}(k_2) = (\bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i})) \wedge (\bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i-1})) = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} (\mathrm{coz}(h_{2i}) \wedge \mathrm{coz}(h_{2j-1})) = 0$ and consequently $k_1k_2 = \mathbf{0} \in P$. But it is not hard to prove that neither k_1 nor k_2 belongs to P and this contradicts that P is a prime ideal of $\mathcal{R}(L)$. We show only that $k_1 \notin P$, the fact that $k_2 \notin P$ can be shown analogously. Indeed if $k_1 \in P$, then we can write, $k_1 = \sum_{i=1}^{n} \phi_{\alpha_i}(k_1) f_{\alpha_i}$, for a finite subset $\{f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_n}\}$ of the original family $\{f_\alpha\}_{\alpha\in\Lambda}$. But the last family is star finite, hence there exists an $i_0 \in \mathbb{N}$ such that $f_{\alpha_i} f_{2i_0} = \mathbf{0}$ for each $i = 1, 2, \ldots, n$. This yields $\bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i}) = \mathrm{coz}\, k_1 \leq \mathrm{coz}(f_{\alpha_1}) \vee \mathrm{coz}(f_{\alpha_2}) \vee \cdots \vee \mathrm{coz}(f_{\alpha_n}) \leq (\mathrm{coz}(f_{2i_0}))^*$, which in turn implies that $0 = \mathrm{coz}(f_{2i_0}) \wedge \bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i}) = \bigvee_{i=1}^{\infty} \mathrm{coz}(h_{2i} f_{2i_0})$ and hence $\mathrm{coz}(h_{2i_0} f_{2i_0}) = 0$. This relation, in conjunction with the choice $0 \neq \mathrm{coz}(h_{2i_0}) \leq \mathrm{coz}(f_{2i_0})$ gives rise to the relation $\mathrm{coz}(h_{2i_0}) = \mathrm{coz}(h_{2i} f_{2i_0}) = 0$ and hence $h_{2i_0} = \mathbf{0}$, a contradiction to the initial choice of h_i's. This completes the proof. □

3. Characterizations of *F*-frames and quasi-*F*-frames

Theorem 3.1. *L is an F-frame if and only if every finitely generated ideal of $\mathcal{R}(L)$ is flat.*

Proof. First suppose that, L is an F-frame. Let I be a finitely generated ideal of $\mathcal{R}(L)$. Since L is an F-frame, $I = \langle f \rangle$ for some $f \in \mathcal{R}(L)$ (see [12, Proposition 3.2]). Now it can be easily checked that

$$0 \longrightarrow K \longrightarrow \mathcal{R}(L) \xrightarrow{\varphi} I \longrightarrow 0$$

is an exact sequence of $\mathcal{R}(L)$-modules, where $K = \{k \in \mathcal{R}(L) : kf = \mathbf{0}\}$ and $\varphi(g) = fg$, $g \in \mathcal{R}(L)$. Let J be another finitely generated ideal. Then J is also principal and so $J = \langle r \rangle$ for some $r \in \mathcal{R}(L)$. We shall show that $K \cap J = KJ$. Firstly, $KJ \subseteq K \cap J$ always, so we must show that $K \cap J \subseteq KJ$. Let $gr \in K$ with $g \in \mathcal{R}(L)$. Then $grf = \mathbf{0}$ and so $\mathrm{coz}(fg) \wedge \mathrm{coz}(r) = 0$. Therefore $\mathrm{coz}(fg)$ and $\mathrm{coz}(r)$ are disjoint cozero elements of L, also since L is an F-frame, they are completely separated (see [5, Proposition 8.4.10]). Hence there exists $h \in \mathcal{R}(L)$ such that $\mathrm{coz}(fg) \wedge \mathrm{coz}(h) = 0$ and $\mathrm{coz}(r) \wedge \mathrm{coz}(\mathbf{1} - h) = 0$. First equality ensures that, $gh \in K$ and second ensures that, $gr = ghr \in KJ$. Therefore by Proposition 3.60 of [27] and taking care of the fact that $\mathcal{R}(L)$ is flat as it is free and hence projective and every projective module is flat (see [27]), we can conclude that I is flat.

Conversely suppose that, every finitely generated ideal of $\mathcal{R}(L)$ is flat. To show that L is an F-frame, it is sufficient to show that disjoint cozero elements of L are completely separated (see [5, Proposition 8.4.10]).

Let $\mathrm{coz}(f) \wedge \mathrm{coz}(r) = 0$ with $f, r \in \mathcal{R}(L)$, from which it follows that $fr = \mathbf{0}$. Consider the principal ideals $I = \langle f \rangle$, $J = \langle r \rangle$ and the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{R}(L) \xrightarrow{\varphi} I \longrightarrow 0$$

of $\mathcal{R}(L)$-modules with $K = \{k \in \mathcal{R}(L) : kf = \mathbf{0}\}$ and $\varphi(g) = fg$, $g \in \mathcal{R}(L)$. Then since I and $\mathcal{R}(L)$ are both flat we have $K \cap J = KJ$, by Proposition 3.60 of [27]. Since $r \in K \cap J$ (as $rf = \mathbf{0}$), it follows that $r = kr$ for some $k \in K$. So $\mathrm{coz}(f) \wedge \mathrm{coz}(k) = 0$ and $\mathrm{coz}(r) \wedge \mathrm{coz}(\mathbf{1} - k) = 0$ and hence $\mathrm{coz}(f)$ and $\mathrm{coz}(r)$ are completely separated by k. □

Corollary 3.2. *L is an F-frame if and only if every ideal of $\mathcal{R}(L)$ is flat.*

Proof. Follows from Proposition 3.48 of [27] and Theorem 3.1. □

Lemma 3.3. *Let $f \in \mathcal{R}(L)$. Then the ideal $\langle f, |f| \rangle$ is principal if and only if $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated.*

Proof. First suppose that, $\langle f, |f| \rangle$ is principal and generated by $d \in \mathcal{R}(L)$, i.e., $\langle f, |f| \rangle = \langle d \rangle$. Then $f = gd$, $|f| = hd$ and $sf + t|f| = d$, for some $g, h, s, t \in \mathcal{R}(L)$. Now $\mathrm{pos}(f) = \mathrm{coz}(f^+) = \mathrm{coz}(2f^+) = \mathrm{coz}(f + |f|) = \mathrm{coz}(g+h) \wedge \mathrm{coz}(d)$ and $\mathrm{neg}(f) = \mathrm{coz}(f^-) = \mathrm{coz}(-f^-) = \mathrm{coz}(-2f^-) = \mathrm{coz}(f - |f|) = \mathrm{coz}(g-h) \wedge \mathrm{coz}(d)$. Also $d = sf + t|f| = sgd + thd$ implies $\mathrm{coz}(\mathbf{1} - sg - th) \wedge \mathrm{coz}(d) = 0$. So $\mathrm{pos}(f) \wedge (\mathrm{coz}(g-h) \vee \mathrm{coz}(\mathbf{1} - sg - th)) = \mathrm{pos}(f) \wedge \mathrm{coz}(g-h) = \mathrm{pos}(f) \wedge \mathrm{neg}(f) = 0$ and $\mathrm{neg}(f) \wedge (\mathrm{coz}(g+h) \vee \mathrm{coz}(\mathbf{1} - sg - th)) = \mathrm{neg}(f) \wedge \mathrm{coz}(g+h) = \mathrm{neg}(f) \wedge \mathrm{pos}(f) = 0$. But $\mathrm{coz}(g-h) \vee \mathrm{coz}(g+h) \vee \mathrm{coz}(\mathbf{1} - sg - th) = 1$. Indeed $\mathrm{coz}(g-h) \vee \mathrm{coz}(g+h) \vee \mathrm{coz}(\mathbf{1} - sg - th) \geq \mathrm{coz}(2g) \vee \mathrm{coz}(2h) \vee \mathrm{coz}(\mathbf{1} - sg - th) \geq \mathrm{coz}(2sg) \vee \mathrm{coz}(2th) \vee \mathrm{coz}(\mathbf{2} - 2sg - 2th) \geq \mathrm{coz}(\mathbf{2}) = 1$. Therefore $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated.

Conversely suppose that, $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated. Then there exists $g \in \mathcal{R}(L)$ such that $\mathrm{pos}(f) \wedge \mathrm{coz}(\mathbf{1} - g) = 0$ and $\mathrm{neg}(f) \wedge \mathrm{coz}(\mathbf{1} + g) = 0$. So $f^+ = gf^+$ and $f^- = -gf^-$ and hence $f = g|f|$. Therefore $\langle f, |f| \rangle = \langle |f| \rangle$. Hence the theorem is proved. □

Theorem 3.4. *For a frame L, $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated, for each $f \in \mathcal{R}(L)$ if and only if it is an F-frame.*

Proof. First suppose that L is an F-frame and $f \in \mathcal{R}(L)$. Then $\mathrm{pos}(f) \wedge \mathrm{neg}(f) = \mathrm{coz}(f^+) \wedge \mathrm{coz}(f^-) = \mathrm{coz}(f^+ f^-) = 0$ implies $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated (see [5, Proposition 8.4.10]).

Conversely suppose that, $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated for each $f \in \mathcal{R}(L)$. To show L is an F-frame, it is sufficient to show in view of [12, Corollary 3.3 and Proposition 3.4] that, $O^I = \{g \in \mathcal{R}(\lambda L) : \mathrm{coz}(g) \in I\}$ is a prime ideal of $\mathcal{R}(\lambda L)$, for each prime element I of $\beta(\lambda L)$, where λL is the Lindelöf coreflection of L (see [5] for details). Since O^I is a z-ideal of $\mathcal{R}(\lambda L)$, it is sufficient to show that, for any $f \in \mathcal{R}(\lambda L)$, there exists $g \in O^I$ such that $\mathrm{pos}(f) \leq \mathrm{coz}(g)$ or $\mathrm{neg}(f) \leq \mathrm{coz}(g)$ (see [1, Lemma 4.8]). So let $f \in \mathcal{R}(\lambda L)$. Then by hypothesis $\mathrm{pos}(\lambda_L \circ f)$ and $\mathrm{neg}(\lambda_L \circ f)$ are completely separated and hence there exist $k, l \in \mathcal{R}(L)$ such that, $\mathrm{pos}(\lambda_L \circ f) \wedge \mathrm{coz}(k) = 0 = \mathrm{neg}(\lambda_L \circ f) \wedge \mathrm{coz}(l)$ and $\mathrm{coz}(k) \vee \mathrm{coz}(l) = 1$, here $\lambda_L : \lambda L \to L$ is the coreflection map. Now since L is a C-quotient of λL (see [5, Corollary 8.2.13]), we must have $\bar{k}, \bar{l} \in \mathcal{R}(\lambda L)$ such that, $k = \lambda_L \circ \bar{k}$ and $l = \lambda_L \circ \bar{l}$. So $0 = \mathrm{pos}(\lambda_L \circ f) \wedge \mathrm{coz}(k) = \lambda_L(\mathrm{pos}(f)) \wedge \lambda_L(\mathrm{coz}(\bar{k})) = \lambda_L(\mathrm{pos}(f) \wedge \mathrm{coz}(\bar{k}))$ implies $\mathrm{pos}(f) \wedge \mathrm{coz}(\bar{k}) = 0$, as λ_L is dense. Similarly $\mathrm{neg}(f) \wedge \mathrm{coz}(\bar{l}) = 0$ and $1 = \mathrm{coz}(k) \vee \mathrm{coz}(l) = \lambda_L(\mathrm{coz}(\bar{k}) \vee \mathrm{coz}(\bar{l}))$ implies $\mathrm{coz}(\bar{k}) \vee \mathrm{coz}(\bar{l}) = 1$, as λ_L is coz-codense (see [5, Theorem 8.2.12]). So $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated in λL and hence $(\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^* = 1$. Therefore $r_{\lambda L}((\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^*) = r_{\lambda L}((\mathrm{pos}(f))^*) \vee r_{\lambda L}((\mathrm{neg}(f))^*) = \lambda L$ (see [4, Lemma 3.1]) and hence $r_{\lambda L}((\mathrm{pos}(f))^*) \nsubseteq I$ or $r_{\lambda L}((\mathrm{neg}(f))^*) \nsubseteq I$, as I is prime. If $r_{\lambda L}((\mathrm{pos}(f))^*) \nsubseteq I$

then $r_{\lambda L}((\mathrm{pos}(f))^*) \vee I = \lambda L$, as I is a maximal element of $\beta(\lambda L)$ and so there exists $x \in \lambda L$ with $x \prec\!\prec (\mathrm{pos}(f))^*$ and $y \in I$ such that $x \vee y = 1$. But $y \in I$ implies $y \leq \mathrm{coz}(g) \in I$ with $g \in \mathcal{R}(\lambda L)$. So $x \vee \mathrm{coz}(g) = 1$ and $\mathrm{pos}(f) \leq (\mathrm{pos}(f))^{**} \leq x^* \leq \mathrm{coz}(g)$ with $g \in O^I$. □

Theorem 3.5. *For any frame L, every ideal of $\mathcal{R}(L)$ is convex if and only if it is an F-frame.*

Proof. Since $\mathcal{R}(L)$ is a semiprime f-ring with bounded inversion property (see [6, Proposition 11]), we have from Theorem 1 of [24] that, every ideal of $\mathcal{R}(L)$ is convex if and only if it is a Bézout ring.[6] Also Proposition 3.2 of [12] tells that, L is an F-frame if and only if $\mathcal{R}(L)$ is a Bézout ring. Thus we get the required result. □

From Lemma 3.3, Theorem 3.4 and Theorem 3.5 we get the following result.

Theorem 3.6. *The following statements are equivalent for any frame L:*

(1) *L is an F-frame.*
(2) $\mathrm{pos}(f)$ *and* $\mathrm{neg}(f)$ *are completely separated, for each $f \in \mathcal{R}(L)$.*
(3) *$\langle f, |f|\rangle$ is a principal ideal of $\mathcal{R}(L)$, for each $f \in \mathcal{R}(L)$.*
(4) *Every ideal of $\mathcal{R}(L)$ is convex.*

Theorem 3.7. *Let L be a normal, $\mathcal{P}$-continuous frame. Then L is an F-frame if and only if* $\mathrm{pos}(f)$ *and* $\mathrm{neg}(f)$ *are completely separated, for each $f \in \mathcal{R}_{\mathcal{P}}(L)$.*

Proof. If L is an F-frame then by Theorem 3.4, $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated, for each $f \in \mathcal{R}(L)$ and hence $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated, for each $f \in \mathcal{R}_{\mathcal{P}}(L)$ as $\mathcal{R}_{\mathcal{P}}(L) \subseteq \mathcal{R}(L)$.

Conversely suppose that, L is not an F-frame. Then again by Theorem 3.4, there exists $f \in \mathcal{R}(L)$ such that $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are not completely separated. We claim that, $(\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^* \neq 1$. Indeed otherwise by normality of L, there exists $g \in \mathcal{R}(L)$ such that $\mathrm{coz}(g) \leq (\mathrm{pos}(f))^*$ and $\mathrm{coz}(\mathbf{1} - g) \leq (\mathrm{neg}(f))^*$ (see [5, Proposition 8.3.1]), which implies $\mathrm{pos}(f)$ and $\mathrm{neg}(f)$ are completely separated, a contradiction. So $(\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^* \neq 1$ and hence there exists $h \in \mathcal{R}_{\mathcal{P}}(L)$ such that $\mathrm{coz}(h) \nleq (\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^*$, as L is $\mathcal{P}$-continuous ensures that $\mathcal{R}_{\mathcal{P}}(L)$ is a free ideal[7] of $\mathcal{R}(L)$ (see [1, Theorem 4.11]). We assert that $\mathrm{pos}(f|h|)$ and $\mathrm{neg}(f|h|)$ are not completely separated. Otherwise $(\mathrm{pos}(f|h|))^* \vee (\mathrm{neg}(f|h|))^* = 1$. So $\mathrm{coz}(h) = \mathrm{coz}(|h|) = \mathrm{coz}(|h|) \wedge ((\mathrm{pos}(f|h|))^* \vee (\mathrm{neg}(f|h|))^*) = (\mathrm{coz}(|h|) \wedge (\mathrm{pos}(f|h|))^*) \vee (\mathrm{coz}(|h|) \wedge (\mathrm{neg}(f|h|))^*)$. Now $\mathrm{coz}(|h|) \wedge \mathrm{pos}(f) \wedge (\mathrm{pos}(f|h|))^* = \mathrm{coz}(|h|f^+) \wedge (\mathrm{pos}(f|h|))^* = \mathrm{coz}((|h|f)^+) \wedge (\mathrm{coz}((|h|f)^+))^* = 0$ and hence, $(\mathrm{coz}(|h|) \wedge (\mathrm{pos}(f|h|))^*) \leq (\mathrm{pos}(f))^*$, similarly we can show that $(\mathrm{coz}(|h|) \wedge (\mathrm{neg}(f|h|))^*) \leq (\mathrm{neg}(f))^*$. Therefore $\mathrm{coz}(h) \leq (\mathrm{pos}(f))^* \vee (\mathrm{neg}(f))^*$, a contradiction. So we have that $\mathrm{pos}(f|h|)$ and $\mathrm{neg}(f|h|)$ are not completely separated, but since $h \in \mathcal{R}_{\mathcal{P}}(L)$ and $\mathrm{coz}(h) = \mathrm{coz}(|h|)$ we also have that $f|h| \in \mathcal{R}_{\mathcal{P}}(L)$, which provides another contradiction. Hence the theorem is proved. □

The following are simple corollaries of the preceding theorem, taking for $\mathcal{P}$ the ideal $\mathcal{K}$, resp. $\mathcal{L}$ of all compact, resp. Lindelöf, closed quotients of L. We like to mention in view of [1, Theorem 4.2], that $\mathcal{K}$-continuous means continuous.

Corollary 3.8. *For a normal continuous frame L, it is an F-frame if and only if* $\mathrm{pos}(f)$ *and* $\mathrm{neg}(f)$ *are completely separated, for each $f \in \mathcal{R}_{\mathcal{K}}(L)$.*

[6] A ring R is called a *Bézout ring* if every finitely generated ideal of it is principal.
[7] An ideal I of $\mathcal{R}(L)$ is called *free* if $\bigvee\{\mathrm{coz}(f) : f \in I\} = 1$.

Corollary 3.9. *For a normal $\mathcal{L}$-continuous frame L, it is an F-frame if and only if* $\text{pos}(f)$ *and* $\text{neg}(f)$ *are completely separated, for each* $f \in \mathcal{R}_{\mathcal{L}}(L)$.

Theorem 3.10. *Let L be a normal $\mathcal{P}$-continuous frame. Then it is an F-frame if $\mathcal{R}_{\mathcal{P}}(L)$ is a Bézout ring.*

Proof. Let $\mathcal{R}_{\mathcal{P}}(L)$ be a Bézout ring and $f \in \mathcal{R}_{\mathcal{P}}(L)$. Then $\langle f, |f|\rangle = \langle g\rangle$, for some $g \in \mathcal{R}_{\mathcal{P}}(L)$ (here $\langle,\rangle$ denotes the ideal generation in $\mathcal{R}_{\mathcal{P}}(L)$). Now if we adopt the same technique as in Lemma 3.3, we get $\text{pos}(f)$ and $\text{neg}(f)$ are completely separated and hence by Theorem 3.4, L is an F-frame. □

Remark 3.11. We don't know whether the converse part of the above theorem is true or not. But the converse part is true for continuous frames (not necessarily normal), i.e., if a continuous frame L is an F-frame then $\mathcal{R}_{\mathcal{K}}(L)$ is a Bézout ring. Indeed for a continuous frame L, $\mathcal{R}_{\mathcal{K}}(L)$ is a free ideal (see [13, Corollary 4.14]) and hence a pure ideal[8] (see [13, Proposition 4.18]) of $\mathcal{R}(L)$. Let $\langle f_1, f_2, \ldots, f_n\rangle_{\mathcal{R}_{\mathcal{K}}(L)}$ be a finitely generated ideal of $\mathcal{R}_{\mathcal{K}}(L)$. Since L is an F-frame, $\langle f_1, f_2, \ldots, f_n\rangle_{\mathcal{R}(L)} = \langle f\rangle_{\mathcal{R}(L)}$ for some $f \in \mathcal{R}(L)$ (see [12, Proposition 3.2]). Then $f, f_i \in \mathcal{R}_{\mathcal{K}}(L)$, $i = 1, 2, \ldots, n, n \in \mathbb{N}$ and since it is pure, $f = fk$ and $f_i = f_i k_i$ for some $k, k_i \in \mathcal{R}_{\mathcal{K}}(L)$, $i = 1, 2, \ldots, n$. Now it can be easily checked that $\langle f_1, f_2, \ldots, f_n\rangle_{\mathcal{R}_{\mathcal{K}}(L)} = \langle f\rangle_{\mathcal{R}_{\mathcal{K}}(L)}$.

Theorem 3.12. *The following statements are equivalent for any frame L:*

(1) *L is a quasi-F-frame.*
(2) *$\mathcal{R}(L)$ is a Prüfer ring.*
(3) *$\mathcal{R}^*(L)$ is a Prüfer ring.*

Proof. (1) ⇔ (2): Since L is a quasi-F-frame if and only if $\mathcal{R}(L)$ is a quasi-Bézout ring (see [14, Proposition 3.14]) if and only if $\mathcal{R}(L)$ is a Prüfer ring (see [24, Theorem 2]), as it is a semiprime f-ring with bounded inversion property (see [6, Proposition 11]).

(1) ⇔ (3): Since L is a quasi-F-frame if and only if βL is a quasi-F-frame (see [14, Proposition 3.6]) if and only if $\mathcal{R}(\beta L)$ is a Prüfer ring if and only if $\mathcal{R}^*(L)$ is a Prüfer ring as $\mathcal{R}(\beta L) \cong \mathcal{R}^*(L)$. □

4. Characterizations of *P*-frames and basically disconnected frames

Theorem 4.1. *L is a P-frame if and only if each prime ideal of $\mathcal{R}(L)$ is a z-ideal.*

Proof. If L is a P-frame then each ideal and hence each prime ideal of $\mathcal{R}(L)$ is a z-ideal (see [11, Proposition 3.9]).

Conversely, if L is not a P-frame then there is non-maximal prime ideal P of $\mathcal{R}(L)$ (see [11, Proposition 3.9]). Let M be the unique maximal ideal containing P, then there is an upper ideal U in between P and M (see [1, Theorem 3.7]). Therefore we get a prime ideal U of $\mathcal{R}(L)$ which is not a z-ideal, as it is an upper ideal (see [1, Theorem 3.6]). Hence the theorem is proved. □

Remark 4.2. The 'if'-part of the above theorem was also established by Dube and Ighedo without using the upper ideals of $\mathcal{R}(L)$, and this we have come to learn from a personal communication.

Theorem 4.3. *L is a P-frame if and only if every $\mathcal{R}(L)$-module is flat.*

Proof. Since we know that, L is a P-frame if and only if $\mathcal{R}(L)$ is a von Neumann regular ring (see [11, Proposition 3.9]), the theorem follows from Theorem 4.9 of [27]. □

[8] An ideal I of $\mathcal{R}(L)$ is called *pure* if $f \in I$ implies $f = fg$ for some $g \in I$.

Theorem 4.4. *Let I be a z-ideal of $\mathcal{R}(L)$. Then $\mathcal{R}(L)/I$ is a totally ordered ring if and only if I is prime ideal of $\mathcal{R}(L)$.*

Proof. Let $\mathcal{R}(L)/I$ be a totally ordered ring. To show I is prime, it is sufficient to show that, for any $f \in \mathcal{R}(L)$, there is a cozero element $a \in \mathrm{Coz}[I]$ such that, $f(-,0) \leq a$ or $f(0,-) \leq a$ (see [1, Lemma 4.8]). So let $f \in \mathcal{R}(L)$, and hence by hypothesis, $I(f) \geq 0$ or $I(f) \leq 0$, where $I(f)$ is the residue class of f in the residue class ring $\mathcal{R}(L)/I$. Therefore, if $I(f) \geq 0$ then there exists $g \geq \mathbf{0}$ in $\mathcal{R}(L)$ such that $f - g \in I$. Now, $f - g \leq f$ as $g \geq \mathbf{0}$, implies $f(-,0) \leq (f-g)(-,0) \leq \mathrm{coz}(f-g) = a \in \mathrm{Coz}[I]$. If $I(f) \leq 0$, then the other possibility holds. The converse implication trivially holds (see [1], the explanations preceding Lemma 3.3). □

Corollary 4.5. *Let L be a P-frame. Then for any ideal I of $\mathcal{R}(L)$, $\mathcal{R}(L)/I$ is a totally ordered ring if and only if I is maximal ideal of $\mathcal{R}(L)$.*

Proof. Since L is a P-frame, every ideal of $\mathcal{R}(L)$ is a z-ideal and every prime ideal of it is maximal [11, Proposition 3.9]. So the result follows from the above theorem. □

Theorem 4.6. *The following statements are equivalent for a frame L:*

(1) *L is basically disconnected.*
(2) *$\mathcal{R}(L)$ is a semihereditary ring.*
(3) *$\mathcal{R}(L)$ is a coherent ring.*

Proof. Since the equivalence of (1) and (3) is already established in our paper (see [1, Theorem 5.1]) and every semihereditary ring is well-known to be coherent also [27], it suffices to establish the relation (1) ⇒ (2) only. So let L be basically disconnected and I be a finitely generated ideal of $\mathcal{R}(L)$. We have to prove that I is projective. Since every basically disconnected frame is an F-frame (see [5, page 60]) and every finitely generated ideal of $\mathcal{R}(L)$ for an F-frame L is principal (see [12, Proposition 3.2]), there exists $f \in \mathcal{R}(L)$ such that $I = \langle f \rangle$, and $(\mathrm{coz}(f))^*$ is complemented as L is basically disconnected. Then the characteristic function $\chi_{(\mathrm{coz}(f))^*}$ of $(\mathrm{coz}(f))^*$ is in $\mathcal{R}(L)$ and $\mathrm{coz}(\chi_{(\mathrm{coz}(f))^*}) = (\mathrm{coz}(f))^*$ (see [8, Example 2]). Also since disjoint cozero elements of an F-frame are completely separated (see [5, Proposition 8.4.10]), there exists $k \in \mathcal{R}(L)$ such that, $\mathrm{coz}(f) \wedge \mathrm{coz}(\mathbf{1} - k) = 0 = (\mathrm{coz}(f))^* \wedge \mathrm{coz}(k)$. Define $\varphi_1 : I \to \mathcal{R}(L)$ by $\varphi_1(gf) = gk$, for all $g \in \mathcal{R}(L)$. Then φ_1 is well defined: indeed if $gf = hf$ with $g, h \in \mathcal{R}(L)$ then $\mathrm{coz}(g - h) \leq (\mathrm{coz}(f))^*$, which implies $\mathrm{coz}(g-h) \wedge \mathrm{coz}(k) \leq \mathrm{coz}(k) \wedge (\mathrm{coz}(f))^* = 0$ and so $gk = hk$. Now clearly, φ_1 is a module homomorphism and $\varphi_1(gf)f = gkf = gf$ (as $f(\mathbf{1} - k) = \mathbf{0}$), for all $g \in \mathcal{R}(L)$. Therefore $\{a_1 = f, \varphi_1\}$ is a projective basis for I and hence I is projective. □

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