

Probability distribution and sizes of spanning clusters at the percolation thresholds

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Abstract

For random percolation at p_c , the probability distribution $P(n)$ of the number of spanning clusters (n) has been studied in large scale simulations. The results are compatible with $P(n) \sim \exp(-an^2)$ for all dimensions. We also study the variation of the average size (mass) of the spanning clusters when there are more than one spanning cluster. While the average size of the spanning clusters scales as usual with L^D where $D = d - \beta/\nu$ for any number of clusters, it shows a smooth decrease as the number of spanning clusters increases.

More than one spanning cluster (when the spanning is considered in one direction) at the percolation threshold for dimensions below 6 have been shown to exist in a number of recent studies [1-6]. The numerical evidence for nonunique spanning cluster was already there in five dimensions in de Arcangelis' study of spanning clusters [1] nearly a decade ago, which was, however, interpreted as a finite size effect.

Recently, there have been some controversies regarding the behaviour of the probability distribution $P(n)$ of the number of spanning clusters (n). Using mathematical arguments, Aizenman has shown [2,5] that $P(n) \sim \exp(-an^2)$ in two dimensions. Aizenman [2] also conjectures that in any dimension $P(n) \sim \exp(-an^{\frac{d}{d-1}})$. In simulations reported earlier [3], we had remarked that the distribution appears to be a simple exponential (for 4 and 5 dimensions, only for which considerable data was available). But for reasons given below and on the basis of our present results, this now seems less clear. In the earlier study ([3], referred to as I henceforth), we obtained these probabilities by varying the site concentration p (the initial value taken close to the known value of p_c) until a spanning cluster occurs and then counted the number n of such clusters. Here we use the p_c for the infinite lattice thereby increasing the efficiency of the program. However, the numerical results are then different. We also later realised that the averaging in the previous data in I was actually being taken over a number of configurations lesser by a factor of 136 than the intended number due to an error in the parallelisation of the program. However, the main results of I have been verified to be qualitatively correct; the probability of getting more than one spanning cluster does remain finite. In fact, the present definition indicates a higher probability with lesser error and as an example we have presented the result for two dimensions in fig. 1 for this probability. Here, as in I, the probabilities have been normalised by a factor of $(1 - P(0))$, which is the spanning probability.

Here also, we have simulated hypercubic lattices with L^d sites in d dimensions with helical boundary condition. The number of configurations over which averaging is done varies from 10^4 to 10^6 according to the sizes of the lattice. Simulations are carried on for 2 to 5 dimensions.

We have plotted (fig. 2 and 3) the logarithm of the probabilities against (a) n and (b) n^2 for dimensions 2 to 5 and also against (c) $n^{\frac{d}{d-1}}$ for dimensions 4 and 5. It is observed that plot (b) gives a straight line over a wider range of n in comparison to (a) and (c), the latter correspond to a simple exponential and the exponent conjectured in [2] respectively. We also get support for an exponent equal to 2 by a different analysis. Assuming that

$P(n) \sim \exp(-an^\gamma)$, let

$$z = \ln(P(n))/\ln(P(n+1)) \sim (n/(n+1))^\gamma \quad (1)$$

We have first plotted double-logarithmically z against $\frac{n}{n+1}$ to see whether we really get a straight line. This has been done for five dimensions only, where we have obtained the maximum number of spanning clusters. It is difficult to determine from fig. 4a which straight line of the three (with slopes = 1.00, 1.25 and 2.00) is the best fit to the data (for $\frac{n}{n+1} \rightarrow 1$). This is compatible with the fact that there are some regions in each of the figures 2a, 2b, 3a, 3b and 3c where a straight line can be fitted. However, it is also true that the data represented by the \square and \times in fig. 4a, have a slope close to $\gamma = 2$, for which the we have the best statistics ($L = 15$ and $L = 13$).

We have also calculated $x = \ln(z)/(\ln(n/n+1))$ for different sizes which should approach γ for $n \rightarrow \infty$ according to (1). In two dimensions, we do not have enough clusters to calculate this quantity. In order to check whether the values are affected by the presence of a prefactor (which has been neglected in (1)), we have also calculated x with the absolute values $P(n)$ in eqn (1) replaced by the ratios $P(n)/(1 - P(0))$. We find that x approaches 2 in dimensions 3 to 5. The results are shown in fig. 4b-d, where we show the values of x for both cases. The \diamond represents the case when the prefactor is included. For three dimensions, in fig. 4b, the data close to $x = 6$ correspond to $n = 1$ and the one close to 2 to $n = 2$. With no prefactor, the data is above 2 for $n = 1$ and slightly below 2 for $n = 2$. For $d = 4$ and 5 (fig. 4c and fig. 4d), the values of x approach 2 from below as n is increased when there is no prefactor and from above when there is a prefactor (in general). We notice that the values of x from the two different measures come closer as n is increased in all dimensions and it also approaches 2 clearly in 4 and 5 dimensions where we have data for a larger number of clusters. In fig. 4c, data for $n = 1, 2$ and 3 are shown while in fig. 4d, data for $n = 2, 3, 4, 5$ and 6 are shown. Although there are some fluctuations in the data, apparently the exponent is closer to 2 than to 1. Our conclusion is therefore that for intermediate n the probability distribution behaves as a stretched exponential $P(n) = \exp(-an^\gamma)$, with the exponent close to 2 in all dimensions. Apparently, the value of a decreases with dimensionality (~ 0.5 for $d = 4$ and ~ 0.2 for $d = 5$).

We have also studied the sizes or masses (i.e., the number of occupied sites in the cluster) of the spanning clusters. The average size of the spanning clusters indicates that even when there are more than one cluster, the size still scales as L^D , with $D = d - \beta/\nu$, as in the

case of unique cluster [7]. However, there is a decrease in the average size as more spanning clusters appear and this is also expected. In two or three dimensions, where we have at the most 2 or 3 clusters respectively in general, the average size of spanning clusters versus the system sizes L data is shown in fig. 5a. For 4 and 5 dimensions, where more clusters are obtained, the scaled data (\bar{m}/L^D) , where \bar{m} is the average mass, against the cluster number is shown in a log-log plot (fig.5b). It is difficult to verify whether the variation with n is a power law because of the fluctuations for large number of clusters. These fluctuations are unavoidable as the averaging for cluster sizes for larger n is done over a smaller number of cases (as the probability decreases with number of clusters). However, the smooth decrease is clearly indicated.

We also calculated the second and third moments of the masses where the q th moment is defined as $m_q = \frac{1}{n} \sum_i^n (m_i - \bar{m})^q$, to get an idea of the probability distribution of the sizes. While m_2/\bar{m}^2 remains more or less a constant for different system sizes in five dimensions for six clusters, m_3/\bar{m}^3 remains positive and fairly constant indicating an asymmetric distribution : possibly one large cluster and several smaller clusters exist.

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Figure Captions

Fig. 1 The variation of the probability of getting more than one spanning cluster is shown against different system sizes in two dimensions.

Fig. 2 Probability distribution ($P(n)$) of the number of spanning clusters (n) for 2 (\diamond $L = 150$) and 3 (upper data points) dimensions against (a) n and (b) n^2 . The different symbols are for sizes : + $L = 100$, \square $L = 125$, \times $L = 150$, \triangle $L = 175$, * $L = 200$ in 3 dimensions.

Fig. 3 Probability distribution ($P(n)$) of the number of spanning clusters (n) for 4 (lower data points) and 5 (upper data points) dimensions against (a) n and (b) n^2 and (c) against $n^{\frac{d}{d-1}}$. The different symbols correspond to different system sizes : \diamond for $L = 21$, + for $L = 19$, \square for $L = 17$, \times for $L = 15$ and \triangle for $L = 13$ for $d = 5$ and * for $L = 39$, filled \square for $L = 35$, filled \triangle for $L = 30$, \diamond for $L = 27$, and + for $L = 23$ for $d = 4$.

Fig. 4a The values of z against $\frac{n}{n+1}$ as found for 5 dimensions. The different symbols represent different sizes : \diamond $L = 21$, + $L = 19$, \square $L = 15$ and \times $L = 13$. The three straightlines have slope = 1.00, 1.25 and 2.00 (from left to right).

Fig. 4b The values of x vs. L as found for 3 dimensions. The \diamond are for the ratios and the + for the absolute values of $P(n)$. n increases as the value of x decreases.

Fig. 4c The values of x vs. L as found for 4 dimensions. The \diamond are for the ratios and the + for the absolute values of $P(n)$. n increases as the value of x decreases for \diamond and vice versa for the +.

Fig. 4d The values of x vs. n as found for 5 dimensions. The \diamond are for the ratios and the + for the absolute values of $P(n)$. The data correspond to different sizes $L = 13$ to $L = 21$.

Fig 5a The variation of the average mass of the spanning clusters against the system sizes for $d = 2$ (\diamond for no. of spanning clusters $n = 1$ and + for $n = 2$) and $d = 3$ (\square for $n = 1$, \times for $n = 2$ and \triangle for $n = 3$). The system sizes in $d = 3$ have been multiplied by 10 in order to be shown in the same range.

Fig 5b The variation of the scaled average mass (\bar{m}/L^D) of the spanning clusters against the the number of clusters in 4 and 5 dimensions. The average masses in four dimensions have been divided by 2 for better viewing. The different symbols correspond to different

system sizes : \diamond for $L = 21$, $+$ for $L = 19$, \square for $L = 17$, \times for $L = 15$ and \triangle for $L = 13$ for $d = 5$ and $*$ for $L = 39$, filled \square for $L = 35$, filled \triangle for $L = 30$, \diamond for $L = 27$, and $+$ for $L = 23$ for $d = 4$.











