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RINGS AND SUBRINGS OF CONTINUOUS FUNCTIONS WITH COUNTABLE RANGE

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Abstract. Intermediate rings of real valued continuous functions with countable range on a Hausdorff zero-dimensional space X are introduced in this article. Let $\Sigma_c(X)$ be the family of all such intermediate rings $A_c(X)$'s which lie between $C_c^*(X)$ and $C_c(X)$. It is shown that the structure space of each $A_c(X)$ is $\beta_0 X$, the Banaschewski compactification of X. X is shown to be a P-space if and only if each ideal in $C_c(X)$ is closed in the m_c-topology on it. Furthermore X is realized to be an almost P-space when and only when each maximal ideal/ z-ideal in $C_c(X)$ becomes a z^0-ideal. Incidentally within the family of almost P-spaces, $C_c(X)$ is characterized among all the members of $\Sigma_c(X)$ by virtue of either of these two properties. Equivalent descriptions of pseudocompact condition on X are given via U_c-topology, m_c-topology and norm on $C_c(X)$. The article ends with a result which essentially says that z^0-ideals in a typical $A_c(X) \in \Sigma_c(X)$ are precisely the contraction of z^0-ideals in $C_c(X)$.

1. Introduction

In what follows X stands for a completely regular Hausdorff topological space and $C(X)$ as usual denotes the ring of all real valued continuous functions on X. $C^*(X)$ designates the subring of $C(X)$ containing all those members which are bounded over X. Suppose $C_c(X)$ is the subset of $C(X)$ consisting of those functions f for which $f(X)$ is a countable subset of $\mathbb{R}$ and $C_c^*(X)$=$C_c(X) \cap C^*(X)$. It is well known that $C_c(X)$ (respectively $C_c^*(X)$) is a subring as well as a sublattice of $C(X)$ (respectively $C^*(X)$). These two rings $C_c(X)$ and $C_c^*(X)$ have received the attention of a few experts in this area only recently. We refer to the reader the articles [5],[11],[21],[16] in this connection. A natural expectation has cropped up as a bye-product of these recent investigations that, there is a hidden interplay existing between the topological structure of X and the ring and the lattice structure of $C_c(X)$ and $C_c^*(X)$. To study this interaction in an efficient manner the authors in [14] have already discovered that one can stick to a well chosen class of spaces viz. the zero-dimensional Hausdorff topological space X. Indeed it is proved in ([14], Theorem 4.6) that starting from any topological space X (not necessarily even completely regular), one can construct a Hausdorff zero-dimensional space Y such that the ring $C_c(Y)$ is isomorphic to the ring $C_c(X)$. This may be called the analogous fact for its classical antecedent in the theory of $C(X)$ which says that any topological space X can give rise to a completely regular Hausdorff space Y

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for which $C(X)$ is isomorphic to $C(Y)$ ([13], Theorem 3.9). Therefore in the study of $C_c(X)$ and $C_c^*(X)$ vis-a-vis the space X the ambient topological space X may well chosen to be Hausdorff and zero-dimensional in the sense that clopen sets make base for the topology on X. We will stick to this convention throughout this article. Furthermore an ideal I unmodified in any ring R in this paper will always stand for a proper ideal.

It is a standard result in the theory of Rings of Continuous functions that the structure space of $C(X)$ and $C^*(X)$ are both βX, the Stone-Čech compactification of X (7N, [13]). As a countable counterpart of the result, it is proved in ([5], Remark 3.6) that the structure space of $C_c(X)$ is $\beta_0 X$, the largest zero-dimensional compactification of a zero-dimensional Hausdorff space X, also known as Banaschewski compactification of X. The structure space of a commutative ring R with unity stands for the set of all maximal ideals of R equipped with the familiar hull-kernel topology. In the present article we have initiated the study on intermediate rings viz those rings that lie between $C_c^*(X)$ and $C_c(X)$. Let $\Sigma_c(X)$ stand for the aggregate of all such intermediate rings. In section 2 of the present article we establish that if $A_c(X) \in \Sigma_c(X)$ then the structure space of $A_c(X)$ is also $\beta_0 X$ (Theorem 2.7). This generalizes the Proposition mentioned in [5]. This is incidentally the first important technical result in this article. A space X is termed as a CP-space in [14] if the ring $C_c(X)$ is regular in the sense of Von-Neumann and several equivalent versions of this property are recorded in ([14], Theorem 5.8). These are natural counterparts of the corresponding equivalent descriptions of a P-space in the classical setting of $C(X)$ as mentioned in (4J, [13]). It is also proved in the same article ([14], Corollary 5.7) that a zero-dimensional space X is a CP-space if and only if it is a P-space. In section 3 of the present article we introduce m_c-topology on $C_c(X)$ as a counterpart for the present set up of the well known m-topology on $C(X)$, introduced longtime back by Hewitt in 1948 [15]. We prove that if I is an ideal of $C_c(X)$, then the closure of I in the m_c-topology coincides with the intersection of all the maximal ideals of $C_c(X)$ which contain I (Theorem 3.8). From this it follows that a zero-dimensional space X is P-space if and only if each ideal in $C_c(X)$ is closed in m_c-topology (Theorem 3.10). We further establish that if $A_c(X) \in \Sigma_c(X)$ is properly contained in $C_c(X)$ then it is never Von-Neumann regular (Theorem3.16). Thus within the class of P-spaces X, $C_c(X)$ is characterized amongst all the intermediate rings by the property that it is Von-Neumann regular.

A Tychonoff space X is called almost P if the interior of each non empty zero set in X is open. These spaces are introduced in [18] as a generalization of P-spaces. In section 4 of this article we make some query about when a zero-dimensional space X becomes an almost P-space. We establish that X is almost P if and only if each maximal ideal of $C_c(X)$ is a z^0-ideal and this happens when and only when each z-ideal in $C_c(X)$ becomes a z^0-ideal (Theorem 4.10). It turns out that within the class of almost P-space X , $C_c(X)$ is the unique ring amongst all the intermediate rings that lie between $C_c^*(X)$ and $C_c(X)$ which enjoys either of these two properties (Theorem 4.11,4.12).

A space X is called pseudocompact if $C(X) = C^*(X)$. It is established by the authors in ([16], Theorem 6.3) that a zero-dimensional space X is pseudocompact if and only if $C_c(X) = C_c^*(X)$. In section 5 of this article we find out a few equivalent versions of pseudocompactness in terms of both the m_c-topology on $C_c(X)$ and U_c-topology on $C_c(X)$ (Theorem 5.2, 5.3). The U_c-topology on $C_c(X)$ may be called the countable counterpart of the well known U-topology or the topology of uniform convergence on $C(X)$ (See 2M, 2N, [13]).

In section 6 of this article we examine when do a few chosen subrings of $C_c(X)$ become Noetherian/ Artinian (Theorem 6.4). It follows as special cases that a zero-dimensional space X is finite if and only if $C_c(X)$ is Noetherian if and only if $C_c(X)$ is Artinian. Furthermore a locally compact zero-dimensional space X is seen to be finite if and only if $C_c(X) \cap C_K(X)$ becomes Noetherian/Artinian if and only if $C_c(X) \cap C_\infty(X)$ becomes Noetherian/ Artinian. Here $C_\infty(X)$ stands for the rings of all real valued continuous functions on X which vanish at infinity and $C_K(X)$ is the subring of $C_\infty(X)$ containing those functions which have compact support.

In the final section 7 of this article we give an explicit formula for z^0-ideals in a typical intermediate ring $A_c(X) \in \Sigma_c(X)$ (Theorem 7.1). From this it follows that z^o-ideals of $A_c(X)$, in particular z^0-ideals of $C_c(X)$ or $C_c^*(X)$ are the contraction of z^0-ideals in $C(X)$.

2. STRUCTURE SPACES OF INTERMEDIATE RINGS

We recall from (7M [13]) that if A is a commutative ring with unity and $\mathcal{M}(A)$ the set of all maximal ideals of A and for each $a \in A$ if we set $\mathcal{M}_a = \{M \in \mathcal{M}(A) : a \in M\}$, then the family $\{\mathcal{M}_a\colon a \in A\}$ turns out to be a closed base for the hull-kernel topology on $\mathcal{M}(A)$. For any $\mathcal{M}_0 \subseteq \mathcal{M}(A)$ the closure of $\mathcal{M}_0 = \{M \in \mathcal{M}(A) : M \supset \cap \mathcal{M}_0\}$. $\mathcal{M}(A)$ equipped with this topology known as the structure space of A is a compact T_1 topological space and is Hausdorff if and only if given any two distinct maximal ideals M_1 , M_2 in A, there exist points a_1, a_2 in A such that $a_1 \notin M_1$ and $a_2 \notin M_2$ and $a_1 a_2 \in \cap \mathcal{M}(A)$. In what follows we will let $A_c(X)$ stand for a typical intermediate ring lying between the two rings $C_c^*(X)$ and $C_c(X)$. Suppose $Max(A_c(X))$ denotes the structure space of $A_c(X)$.

Theorem 2.1. *$Max(A_c(X))$ is a (compact) Hausdorff space.*

Proof. We shall prove the Hausdorffness of $Max(A_c(X))$ only. For any $f \in A_c(X)$, set $\mathcal{Z}_A(f) = \{Z \in Z_c(X) :$ there exists $g \in A_c(X)$ such that for each $x \in X \setminus Z, f(x)g(x) = 1\}$. Here $Z_c(X) = \{Z(f) : f \in C_c(X)\}$, the family of all zero sets in X of functions lying in $C_c(X)$. For any ideal I in $A_c(X)$, let $\mathcal{Z}_A[I] = \bigcup\limits_{f \in I} \mathcal{Z}_A(f)$. Then it can be proved by following the technique adopted in [10],[20],[21] that $\mathcal{Z}_A(f)$ and $\mathcal{Z}_A[I]$ are both z_c-filter on X. A z_c-filter on X is a subfamily of $Z_c(X) - \{\emptyset\}$ which is closed under finite intersection and formation of supersets (see [14]). Furthermore if $\mathcal{F}$ is a z_c-filter on X, then it can be checked by using the methods in [10],[20],[21] that $\mathcal{Z}_A^{-1}[\mathcal{F}] = \{f \in A_c(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$ is a (proper) ideal in $A_c(X)$. Now let M_1 and M_2 be two distinct members of $A_c(X)$. It is sufficient to produce h_1, h_2 in $\mathcal{M}_c(X)$ with $h_1 \notin M_1$, $h_2 \notin M_2$ such that $h_1 h_2 = 0$. To this end we assert that there exists $Z_1 \in \mathcal{Z}_A[M_1]$, $Z_2 \in \mathcal{Z}_A[M_2]$ with $Z_1 \cap Z_2 = \emptyset$. For otherwise each member of $\mathcal{Z}_A[M_1]$ meets any member of $\mathcal{Z}_A[M_2]$ and hence $\mathcal{Z}_A[M_1] \cup \mathcal{Z}_A[M_2]$ becomes a subfamily of $Z_c(X)$ with finite intersection property. Consequently there exists a z_c-filter $\mathcal{F}$ on X such that $\mathcal{Z}_A[M_1] \cup \mathcal{Z}_A[M_2] \subseteq \mathcal{F}$ which yields that $M_1 \cup M_2 \subseteq \mathcal{Z}_A^{-1}[\mathcal{F}]$= a proper ideal in $A_c(X)$, a contradiction since M_1 and M_2 are distinct maximal ideals in $A_c(X)$. So choose $f \in M_1$ and $g \in M_2$ such that $Z_1 \in \mathcal{Z}_A(f)$,

$Z_2 \in \mathcal{Z}_A(g)$ and $Z_1 \cap Z_2 = \emptyset$. This means that there exist $f_1, g_1 \in A_c(X)$ such that for each $x \in X - Z_1$, $f(x)f_1(x) = 1$ and for any $x \in X - Z_2$, $g(x)g_1(x) = 1$. Now since $ff_1 \in M_1$ and $gg_1 \in M_2$, it follows that $1 - ff_1 \notin M_1$ and $1 - gg_1 \notin M_2$. Since $(X - Z_1) \cup (X - Z_2) = X - (Z_1 \cap Z_2) = X - \emptyset = X$, it implies that $(1 - ff_1).(1 - gg_1) = 0$. □

We set for any $x \in X$, $M_{A,x} = \{f \in A_c(X) : f(x) = 0\}$. Then it is easy to prove on applying the first isomorphism theorem of algebra, taking care of the presence of constant functions in $A_c(X)$ that the complete list of fixed maximal ideals of $A_c(X)$ is given by $\{M_{A,x} : x \in X\}$. An ideal I in $A_c(X)$ is called fixed if there exists a point on X at which all the functions in I vanish. For any $f \in A_c(X)$ we set $(\mathcal{M}_A)_f = \{M \in Max(A_c(X)) : f \in M\}$. Then $\{(\mathcal{M}_A)_f : f \in A_c(X)\}$ is the family of basic closed sets in the structure space $Max(A_c(X))$ of $A_c(X)$. For $f \in A_c(X)$ and $x \in X$, $x \in Z(f)$ if and only if $M_{A,x} \in (\mathcal{M}(A))_f \cap \{M_{A,y} : y \in X\}$. On the other hand since X is zero-dimensional it follows from Proposition 4 in [5] that $\{Z(f) : f \in C_c^*(X)\} = \{Z(f) : f \in A_c(X)\} = \{Z(f) : f \in C_c(X)\}$ constitutes a base for the closed sets of X. These two facts therefore yield that the map $\psi_A : X \to Max(A_c(X))$ given by $\psi_A(x) = M_{A,x}$ which is obviously one-to-one exchanges the basic closed sets of the space X and the subspace $\psi_A(X)$ of $Max(A_c(X))$. Furthermore the closure of $\psi_A(X)$ in $Max(A_c(X))$ is given by $\{M \in Max(A_c(X)) : M \supseteq \bigcap \psi_A(X)\} = \{M \in Max(A_c(X)) : M \supseteq \{0\}\} = Max(A_c(X))$, thus demonstrating that $\psi_A(X)$ is dense in $Max(A_c(X))$. The above observations therefore lead to the following proposition.

Theorem 2.2. *The pair $(\Psi_A, Max(A_c(X)))$ is a Hausdorff compactification of X in the following sense, which we reproduce from the monograph* [12].

Definition 2.3. A (Hausdorff) compactification of a Tychonoff space X stands for a pair $(\alpha, \alpha X)$, where αX is a compact Hausdorff space and $\alpha : X \to \alpha X$ is a topological embedding with $\alpha(X)$ dense in αX. For simplicity we often write αX instead of $(\alpha, \alpha X)$. Let $K(X)$ be the family of all Hausdorff compactifications of X.

Definition 2.4. For αX, $\gamma X \in K(X)$, we write $\alpha X \geqq \gamma X$ if there is a continuous map $t : \alpha X \to \gamma X$ with the property $t \circ \alpha = \gamma$. If in this definition $'t'$ is a homeomorphism then we say that αX is topologically equivalent to γX and we write $\alpha X \approx \gamma X$. It can be proved without difficulty that for αX, $\gamma X \in K(X)$, $\alpha X \approx \gamma X$ when and only when $\alpha X \geqq \gamma X$ and $\gamma X \geqq \alpha X$. Furthermore $(K(X), \geqq)$ becomes a complete upper semilattice, which has definitely then a largest member,which is incidentally βX the Stone-Čech compactification of X. If in addition X is zero-dimensional then there is a largest zero-dimensional member of $K(X)$, designated by $\beta_0 X$, called the Banaschewski compactification of X. For more information on these topics see [19].

Definition 2.5. For a zero-dimensional space X, $\alpha X \in K(X)$ is said to enjoy C-extension property if given any compact Hausdorff zero-dimensional space Y and a continuous map $f : X \to Y$ there exists a unique continuous map $f^\alpha : \alpha X \to Y$ such that $f^\alpha \circ \alpha = f$.

It is clear from the above definition that if $\alpha X \in K(X)$ possesses C-extension property then $\alpha X \geqq \beta_0 X$ and if in addition αX is zero-dimensional then $\beta_0 X \geqq \alpha X$

and consequently $\alpha X \approx \beta_0 X$. We need the following subsidiary result before stating the first principal technical result of this section.

Theorem 2.6. *Let X be zero-dimensional and $A_c(X) \in \Sigma_c(X)$. Then given $f \in A_c(X)$, there exists an idempotent e in $A_c(X)$ such that e is multiple of f and $(1-e)$ is a multiple of $(1-f)$ in this ring. [A special case of this result with $A_c(X) = C_c(X)$ is proved in Remark 3.6 in* [5] *].*

Proof. There exists $r,\ 0 < r < 1$ such that $r \notin f(X)$. Let $W = f^{-1}(-\infty, r) = f^{-1}((-\infty, r])$. So W and $X - W$ are both clopen sets in X. The function $e : X \to R$ defined by the rule : $e(W) = 0$ and $e(X - W) = 1$ is clearly an idempotent in the ring $A_c(X)$. Define the functions $h : X \to R$ and $k : X \to R$ as follows: $h(W) = 0$ and $h(x) = \frac{1}{f(x)}$ if $x \in X - W$. $k(X - W) = 0$ and $k(x) = \frac{1}{1-f(x)}$ if $x \in W$. Clearly h and k are both bounded functions in $C_c(X)$ and hence both are members of the ring $A_c(X)$. It is easy to see that $e = h.f$ and $1 - e = k(1 - f)$. □

Theorem 2.7. *$Max(A_c(X))$ is a (compact Hausdorff) zero-dimensional space. Furthermore the pair $(\Psi_A, Max(A_c(X))$ is topologically equivalent to $\beta_0 X$. If in addition X is strongly zero-dimensional meaning that βX is zero-dimensional, then $(\Psi_A, Max(A_c(X)))$ is topologically equivalent to βX.*

Proof. We first prove (only) the zero-dimensionality of $Max(A_C(X))$, because of Theorem 2.1. We recall the notation that for any $f \in A_c(X)$, $(\mathcal{M}_A)_f = \{M \in Max(A_c(X)) : f \in M\}$. So let $M \in Max(A_c(X))$ and $f \in A_c(X)$ be such that $M \in Max(A_c(X)) \setminus (\mathcal{M}_A)_f$. It suffices to find out a clopen set in $Max(A_c(X))$ which contains M and is contained in $Max(A_c(X)) \setminus (\mathcal{M}_A)_f$. We first observe that $M \notin (\mathcal{M}_A)_f$ implies that $f \notin M$ which in turn implies that there exist $h \in A(_c(X))$ and $g \in M$ such that $1 - g = hf$. By Theorem 2.6, there exists an idempotent $'e'$ in $A_c(X)$ such that e is a multiple of g and $(1 - e)$ is a multiple of $(1 - g)$ in the ring $A_c(X)$. Since $g \in M$, this implies that $e \in M$, in other words $M \in (\mathcal{M}_A)_e$. On the otherhand if $N \in (\mathcal{M}_A)_f$ then $f \in N$, hence $1 - g = hf \in N$ consequently $1 - e \in N$ and therefore $e \notin N$ (as N is a maximal ideal in $A_c(X)$) which means that $N \notin (\mathcal{M}_A)_e$. Thus we get that $M \in (\mathcal{M}_A)_e \subseteq Max(A_c(X)) \setminus (\mathcal{M}_A)_f$. We now assert that $(\mathcal{M}_A)_e = Max(A_c(X)) \setminus (\mathcal{M}_A)_{1-e}$ and have $(\mathcal{M}_A)_e$ is clopen in $Max(A_c(X)$. Indeed if $M \in (\mathcal{M}_A)_e$ then $e \in M$, which implies that $1 - e \notin M$ and hence $M \notin (\mathcal{M}_A)_{1-e}$ *i.e*; $M \in Max(A_c(X)) \setminus (\mathcal{M}_A)_{1-e}$. Thus $(\mathcal{M}_A)_e \subseteq Max(A_c(X)) \setminus (\mathcal{M}_A)_{1-e}$. From symmetry it follows that, as $(1 - e)$ is an idempotent of $A_c(X)$. $(\mathcal{M}_A)_{1-e} \subseteq Max(A_c(X)) \setminus (\mathcal{M}_A)_e$, hence $(\mathcal{M}_A)_e = Max(A_c(X)) \setminus (\mathcal{M}_A)_{1-e}$.

Now that we have proved that $Max(A_c(X))$ is zero-dimensional, to prove the second part of the present theorem, it is sufficient to prove that $(\Psi_A, Max(A_c(X))$ enjoys the C-extension property. So let Y be a compact Hausdorff zero-dimensional space and $f : X \to Y$ a continuous map. It is sufficient to define a continuous map $f^{\mathcal{M}} : Max(A_c(X)) \to Y$ with the following property: $f^{\mathcal{M}} \circ \Psi_A = f$. To that end choose $M \in Max(A_c(X))$ i.e; M is a maximal ideal in $A_c(X)$. Set $\widetilde{M} = \{g \in C_c(Y) : g \circ f \in M\}$. Note that if $g \in C_c(Y)$ then $g \circ f \in C_c(X)$. Furthermore since Y is compact and $g \in C_c(Y)$ then $g(Y)$ is a bounded subset of $\mathbb{R}$, consequently $(g \circ f)(X)$ is a bounded subset of $\mathbb{R}$ and hence $(g \circ f) \in C_c^*(X)$ and therefore $g \circ f \in A_c(X)$. Thus the definition of $\widetilde{M}$ is without any ambiguity. Since M is a maximal ideal of $A_c(X)$ it follows that $\widetilde{M}$ is a prime ideal of $C_c(Y)$. Now it is already proved in ([14], Corollary 2.14) that every prime ideal in $C_c(Y)$

is contained in a unique maximal ideal. Thus $\widetilde{M}$ extends to a unique maximal ideal in$C_c(Y)$ which is fixed because Y is compact. Thus there exists a unique point $y \in Y$ such that for each $g \in \widetilde{M}$, $g(y) = 0$ and hence $\bigcap_{g\in\widetilde{M}} Z(g) = \{y\}$. We set $f^{\mathcal{M}}(M) = y$. Thus $\{f^{\mathcal{M}}(M)\}= \bigcap_{g\in\widetilde{M}} Z(g)$ ...(1). We note that if $x \in X$ and $g \in \widetilde{M}_{A,x}$, then $g \circ f \in M_{A,x}$ and hence $(g \circ f)(x) = 0$, which implies that $f(x) \in Z(g)$. This proves that $\bigcap_{g\in\widetilde{M}_{A,x}} Z(g) = \{f(x)\}$. This implies in view of the definition (1) above that $f^{\mathcal{M}}(M_{A,x}) = f(x)$, in other words: $f^{\mathcal{M}} \circ \Psi_A(x) = f(x)$. Thus $f^{\mathcal{M}} \circ \Psi_A = f$. To ensure the continuity of the map $f^{\mathcal{M}} : Max(A_c(X) \to Y$ defined in (1) at an arbitrary $M \in Max(A_c(X))$, let W be a neighbourhood of $f^{\mathcal{M}}(M)$ in the space Y. Since Y is zero-dimensional, each neighbourhood of a point $'y'$ in this space contains a co-zero set neighbourhood of y of the form $Y \setminus Z(g_1)$ for some $g_1 \in C_c(Y)$ and also a zero set neighbourhood of y of the form $Z(g_2)$ for an appropriate $g_2 \in C_c(Y)$ (see Proposition 4.4, [14]). Thus there exist $g_1, g_2 \in C_c(Y)$ such that $f^{\mathcal{M}} \in Y \setminus Z(g_1) \subset Z(g_2) \subset W$ ...(2). As $f^{\mathcal{M}}(M) \notin Z(g_1)$, it follows from (1) that $g_1 \notin \widetilde{M}$ which implies that $g_1 \circ f \notin M$, in other words $M \notin (\mathcal{M}_A)_{g_1\circ f}$. Thus $Max(A_c(X)) \setminus (\mathcal{M}_A)_{g_1\circ f}$ becomes an open neighbourhood of M in the space $Max(A_c(X))$. We assert that $f^{\mathcal{M}}(Max(A_c(X)) \setminus (\mathcal{M}_A)_{g_1\circ f})) \subseteq W$ and this settles the continuity of $f^{\mathcal{M}}$ at the point M.
Proof of the last assertion: Let $N \in (Max(A_c(X)) \setminus (\mathcal{M}_A)_{g_1\circ f}$, then $g_1 \circ f \notin N$, hence $g_1 \notin \widetilde{N}$. Since $g_1 g_2 = 0$ as is evident from the relation (2) above and $\widetilde{N}$ is a prime ideal in C(Y), it follows therefore that $g_2 \in \widetilde{N}$. This implies in view of the relation (1) that $f^{\mathcal{M}}(N) \in Z(g_2)$ and hence from (2) we get that $f^{\mathcal{M}}(N) \in W$.

The part three of the theorem follows from the simple observation that if βX is zero-dimensional , then $\beta_0 X \geqq \beta X$ and consequently $\beta_0 X \approx \beta X$.

□

3. P-spaces X versus the m_c-topology on $C_c(X)$

Notation 3.1. For any $g \in C_c(X)$ and a positive unit u of this ring set $M(g, u) = \{f \in C_c(X) : |f(x) - g(x)| < u(x)\ for\ each\ x \in X\}$. Then it needs a routine calculation to conclude that $\mathcal{B} = \{M(g, u) : g \in C_c(X), u \text{ a positive unit of } C_c(X)\}$ is an open base for some topology, which we call the m_c-topology on $C_c(X)$. It is also not at all hard to show by employing stereotyped routine arguments that $C_c(X)$ with m_c-topology is a topological ring as well as a topological vector space over $\mathbb{R}$. Let U stand for the set of all units in $C_c(X)$. Then for each $u \in U$, it is easy to prove that $M(u, \frac{1}{2}|u|) \subseteq U$. It follows that U is an open set in $C_c(X)$ in the m_c-topology. It is a standard result that in a topological ring the closure of an ideal is either an ideal or the whole of the ring (2M1, [13]). This implies that if I is a proper ideal of $C_c(X)$ then the closure of I in the m_c-topology is also a proper ideal in $C_c(X)$. We therefore get the following result:

Theorem 3.2. *Each maximal ideal in $C_c(X)$ is closed in the m_c-topology.*

Before proceeding further in this technical section on m_c-topology on $C_c(X)$ we recall that the structure space of $C_c(X)$ is $\beta_0 X$. Hence the maximal ideals of $C_c(X)$ can be indexed by virtue of the points of $\beta_0 X$. Indeed the complete list of maximal ideals in $C_c(X)$ is given in ([5], Theorem 4.2) by the family $\{M_c^p : p \in \beta_0 X\}$, where

$M_c^p = \{f \in C_c(X) : p \in cl_{\beta_0 X} Z(f)\}$. This is the C-analogue of the well known Gelfand-Kolmogoroff theorem ([13], Theorem 7.3).

Notation 3.3. For any ideal I in $C_c(X)$ set $Q_c(I) = \{p \in \beta_0 X : M_c^p \supseteq I\}$. Then the following result turns out as a simple consequence of the above formula for the maximal ideals M_c^p's in $C_c(X)$.

Theorem 3.4. *$Q_c(I) = \bigcap_{f\in I} cl_{\beta_0 X} Z(f)$, which is set of all cluster points of the z_c-ultrafilters $Z(I)$ in the space $\beta_0 X$.*

We need to use the following three subsidiary results to prove the first important technical result in this section.

Theorem 3.5. *Let $f \in C_c(X)$ and I be an ideal in $C_c(X)$ such that $cl_{\beta_0 X} Z(f)$ is a neighbourhood of $Q_c(I)$ in $\beta_0 X$. Then $f \in I$.*

Proof. The hypothesis tells that there exists an open subset W of $\beta_0 X$ such that $cl_{\beta_0 X} Z(f) \supseteq W \supseteq Q_c(I)$. We can rewrite this relation in view of Theorem 3.4 in the manner: $cl_{\beta_0 X} Z(f) \supseteq W \supseteq \bigcap\limits_{f\in I} cl_{\beta_0 X} Z(f)$. This implies that $\beta_0 X \setminus cl_{\beta_0 X} Z(f) \subseteq \beta_0 X \setminus W \subseteq \bigcup\limits_{f\in I} (\beta_0 X \setminus cl_{\beta_0 X} Z(f))$. Since the closed subset $\beta_0 X \setminus W$ of $\beta_0 X$ is compact, the last relation yields : $\beta_0 X \setminus cl_{\beta_0 X} Z(f) \subseteq \beta_0 X \setminus W \subseteq \beta_0 X \setminus \bigcap\limits_{i=1}^{n} cl_{\beta_0 X} Z(f_i)$ for a suitable finite subset $\{f_1, f_2 ... f_n\}$ of I. Consequently we have $cl_{\beta_0 X} Z(f) \supseteq W \supseteq \bigcap\limits_{i=1}^{n} cl_{\beta_0 X} Z(f_i)$, which further implies that $cl_{\beta_0 X} Z(f) \cap X \supseteq W \cap X \supseteq Z(\sum\limits_{i=1}^{n} f_i^2) = Z(h)$ say, writing $h = f_1^2 + f_2^2 + ... + f_n^2$. The last relation says that with $h \in I$, $Z(f)$ is a neighbourhood of $Z(h)$ in the space X. It follows from Lemma 2.4 in [14] that f is a multiple of h in the ring $C_c(X)$. Since $h \in I$, we have $f \in I$.

□

Theorem 3.6. *Given $g \in C_c(X)$ and a positive unit u in this ring, there exists $f \in C_c(X)$ such that $|g - f| \leq u$ and $cl_{\beta_0 X} Z(f)$ is a neighbourhood of $cl_{\beta_0 X} Z(g)$ in the space $\beta_0 X$.*

Proof. Let the map $f : X \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & if\ |g(x)| \leq u(x) \\ g(x) + u(x) & if\ g(x) \leq -u(x) \\ g(x) - u(x) & if\ g(x) \geq u(x) \end{cases}$$

It is clear that f is a continuous function and of course $f \in C_c(X)$. It is easily seen that $|f - g| \leq u$ on X. Let $F = \{x \in X : |g(x)| \geq u(x)\}$. Then $F \in Z_c(X)$ so that we can write $F = Z_c(h)$ for some $h \in C_c(X)$. Hence $Z(g) \subseteq X \setminus Z(h) \subseteq Z(f)$. This implies that $Z(g) \cap Z(h) = \emptyset$ and $Z(f) \cup Z(h) = X$. From this it follows that $cl_{\beta_0 X} Z(g) \cap cl_{\beta_0 X} Z(h) = \emptyset$ and $cl_{\beta_0 X} Z(g) \cup cl_{\beta_0 X} Z(f) = cl_{\beta o X} X = \beta_0 X$ [see Proposition 3.2 and Proposition 3.3 in [5]]. This further yields:
$cl_{\beta_0 X} Z(f) \supseteq \beta_0 X \setminus cl_{\beta_0 X} Z(h) \supseteq cl_{\beta_0 X} Z(g)$ this shows that $cl_{\beta_0 X} Z(f)$ is a neighbourhood of $cl_{\beta_0 X} Z(g)$ in the space $\beta_0 X$. □

Define as in 7Q [13], for an ideal I in $C_c(X)$.
$\overline{I} = \bigcap\{M_c^p : M_c^p \supseteq I\}$= the intersection of all maximal ideals in $C_c(X)$ which contain I.

Theorem 3.7. *For any ideal I in $C_c(X)$ $\overline{I}$ is a closed ideal in the m_c-topology and $\overline{I} = \{g \in C_c(X) : cl_{\beta_0 X} Z(g) \supseteq Q_c(I)\}$.*

Proof. It follows immediately from Theorem 3.2 that $\overline{I}$ is a closed ideal in $C_c(X)$ in the m_c-topology. Let $g \in \overline{I}$, choose $x \in Q_c(I)$, then $M_c^x \supseteq I$. Consequently $g \in M_c^x$ and hence $x \in cl_{\beta_0 X} Z(g)$. This implies that $Q_c(I) \subseteq cl_{\beta_0 X} Z(g)$. To prove the reverse inclusion relation let $g \in C_c(X)$ be such that $Q_c(I) \subseteq cl_{\beta_0 X} Z(g)$. Let M_c^p be any maximal ideal in $C_c(X)$ containing I, $p \in \beta_0 X$. Then $p \in Q_c(I)$ consequently $p \in cl_{\beta_0 X} Z(g)$ hence $g \in M_c^p$. Thus $g \in \overline{I}$. □

Theorem 3.8. *For any ideal I in $C_c(X)$, $\overline{I}$ is essentially the closure of I in the m_c-topology.*

Proof. It follows from the first part of Theorem 3.7 that the closure of I in the m_c-topology is contained in $\overline{I}$. To prove the reverse containment let $g \in \overline{I}$ and u be a positive unit of $C_c(X)$. It suffices to produce an $h \in I$ such that $|g-h| \leq u$. Indeed from Theorem 3.6 there exists an $h \in C_c(X)$ with $|g-h| \leq u$ such that $cl_{\beta_0 X} Z(h)$ is a neighbourhood of $cl_{\beta_0 X} Z(g)$ in the space $\beta_0 X$. But $g \in \overline{I}$ implies by Theorem 3.7 that $cl_{\beta_0 X} Z(g) \supseteq Q_c(I)$. Consequently $cl_{\beta_0 X} Z(h)$ becomes a neighbourhood of $Q_c(I)$ in $\beta_0 X$. Hence we get from Theorem 3.5 that $h \in I$. □

Corollary 3.9. An ideal in $C_c(X)$ is closed in the m_c-topology if and only if it is the intersection of all the maximal ideals in $C_c(X)$ which contain it.

Theorem 3.10. *A zero-dimensional space X is a P-space if and only if each ideal in $C_c(X)$ is closed in the m_c-topology.*

Proof. It follows from Corollary 3.9 that each ideal I in $C_c(X)$ is closed in the m_c-topology if and only if each ideal in $C_c(X)$ is the intersection of all the maximal ideals in $C_c(X)$ containing it. In view of Corollary 5.7 and Theorem 5.8 in [14], the last condition is equivalent to the requirement that X is a P-space. □

Before examining the Von-Neumann regularity of the intermediate rings in the family $\Sigma_c(X)$, we need to further organize our machinery accordingly. A commutative ring R with unity is called reduced if 0 is the only nilpotent element of R. It is trivial that each $A_c(X) \in \Sigma_c(X)$ is a reduced ring. In what follows all the rings that will appear will be assumed to be reduced. An ideal I (proper) in R is called a z^0-ideal in R if for each $a \in I$, $\mathcal{P}_a \subseteq I$, where $\mathcal{P}_a$ is the intersection of all minimal prime ideals in R which contains a. We reproduce the following standard useful formula for the $\mathcal{P}_a$ from ([7], Proposition 1.5).

Theorem 3.11. *For each $a \in R$, $\mathcal{P}_a = \{b \in R : Ann(a) \subseteq Ann(b)\}$, where $Ann(a) = \{c \in R : ac = 0\}$ is the annihilator of a in R. We also reproduce the following standard proposition.*

Theorem 3.12. (*Due to Kist,* [17]): *A prime ideal P in a ring R is a minimal prime ideal if and only if for each $a \in P$ there exists $b \in R \setminus P$ such that $a.b$ is a nilpotent member of R and in particular $a.b = 0$ if the ring R is assumed to to reduced.*

Remark 3.13. Each element of a minimal prime ideal in R is a divisor of zero. Consequently each element of a z^0-ideal in R is a divisor of zero.

The following fact is standard and a simple proof is offered in [9] Theorem 4.1.

Theorem 3.14. *Each proper ideal in a Von-Neumann regular ring is a z^0-ideal.*

Theorem 3.15. *An intermediate ring $A_c(X) \in \Sigma_c(X)$ is an absolutely convex subring of $C_c(X)$ in the following sence: If $|f| \leq |g|$ with $g \in A_c(X)$ and $f \in C_c(X)$ then $f \in A_c(X)$. In particular $A_c(X)$ is a lattice ordered ring.*

Proof. since If $|f| \leq |g|$ it follows that $\frac{f}{1+g^2}$ is a bounded function in $C_c(X)$. Thus $f = \frac{f}{1+g^2}.(1+g^2) \in A_c(X)$. □

The following result tells that no intermediate ring in the family $\Sigma_c(X)\backslash\{C_c(X)\}$ can be ever Von-Neumann regular.

Theorem 3.16. *Suppose $A_c(X) \in \Sigma_c(X)$ is Von-Neumann regular , then $A_c(X) = C_c(X)$.*

Proof. Choose $f \in C_c(X)$. We shall show that $f \in A_c(X)$. Because of the absolute convexity of $A_c(X)$ in $C_c(X)$ in the last theorem it suffices to show that $|f| \in A_c(X)$. We shall indeed show that $\frac{1}{1+|f|}$ is a multiplicative unit of the ring $A_c(X)$ and that will do. Suppose towards a contradiction and let $\frac{1}{1+|f|}$ be not a multiplicative unit of $A_c(X)$. It is clear because of the boundedness of the function $\frac{1}{1+|f|}$ over X that $\frac{1}{1+|f|} \in A_c(X)$. Therefore the principle ideal $< \frac{1}{1+|f|} >= I$ in $A_c(X)$ generated by this function is a proper ideal and is hence by Theorem 3.14 a z^0-ideal in $A_c(X)$. It follows from Remark 3.13 that $\frac{1}{1+|f|}$ is a divisor of zero in $A_c(X)$ -a contradiction. □

Since a zero-dimensional space X is a P-space if and only if $C_c(X)$ is Von-Neumann regular (Corollary 5.7, [14]), the following proposition is immediate from the above theorem.

Theorem 3.17. *Let X be a P-space . Then $A_c(X) \in \Sigma_c(X)$ is Von-Neumann reular if and only if $A_c(X) = C_c(X)$.*

4. Almost P-spaces X vis-a-vis the z^0-ideals in $A_c(X)$.

Since the z^0-ideals in $A_c(X)$ are all divisors of zero, the following formula to determine them will be needed from time to time.

Theorem 4.1. *An $f \in A_c(X)$ is a divisor of zero in this ring if and only if $Int_X Z(f) \neq \emptyset$.*

Proof. Suppose $f \in A_c(X)$ is a divisor of zero. Then $f \neq 0$ and there exists $g \neq 0$ in $A_c(X)$ such that $fg = 0$. This shows that $Z(f) \cup Z(g) = X$ and hence $X - Z(g) \subseteq Z(f)$. As $X \setminus Z(g)$ is a non-empty open set in X, it follows that $Int_X Z(f) \neq \emptyset$.

Conversely let $Int_X Z(f) \neq \emptyset$. Choose p from this nonempty set. Since X is zero-dimensional, functions in $C_c(X)$ with their range contained in $[0, 1]$ can separate points and closed sets in X (Proposition 4.4, [14]). Therefore there exists $g \in C_c(X)$ such that $g(p) = 1$ and $g(X \setminus Int_X Z(f)) = 0$. It is clear that $f.g = 0$ and $g \neq 0$. Thus f is divisor of zero in $A_c(X)$. □

The next proposition will also be useful to us:

Theorem 4.2. *Let X be zero-dimensional and $f, g \in A_c(X)$. Then $Int_X Z(f) \subseteq Int_X Z(g)$ if and only if $Ann(f) \subseteq Ann(g)$ in the ring $A_c(X)$.*

Proof. Let $Int_X Z(f) \subseteq Int_X Z(g)$. Choose $h \in Ann(f)$, then $hg = 0$. This implies that $X \setminus Z(h) \subseteq Z(f)$, which further implies that $X \setminus Z(h) \subseteq Int_X Z(f) \subseteq Int_X Z(g) \subseteq Z(g)$. Hence $g.h = 0$ i.e., $h \in Ann(g)$. Thus $Ann(f) \subseteq Ann(g)$.

Conversely let $Ann(f) \subseteq Ann(g)$. It is sufficient to check that $Int_X Z(f) \subseteq Z(g)$. If possible let there exist a point $p \in Int_X Z(f) \setminus Z(g)$. Since X is zero-dimensional, there exists an $h \in C_c^*(X) \subseteq A_c(X)$ such that $h(p) = 1$ and $h(X \setminus Int_X Z(f)) = 0$. It follows that $h.f = 0$ i.e., $h \in Ann(f)$ but $h(p)g(p) \neq 0$. So that $h.g \neq 0$ and hence $h \notin Ann(g)$. This is a contradiction. □

A combination of Theorem 3.10 and Theorem 4.2 yields the following result:

Theorem 4.3. *For any $f \in A_c(X)$ $\mathcal{P}_f = \{g \in A_c(X) : Ann(f) \subseteq Ann(g)\} = \{g \in A_c(X) : Int_X Z(f) \subseteq Int_X Z(g)\}$.*

We recall that $\mathcal{P}_f$ is the intersection of all the minimal prime ideals in $A_c(X)$ which contain f. Before taking up the problem of characterizing almost P-spaces X via z^0-ideals in $C_c(X)$, we need to recall the notion of z-ideal in an arbitrary commutative ring R with unity.

Definition 4.4. An ideal I in R is called a z-ideal in R if for each $a \in I$, $M_a \subseteq I$, here M_a is the intersection of all maximal ideals in R containing a. Evidently each maximal ideal in R is a z-ideal. This notion of z-ideal is consistent with the notion of z-ideals in $C(X)$. (See 4A, [13])

The following result identifies z-ideals and z_c-ideals in $C_c(X)$. An ideal I in $C_c(X)$ is called a z_c-ideal in [14] if whenever $Z(f) \in Z_c(I) = \{Z(g) : g \in I\}$, $f \in C_c(X)$, then $f \in I$.

Theorem 4.5. *Let X be zero-dimensional . Then an ideal I in $C_c(X)$ is a z-ideal if and only if it is a z_c-ideal.*

Proof. Let I be a z_c-ideal in $C_c(X)$. Let $f \in I$ and $g \in M_f$, this means that if for $p \in \beta_0 X$, $f \in M_c^p$ then $g \in M_c^p$. This implies that $cl_{\beta_0 X} Z(f) \subseteq cl_{\beta_0 X} Z(g)$, which further implies on taking intersection with X that $Z(f) \subseteq Z(g)$. Since $f \in I$ and I is a z_c-ideal in $C_c(X)$ it follows that $g \in I$. Thus $M_f \subseteq I$ and hence I is a z-ideal in $C_c(X)$.

Conversely let I be a z-ideal in $C_c(X)$, $f \in I$ and $Z(f) \subseteq Z(g)$ with $g \in C_c(X)$. We have to show that $g \in I$. Since I is a z-ideal in $C_c(X)$ it suffices to show that $g \in M_f$. So let M_c^p be any maximal ideal in $C_c(X)$ it suffices to show that $g \in M_f$. So let M_c^p be any maximal ideal in $C_c(X)$, $p \in \beta_0 X$ which contains f, we have to show that $g \in M_c^p$. Indeed $f \in M_c^p$ implies that $p \in cl_{\beta_0 X} Z(f)$ which further implies that $p \in cl_{\beta_0 X} Z(g)$ hence $g \in M_c^p$. Thus altogether I becomes a z_c-ideal in $C_c(X)$. □

We next establish the countable analogue of the well-known fact 3.11(b) in [13].

Theorem 4.6. *Let K be a compact set contained in a G_δ-set G in a zero-dimensional Hausdorff space X. Then there exists $Z \in Z_c(X)$ such that $K \subseteq Z \subseteq G$.*

Proof. We can write $G = \bigcap_{n=1}^{\infty} G_n$, where each G_n is open in X. Since $K \subset X$ and $K \setminus G_n$ are disjoint closed set in X with K compact, hence by proposition 4.3 in [14], there exists an $f_n \in C_c(X)$ such that $f_n(K) = 0$ and $f \in (X \setminus G_n) = 1$. This implies that $K \subseteq \bigcap_{n=1}^{\infty} Z(f_n) \subseteq G$. Since $Z_c(X)$ is closed under countable intersection by Lemma 2.2(a) in [5], it follows that $\bigcap_{n=1}^{\infty} Z(f_n) = Z(f)$ for some $f \in C_c(X)$.This implies that $K \subseteq Z(f) \subseteq G$. □

Before seriously embarking on almost P-spaces, we introduce the following localized version of this requirement.

Definition 4.7. A point $p \in X$ is called an almost P-point on X if for any zero set Z in X containing p, $Int_X Z \neq \emptyset$. Thus X is an almost P-space if and only if each point on X is an almost P-point.

Theorem 4.8. *The following statements are equivalent for a point p on a zero-dimensional Hausdorff space X.*

1. *p is an almost P-point on X.*
2. *For any any G_δ-set G containing p, $Int_X G \neq \emptyset$.*
3. *For any $Z \in Z_c(X)$ containing p, $Int_X Z \neq \emptyset$.*

Proof. (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (1) are trivial.
(3) $\Longrightarrow$ (2): Let (3) hold. Let G be a G_δ set in X containing p. Then by Theorem 4.6, there exists $Z \in Z_c(X)$ such that $p \in Z \subset G$. Since $Int_X Z \neq \emptyset$ it follows from (3) that $Int_X G \neq \emptyset$. □

Corollary 4.9. A zero-dimensional space X is an almost P-space if and only if for any nonempty $Z \in Z_c(X)$, $Int_X Z \neq \emptyset$.

We are now ready to offer the following comprehensive theorem giving several characterization of almost P-space.

Theorem 4.10. *The following statements are equivalent for a zero-dimensional space X.*

1. *X is almost P.*
2. *Every maximal ideal in $C_c(X)$ is a z^0-ideal.*
3. *Every fixed maximal ideal in $C_c(X)$ is a z^0-ideal.*
4. *Every z-ideal in $C_c(X)$ is a z^0-ideal.*

Proof. (1) $\Longrightarrow$ (2): Let X be almost P-space and M be a maximal ideal in $C_c(X)$. Then we can write $M = M_c^p = \{f \in C_c(X) : p \in cl_{\beta_0 X} Z(f)\}$ for some point $p \in \beta_0 X$. Choose $f \in M$ we shall show that $\mathcal{P}_f \subseteq M$ and hence M is a z^0-ideal in $C_c(X)$. So let $g \in \mathcal{P}_f$. Then from Theorem 4.3 we get $Int_X Z(f) \subseteq Int_X Z(g)$. But since X is almost P, each zero set in X is regular closed see [18]. This implies that $Z(f) = cl_X(Int_X Z(f)) \subseteq cl_X(Int_X Z(g)) = Z(g)$. But $f \in M$ implies that $p \in cl_{\beta_0 X} Z(f)$, consequently $p \in cl_{\beta X} Z(g)$ and hence $g \in M_c^p = M$. Thus $\mathcal{P}_f \subseteq M$.
(2) $\Longrightarrow$ (3): is trivial.
(3) $\Longrightarrow$ (1): Let (3) be true. It is sufficient to show in view of Corollary 4.9 that, for a non-empty $Z \in Z_c(X)$, $Int_X Z \neq \emptyset$. Indeed $Z = Z(f)$ for some $f \in C_c(X)$. Choose a point $p \in Z$, then $f \in M_{p,c} = \{g \in C_c(X) : g(p) = 0\}$. Now by (3), $M_{p,c}$ is a z^0-ideal, consequently by Remark 3.13, f is a divisor of zero in $C_c(X)$. This implies by Theorem 4.1 that , $Int_X Z(f) \neq \emptyset$.
(4) $\Longrightarrow$ (2): is trivial because each maximal ideal in a ring R is a z-ideal.

(1) $\implies$ (4): Let X be almost P-space and I be a z-ideal in $C_c(X)$. Then by Theorem 4.5, I is a z_c-ideal in $C_c(X)$. Let $f \in I$ we need to verify that $\mathcal{P}_f \subseteq I$ in order to show that I is a z^0-ideal in $C_c(X)$. Choose $g \in \mathcal{P}_f$ then it follows from Theorem 4.3 that $Int_X Z(f) \subseteq Int_X Z(g)$. As X is almost P we can therefore write: $Z(f) = cl_X(Int_X Z(f)) \subseteq cl_X(Int_X Z(g)) = Z(g)$. Since $f \in I$ and I is a z_c-ideal, it follows that $g \in I$. Thus $\mathcal{P}_f \subseteq I$. □

We now show that on choosing $A_c(X) \in \Sigma_c(X) \setminus \{C_c(X)\}$ Theorem 4.10 can not be improved by writing that X is almost P if and only if each maximal ideal in $A_c(X)$ is a z^0-ideal (respectively each z-ideal in $A_C(X)$ is a z^0-ideal).

Theorem 4.11. *Let $A_c(X)$ be an intermediate ring in $\Sigma_c(X)$ properly contained in $C_c(X)$. Then there exists a maximal ideal M in $A_c(X)$ which is not a z^0-ideal (clearly M is also a z-ideal in $A_c(X)$ which is not a z^0-ideal).*

Proof. We select $f \in C_c(X)$ such that $f \notin A_c(X)$. Take $g = \frac{1}{1+|f|}$, then $g \in C_c^*(X) \subseteq A_c(X)$. It follows from absolute convexity of $A_c(X)$ in $C_c(X)$ (Theorem 3.15) that $1 + |f| \notin A_c(X)$. Hence g is not invertible in $A_c(X)$. So there exists a maximal ideal M in $A_c(X)$ such that $g \in M$. Since g is not a divisor of zero in $A_c(X)$ (Theorem 4.1). It follows from Remark 3.13 that M is not a z^0-ideal in $A_c(X)$. □

Theortem 4.10 and Theorem 4.11 combined together yield the following characterization of $C_c(X)$ among members of $\Sigma_c(X)$.

Theorem 4.12. *Let X be almost P. Then the following three statements are equivalent for an $A_c(X) \in \Sigma_c(X)$.*

1. *Each maximal ideal of $A_c(X)$ is a z^0-ideal*
2. *Each z-ideal of $A_c(X)$ is a z^0-ideal*
3. *$A_c(X) = C_c(X)$.*

Compare with similar kind of characterizations in [9],[22], [23].

5. Pseudocompact spaces X via U_c-topologies/m_c-topologies on $C_c(X)$

Notation 5.1. For $f \in C_c(X)$ and $\epsilon > 0$ in $\mathbb{R}$. Let $U_c(f, \epsilon) = \{g \in C_c(X) : Sup_{x \in X}|f(x) - g(x)| < \epsilon\}$

It is easy to check that the family $\{ U_c(f, \epsilon) : f \in C_c(X), \epsilon > 0\}$ is an open base for some topology on $C_c(X)$ which we call the U_c-topology on $C_c(X)$ and $C_c(X)$ becomes an additive topological group in this topology. The following proposition shows that $C_c(X)$ neither a topological ring nor a topological vector space unless X is pseudocompact.

Theorem 5.2. *For a zero-dimensional Hausdorff space X, the following statements are equivalent:*

1. *X is pseudocompact.*
2. *$C_c(X)$ with U_c-topology is a topological ring.*
3. *$C_c(X)$ with U_c-topology is a topological vector space.*
4. *The set W of all units in $C_c(X)$ is open in U_C-topology.*

Proof. First assume that X is pseudocompact ie; $C_c(X) = C_c^*(X)$. Then the U_C-topology on $C_c(X)$ coincides with the uniform norm topology on it and $C_c(X)$

becomes a real normed algebra. It is a standard result in Functional Analysis that a real normed algebra is a topological ring as well as a real topological vector space, where the units W make an open subset of $C_c(X)$. Conversely, let X be not pseudocompact. Then there exists an $f \in C_c(X) \setminus C_c^*(X)$ with $f \geqq 1$. Take $g = \frac{1}{f}$. Then $g \in C_c^*(X)$ and it takes values arbitrarily near to zero on X. We note that for arbitrary $\epsilon > 0$, $\delta > 0$ in $\mathbb{R}$, $\underline{\frac{\epsilon}{2}}$ (the constant function on X with value $\frac{\epsilon}{2}$) $\in U_c(0, \epsilon)$ and $f \in U_c(f, \delta)$ while $\underline{\frac{\epsilon}{2}}.f \notin U_c(0, 1)$. This proves that the function: $C_C(X) \times C_c(X) \to C_C(X)$ defined as follows $(k, l) \mapsto k.l$ is not continuous at the point $(0, f)$. It can be proved analogously that the scalar multiplication function:

$$\mathbb{R} \times C_c(X) \to C_c(X)$$
$$(r, f) \mapsto r.f$$

is not continuous at $(0, f)$. Thus $C_c(X)$ neither a topological ring nor a topological vector space over R.

Finally we observe that g is a unit of $C_c(X)$ i.e; $g \in W$. To show that W is not an open set we shall show that g is not an interior point of W. Choose $\epsilon > 0$ in $\mathbb{R}$. Since g takes values arbitrarily near to zero on X, there exists $a \in X$ such that $0 < g(a) < \epsilon$. Take $h = g - g(a)$, then $h \in C_c(X)$ and $h \in U_c(g, \epsilon)$ but h is not a unit of $C_c(X)$ as $h(a) = 0$. Thus $U_c(g, \epsilon)$ is not a subset of W and hence g is not an interior point of W. □

As in the classical scenario with $C(X)$ (see 2N, [13]) it is easy to observe that the relative topology on $C_c^*(X)$ induced by the m_c-topology on $C_c(X)$ is finer that the uniform norm topology on $C_c^*(X)$. The following proposition says that these two topologies coincide when and only when X is pseudocompact.

Theorem 5.3. *The following two statements are equivalent for a Hausdorff zero-dimensional space X.*

(1) *X is pseudocompact.*
(2) *The relative m_c-topology on $C_c^*(X)$ is identical to the uniform norm topology on it.*

Proof. First assume that X is pseudocompact. In view of the above observations, it is sufficient to show that that relative m_c-topology $C_c^*(X)$ is weaker than the uniform norm topology. Choose $f \in C_c^*(X)$ and a positive unit u of this ring. Then u is bounded away from zero so that we can write $u(x) \geq \lambda$ for all $x \in X$ for some $\lambda > 0$. It follows that the closed ball $\{g \in C_c^*(X)) : ||f - g|| \leq \lambda\}$ centered at f with radius λ in the norm topology is contained in $M(f, u)$ and we are through.

To prove the converse let X be not pseudocompact. To show that the relative m_c-topology on $C_c^*(X)$ is not the same as the uniform norm topology on it, we shall show that $C_c^*(X)$ in the former topology is not a topological vector space, Since X is not pseudocompact there exists $k \in C_c^*(X)$ such that k is a positive unit of $C_c(X)$ which takes values arbitrarily near to zero on X. It follows that there does not exist any pair of distinct real numbers r, s with $|\underline{r} - \underline{s}| \leq k$ on X. Hence for any $r \in \mathbb{R}$, $M(\underline{r}, k) \cap \{\underline{s} : s \in \mathbb{R}\} = \{\underline{r}\}$ in other words the set of all constant functions in $C_c^*(X)$ is a discrete subset of $C_c^*(X)$ in the relative m_c-topology. Consequently the scalar multiplication map: $\mathbb{R} \times C^*(X) \to C^*(X))$ defined as follows $(r, f) \to r.f$ is not continuous at the points like $(r, \underline{s})$ with $r, s \in \mathbb{R}$, here $\underline{s}$ stands for for the constant function with value $'s'$ on X.

□

6. Questions of Noetherianness/ Artinianness about $C_c(X)$ and their chosen subrings.

A commutative ring R (with or without identity) is called Noetherian/ Artinian if any ascending sequence of ideals $I_1 \subseteq I_2 \subseteq .....$/ descending sequence of ideals $I_1 \supseteq I_2 \supseteq .....$ terminates at a finite stage. It is established in [2] that for a Tychonoff space X, $C(X)$ (respectively $C^*(X)$) is never Noetherian and also never Artinian unless X is a finite set. Noetherianness/ Artinianness of a selected class of subrings of $C(X)$ are also examined in [2]. In the present section our intention is to record the appropriate counterparts of the problems dealt in [2] in the context of the rings $C_c(X)$ and $C_c^*(X)$ for a zero-dimensional Hausdorff space X.

A family $\mathcal{P}$ of closed set in X is called an ideal of closed sets if

(1) $A \in \mathcal{P}, B \in \mathcal{P} \implies A \cup B \in \mathcal{P}$ and
(2) $A \in \mathcal{P}$ and $K \subseteq A$ with K closed in $X \implies K \in \mathcal{P}$

Notation 6.1. Let $\Omega(X)$ stand for the family of all ideals of closed sets in X with $\mathcal{P} \in \Omega(X)$. We associate the following two subrings of $C(X)$:
$C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$ and
$C_\infty^{\mathcal{P}}(X) = \{f \in C(X) :$ for each $n \in N, \{x \in X : |f(x)| \geqq \frac{1}{n}\} \in \mathcal{P}\}$ with $C_{\mathcal{P}}(X)$ a z-ideal in $C(X)$. X is called locally $\mathcal{P}$ if each point $x \in X$ has an open neighbourhood W with its closure lying on $\mathcal{P}$. Thus the local $\mathcal{P}$ condition reduces to local compactness if $\mathcal{P}$ is the ideal of all compact sets in X and in this case $C_{\mathcal{P}}(X) = C_K(X)$ and $C_\infty^{\mathcal{P}}(X) = C_\infty(X)$. For more information on ideal related problems we refer the articles [3],[4].

The following result is standard and is recorded in [2], Lemma 2.1.

Lemma 6.2. *For any finitely many commutative rings $R_1, R_2, ...R_n$ each with identity, ideals of the direct product $R_1 \times R_2 \times ... \times R_n$ are precisely of the form: $J_1 \times J_2 \times ... \times J_n$ where for $j = 1, 2, ...n,$ J_j is an ideal in R_j.*

We record the following convenient version of the local $\mathcal{P}$ condition for a zero-dimensional space X.

Theorem 6.3. *For a zero-dimensional Hausdorff space X, the following statements are equivalent:*

(1) *X is locally $\mathcal{P}$*
(2) *$\{Z(f) : f \in C_{\mathcal{P}}(X) \cap C_c(X)\}$ is a base for the closed sets in X.*
(3) *$\{Z(f) : f \in C_\infty^{\mathcal{P}}(X) \cap C_c(X)\}$ is a closed base for X.*
(4) *$\{Z(f) : f \in C_{\mathcal{P}}(X) \cap C_c^*(X)\}$ is a closed base for X.*
(5) *$\{Z(f) : f \in C_\infty^{\mathcal{P}}(X) \cap C_c^*(X)\}$ is a closed base for X.*

We omit the proof of this theorem, which can be done by closely following the arguments and making some necessary modifications in the proof of Theorem 4.3 in [2]. We are now ready to enunciate the main theorem in this section.

Theorem 6.4. *Let $\mathcal{P} \in \Omega(X)$ where X is a zero-dimensional Hausdorff space which is further locally $\mathcal{P}$. Then the following statements are equivalent:*

(1) *$C_{\mathcal{P}}(X) \cap C_c(X)$ is a Noetherian Ring.*
(2) *$C_{\mathcal{P}}(X) \cap C_c(X)$ is an Artinian Ring.*
(3) *$C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ is a Noetherian Ring.*
(4) *$C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ is an Artinian Ring.*
(5) *X is a finite set.*

The proof can be accomplished by making a close introspection of the reasonings made in the proof of the Theorem 1.1 in [2]. Nevertheless to make the article self-contained and to highlight a few important remarks regarding the possible dearth of Noetherian Rings/ Artinian rings lying between $C_c^*(X)$ and $C_c(X)$, we wish to provide an alternatively framed regorous proof of the above theorem.

Proof of the Theorem 6.4: First assume that X is a finite set with $'n'$ elements. Then since X is Hausdorff it becomes a discrete space. Consequently $C(X) = \mathbb{R}^X$, which is isomorphic to direct product of $\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$ (n times). On the other hand since X is locally $\mathcal{P}$ it follows that $C_{\mathcal{P}}(X) = C_\infty^{\mathcal{P}}(X) = C_c(X) = C(X)$ consequently $C_{\mathcal{P}}(X) \cap C_c(X) = C_\infty^{\mathcal{P}}(X) \cap C_c(X) = C_c(X) = C(X)$. Since the field $\mathbb{R}$ has just 2 ideals, it follows from Lemma 6.2 that C(X) has just 2^n many ideals. Hence the rings $C_{\mathcal{P}}(X) \cap C_c(X)$ and $C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ are both Noetherian and Artinian.

Conversely, let X be an infinite set. We shall show that thae ring $C_{\mathcal{P}}(X) \cap C_c(X)$ is not a Noetherian ring. Analogous arguments can be made to show that $C_{\mathcal{P}}(X) \cap C_c(X)$ is not an Artinian ring and nor is the ring $C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ Noetherian or Artinian. As X is an infinite Hausdorff space it contains a copy of $\mathbb{N}$ (0.13, [13]), So for each $k \in \mathbb{N}$ there exists an open set W_k in X such that $W_k \cap \mathbb{N} = \{k\}$. Since X is locally $\mathcal{P}$ and zero-dimensional, we can employ Theorem 6.3 to find for each $k \in \mathbb{N}$, an $f_k \in C_{\mathcal{P}}(X) \cap C_c(X)$ such that $k \in X \setminus Z(f_k) \subset W_k$....(1). We now assert that the ideal $I =< f_1, f_2, ...f_k, .. >$ generated by these $f_k's$ in the ring $C_{\mathcal{P}}(X) \cap C_c(X)$ can not be finitely generated and hence $C_{\mathcal{P}}(X) \cap C_c(X)$ is not Noetherian.(A ring R is Noetherian if and only if each ideal in R is finitely generated: A standard result).

Proof of the assertion: Choose $n \in N$. We show that the ideal $< f_1, f_2, ...f_n > \subsetneqq I$ and that will do. Indeed from (0) and (1) it follows that $f_{n+1}(n+1) \neq 0$, while $f_1(n+1) = f_2(n+1) = ... = f_n(n+1) = 0$. Thus there do not exist functions $l_1, l_2, ...l_n \in C_{\mathcal{P}}(X) \cap C_c(X)$ for which we can write: $f_{n+1} = l_1 f_1 + l_2 f_2 + ...l_n f_n$. Hence $f_{n+1} \in I \setminus < f_1, f_2, ...f_k, .. >$.

Remark 6.5. Since for any $\mathcal{P} \in \Omega(X)$, $C_{\mathcal{P}}(X) \subseteq C_\infty^{\mathcal{P}}(X)$ an easy verification, it follows from Theorem 6.3 that for any prescribed ring R lying either between $C_{\mathcal{P}}(X) \cap C_c(X)$ and $C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ or between $C_{\mathcal{P}}(X) \cap C_c(X)$ and $C_{\mathcal{P}}(X) \cap C_c^*(X)$, a zero-dimensional space X is locally $\mathcal{P}$ if and only if $\{Z(f) : f \in R\}$ is a base for the closed sets in X. With this observation in mind, if we make a close scrutiny into the proof of the converse part of Theorem 6.4, we get the following result.

Theorem 6.6. *Given $\mathcal{P} \in \Omega(X)$, if X is an infinite zero-dimensional locally $\mathcal{P}$ space, then no ring lying between $C_{\mathcal{P}}(X) \cap C_c(X)$ and $C_\infty^{\mathcal{P}}(X) \cap C_c(X)$ is Noetherian (respectively Artinian) and also no ring lying between $C_{\mathcal{P}}(X) \cap C_c(X)$ and $C_{\mathcal{P}}(X) \cap C_c^*(X)$ is Noetherian (respectively Artinian).*

We record below two special cases of Theorem 6.6, on choosing $\mathcal{P} \equiv$ the ideal of all compact sets in X in the first part of the Theorem and on choosing $\mathcal{P} \equiv$ the ideal of all closed sets in X in the second part of the theorem.

Theorem 6.7. (1) *If X is an infinite locally compact zero-dimensional space then no ring lying between $C_K(X) \cap C_c(X)$ and $C_\infty(X) \cap C_c^*(X)$ is Notherian/ Artinian.*

(2) *For any infinite zero-dimensional space X, no intermediate ring $A_c(X) \in \Sigma_c(X)$ is Notherian/ Artinian.*

7. Formula for z^0-ideals in intermediate rings.

We first show that ideal I in an intermediate ring $A_c(X) \in \Sigma(X)$ gives rise to an ideal of closed sets in X. Indeed fer any such I, we get $\mathcal{P}_I^{A_c} = \{E \subseteq X : E$ is closed in X and there exists $f \in I$ such that $E \subseteq cl_X(X \setminus Z(f))$. If is easy to verify that $\mathcal{P}_I^{A_c}$ is an ideal of closed sets in X i.e; $\mathcal{P}_I^{A_c} \in \Omega(X)$ and also that $I \subseteq C_{\mathcal{P}_I^{A_c}}(X) \cap A_c(X) \equiv \{f \in A_c(X) : cl_X(X \setminus Z(f)) \in \mathcal{P}_I^{A_c}\}$. The following fact tells decisively when does equality occur in the last inclusion relation. Incidentally we get an explicit formula for z^0-ideals in the intermediate rings.

Theorem 7.1. *Let $A_c(X) \in \Sigma_c(X)$. Then an ideal I in $A_c(X)$ is a z^0-ideal in this ring if and only if there exists $\mathcal{P} \in \Omega(X)$ such that $I = C_{\mathcal{P}}(X) \cap A_c(X)$.*

Proof. First assume that I is a z^0-ideal in $A_c(X)$. In view of the observations foregoing this theorem, it is sufficient to show that $C_{\mathcal{P}_I^{A_c}}(X) \cap A_c(X) \subseteq I$. So let $g \in C_{\mathcal{P}_I^{A_c}}(X) \cap A_c(X)$ then $cl_X(X \setminus Z(g)) \in \mathcal{P}_I^{A_c}$. Consequently there exists $f \in I$ such that $cl_X(X \setminus Z(g)) \subseteq cl_X(X \setminus Z(f))$. This implies on taking complement in X that $Int_X Z(g) \supseteq Int_X Z(f)$, which further implies in view of Theorem 4.3 that $g \in \mathcal{P}_f \equiv$ the intersection of all minimal prime ideals in $A_c(X)$ containing f. Since $f \in I$ and I is a z^0-ideal in $A_c(X)$ it follows that $g \in I$. Thus we get: $I = C_{\mathcal{P}_I^{A_c}}(X) \cap A_c(X)$.

To prove the other part of the theorem we show that for any $\mathcal{P} \in \Omega(X)$, $C_{\mathcal{P}}(X) \cap A_c(X)$ is a z^0-ideal in $A_c(X)$. Choose $f \in C_{\mathcal{P}}(X) \cap A_c(X)$, then $cl_X(X \setminus Z(f)) \in \mathcal{P}$. We need to verify that $\mathcal{P}_f \subseteq C_{\mathcal{P}}(X) \cap A_c(X)$. So choose $g \in \mathcal{P}_f$, then by Theorem 4.3 $Int_X Z(f) \subseteq Int_X(g)$, which implies obviously that $cl_X(X \setminus Z(g)) \subseteq cl_X(X \setminus Z(f))$. Since $f \in C_{\mathcal{P}}(X)$ it follows that $cl_X(X \setminus Z(f)) \in \mathcal{P}$. As $\mathcal{P}$ is an ideal of closed sets in X, this further implies that $cl_X(X \setminus Z(g)) \in \mathcal{P}$ i.e; $g \in C_{\mathcal{P}}(X) \cap A_c(X)$. Thus $\mathcal{P}_f \subseteq C_{\mathcal{P}}(X) \cap A_c(X)$. □

It is established recently in [1], Theorem 5.2 that an ideal I in $C(X)$ with X, Tychonoff is a z^0-ideal in $C(X)$ if and only if there exists $\mathcal{P} \in \Omega(X)$ such that $I = C_{\mathcal{P}}(X)$. Therefore we can make the following comments.

Remark 7.2. z^0-ideals in the intermediate rings $A_c(X) \in \Sigma_c(X)$ with X, zero-dimensional are exactly the contractions of z^0-ideals in $C(X)$.

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