ORDERED FIELD VALUED CONTINUOUS FUNCTIONS WITH COUNTABLE RANGE

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Abstract. For a Hausdorff zero-dimensional topological space X and a totally ordered field F with interval topology, let $C_c(X,F)$ be the ring of all $F-$valued continuous functions on X with countable range. It is proved that if F is either an uncountable field or countable subfield of $\mathbb{R}$, then the structure space of $C_c(X,F)$ is $\beta_0 X$, the Banaschewski Compactification of X. The ideals $\{O_c^{p,F} : p \in \beta_0 X\}$ in $C_c(X,F)$ are introduced as modified countable analogue of the ideals $\{O^p : p \in \beta X\}$ in $C(X)$. It is realized that $C_c(X,F) \cap C_K(X,F) = \bigcap_{p \in \beta_0 X \setminus X} O_c^{p,F}$, this may be called a countable analogue of the well-known formula $C_K(X) = \bigcap_{p \in \beta X \setminus X} O^p$ in $C(X)$. Furthermore, it is shown that the hypothesis $C_c(X,F)$ is a Von-Neumann regular ring is equivalent to amongst others the condition that X is a $P-$space.

1. Introduction

Let F be a totally ordered field equipped with its ordered topology. For any topological space X, suppose $C(X,F)$ is the set of all $F-$valued continuous functions on X. This later set becomes a commutative lattice ordered ring with unity, if the operations are defined pointwise on X. As in classical scenario with $F = \mathbb{R}$, there is already discovered an interplay existing between the topological structure of X and the algebraic ring and order structure of $C(X,F)$ and a few of its chosen subrings. In order to study this interaction, one can stick to a well-chosen class of spaces viz. the so-called completely $F-$regular topological spaces or in brief CFR spaces. X is called CFR space if it is Hausdorff and points and closed sets in X could be separated by $F-$valued continuous functions in an obvious manner. Problems of this kind are addressed in [**1**], [**2**], [**3**], [**4**], [**5**], [**9**]. It turns out that with $F \neq \mathbb{R}$, CFR spaces are precisely zero-dimensional spaces. Thus zero-dimensionality on X can be realized as a kind of separation axiom effected by $F-$valued continuous functions on X. In the present article, we intend to examine the countable analogue of the ring $C(X,F)$ vis-a-vis the corresponding class of spaces X. Towards that end, we let

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$C_c(X,F) = \{f \in C(X,F) : f(X) \textit{ is a countable subset of } F\}$. Then $C_c(X,F)$ is a subring as well as a sublattice of $C(X,F)$. It is interesting to note that spaces X in which points and closed sets can be separated by functions in $C_c(X,F)$ are exactly zero-dimensional also [Theorem 2.9]. Furthermore, the set of all maximal ideals in $C_c(X,F)$ endowed with the well-known Hull-Kernel topology (also known as the structure space of $C_c(X,F)$) turns out to be homeomorphic to the Banaschewski Compactification $\beta_0 X$ of X [Theorem 2.18]. To achieve this result, we have to put certain restriction on the nature of the totally ordered field F viz. that F is either an uncountable field or a countable subfield of $\mathbb{R}$. A special case of this result choosing $F = \mathbb{R}$ reads: the structure space of the ring $C_C(X)$ consisting all real-valued continuous functions on X with countable range is $\beta_0 X$, which is Remark 3.6 in [**7**]. Since the maximal ideals of $C_c(X,F)$ can be indexed by virtue of the points of $\beta_0 X$, it is not surprising that a complete description of these ideals can be given by the family $\{M_c^{p,F} : p \in \beta_0 X\}$, where $M_c^{p,F} = \{f \in C_c(X,F) : p \in cl_{\beta_0 X} Z_c(f)\}$, here $Z_c(f) = \{x \in X : f(x) = 0\}$ stands for the zero set of f [Remark 2.21]. This is analogous to the Gelfand-Kolmogorov Theorem 7.3 [**12**]. Also, this places Theorem 4.8 [**11**] on a wider setting. As a natural companion of $M_c^{p,F}$, we introduce the ideal $O_c^{p,F} = \{f \in M_c^{p,F} : cl_{\beta_0 X} Z_c(f) \textit{ is a neighbourhood of } p \textit{ in } \beta_0 X\}$. Amongst other facts connecting these two classes of ideals in $C_c(X,F)$, we have realized that the ideals that lie between $O_c^{p,F}$ and $M_c^{p,F}$ are precisely those that extend to unique maximal ideals in $C_c(X,F)$ [Theorem 3.1(4)]. This may be called the modified countable counterpart of Theorem 7.13 in [**12**]. Also see Lemma 4.11 in [**7**] in this connection. If $C_K^c(X,F) = \{f \in C_c(X,F) : cl_X(X - Z_c(f)) \textit{ is compact}\}$, then we have found out a formula for this ring in terms of the ideals $O_c^{p,F}$ as follows : $C_K^c(X,F) = \bigcap_{p \in \beta_0 X \setminus X} O_c^{p,F}$ [in Theorem 3.5, compare with the Theorem 3.9, [**4**]]. This we may call the appropriate modified countable analogue of the well-known formula in $C(X)$ which says that $C_K(X) = \bigcap_{p \in \beta X \setminus X} O^p$ [$7E$, [**12**]]. The above-mentioned results constitute technical section 2 and 3 of the present article.

In the final section 4 of this article, we have examined several possible consequences of the hypothesis that $C_c(X,F)$ is a Von-Neumann regular ring with F, either an uncountable field or a countable subfield of $\mathbb{R}$. To aid to this examination, we introduce m_c^F-topology on $C_c(X,F)$ as a modified version of m_c-topology on $C_c(X)$ already introduced in [**6**]. We establish amongst a host of necessary and sufficient conditions that $C_c(X,F)$ is a Von-Neumann regular ring if and only if each ideal in $C_c(X,F)$ is closed in the m_c^F-topology if and only if X is a $P-$space. This places theorem 3.9 in [**6**] on a wider settings, and we may call it a modified countable analogue of the well-known fact that X is a $P-$space when and only when each ideal in $C(X)$ is closed in the $m-$topology [$7Q4$, [**12**]].

2. Duality between ideals in $C_c(X, F)$ and $Z_{F_c}-$filters on X

Notation 2.1. In spite of the difference of notations, we write for $f \in C_c(X, F)$, $Z_c(f) \equiv \{x \in X : f(x) = 0\} \equiv Z(f)$

Let $Z_c(X, F) = \{Z_c(f) : f \in C_c(X, F)\}$.
An ideal unmodified in a ring will always stand for a proper ideal.

Definition 2.2. A filter of zero sets in the family $Z_c(X, F)$ is called a $Z_{F_c}-$filter on X. A $Z_{F_c}-$filter on X is called a $Z_{F_c}-$ultrafilter on X if it is not properly contained in any $Z_{F_c}-$filter on X.

Remark 2.3. A straight forward use of Zorn's Lemma tells that a $Z_{F_c}-$filter on X extends to a $Z_{F_c}-$ultrafilter on X. Furthermore any subfamily of $Z_c(X, F)$ with finite intersection property can be extended to a $Z_{F_c}-$ultrafilter on X.

The following results correlating $Z_{F_c}-$filters on X and ideals in $C_c(X, F)$ can be established by using routine arguments.

Theorem 2.4.

(1) *If I is an ideal in $C_c(X, F)$, then $Z_{F,C}[I] = \{Z_c(f) : f \in I\}$ is a $Z_{F_c}-$filter on X. Dually for a $Z_{F_c}-$filter $\mathcal{F}$ on X, $Z^{-1}_{F,C}[\mathcal{F}] = \{f \in C_c(X, F) : Z_c(f) \in \mathcal{F}\}$ is an ideal in $C_c(X, F)$.*
(2) *If M is a maximal ideal in $C_c(X, F)$, then $Z_{F,C}[M]$ is a $Z_{F_c}-$ultrafilter on X. If $\mathcal{U}$ is a $Z_{F_c}-$ultrafilter on X, then $Z^{-1}_{F,C}[\mathcal{U}]$ is a maximal ideal in $C_c(X, F)$.*

Definition 2.5. An ideal I in $C_c(X, F)$ is called $Z_{F_c}-$ideal if $Z^{-1}_{F,C}[Z_{F,C}[I]] = I$

It follows from Theorem 2.4(2) that each maximal ideal in $C_c(X, F)$ is a $Z_{F_c}-$ideal. Hence the assignment : $M \to Z_{F,C}[M]$ establish a one-to-one correspondence between the maximal ideals in $C_c(X, F)$ and the $Z_{F_c}-$ultrafilters on X.
The following propositions can be easily established on using the arguments adopted in Chapter 2 and Chapter 4 of [**12**] in a straight forward manner.

Theorem 2.6. *A $Z_{F_c}-$ideal I in $C_c(X, F)$ is a prime ideal if and only if it contains a prime ideal. Hence each prime ideal in $C_c(X, F)$ extends to a unique maximal ideal, in other words, $C_c(X, F)$ is a Gelfand ring.*

Theorem 2.7. *The complete list of fixed maximal ideals in $C_c(X, F)$ is given by $\{M^c_{p,F} : p \in X\}$ where $M^c_{p,F} = \{f \in C_c(X, F) : f(p) = 0\}$. An ideal I in $C_c(X, F)$ is called fixed if $\bigcap_{f \in I} Z(f) \neq \phi$.*

Definition 2.8. X is called countably completely $F-$regular or in brief $CCFR$ space if it is Hausdorff and given a closed set K in X and a point $x \in X \backslash K$, there exists $f \in C_c(X, F)$ such that $f(x) = 0$ and $f(K) = 1$.

It is clear that a $CCFR$ space is CFR.
A CFR space with $F \neq \mathbb{R}$ is zero-dimensional by Theorem 2.3 in [**4**]. A $CCFR$ space with $F = \mathbb{R}$ is the same as $C-$completely regular space introduced in [**11**] and is hence zero-dimensional space by Proposition 4.4 in [**11**]. Thus for all choices of the field F, a $CCFR$ space is zero-dimensional. Conversely, it is easy to prove that a zero-dimensional space X is $CCFR$ for any totally ordered field F. Thus, the following result comes out immediately.

THEOREM 2.9. *The statements written below are equivalent for a Hausdorff space X and for any totally ordered field F :*
(1) *X is zero-dimensional.*
(2) *X is $CCFR$.*
(3) *$Z_c(X, F)$ is a base for closed sets in X.*

The following result tells that as in the classical situation with $F = \mathbb{R}$, in the study of the ring $C_c(X, F)$, one can assume without loss of generality that the ambient space X is $CCFR$, i.e., zero-dimensional.

THEOREM 2.10. *Let X be a topological space and F, a totally ordered field. Then it is possible to construct a zero-dimensional Hausdorff space Y such that the ring $C_c(X, F)$ is isomorphic to the ring $C_c(Y, F)$*

We need the following two subsidiary results to prove this theorem.

LEMMA 2.11. *A Hausdorff space X is zero-dimensional if and only if given any ordered field F, there exists a subfamily $\mathcal{S} \subset F_c^X = \{f \in F^X : f(X)$ is countable set$\}$, which determines the topology on X in the sense that, the given topology on X is the smallest one with respect to which each function in $\mathcal{S}$ is continuous.*

The proof of this lemma can be accomplished by closely following the arguments in Theorem 3.7 in [**12**] and using Theorem 2.9.

LEMMA 2.12. *Suppose X is a topological space whose topology is determined by a subfamily $\mathcal{S}$ of F_c^X. Then for a topological space Y, a function $h : Y \to X$ is continuous if and only if for each $g \in \mathcal{S}$, $g \circ h : Y \to F$ is a continuous map.*

The proof of the last lemma is analogous to that of Theorem 3.8 in [**12**].

PROOF. of the main theorem : Define a binary relation '$\sim$' on X as follows : for $x, y \in X$, $x \sim y$ if and only if for each $f \in C_c(X, F)$, $f(x) = f(y)$.
Suppose $Y = \{[x] : x \in X\}$, the set of all corresponding disjoint classes. Let $\tau : X \to Y$ be the canonical map given by $\tau(x) = [x]$. Each $f \in C_c(X, F)$ gives rise to a function $g_f : Y \to F$ as follows :

$g_f[x] = f(x)$.
Let $\mathcal{S} = \{g_f : f \in C_c(X,F)\}$. Then $\mathcal{S} \subset F_c^Y$. Equip Y with the smallest topology, which makes each function in $\mathcal{S}$ continuous. It follows from the Lemma 2.11, that Y is a zero-dimensional space and it is easy to check that Y is Hausdorff. The continuity of τ follows from Lemma 2.12. Now by the following arguments in Theorem 3.9 in [**12**], we can prove that the assignment : $C_c(Y,F) \to C_c(X,F) : g \to g \circ \tau$ is an isomorphism onto $C_c(X,F)$. □

The following result is a countable counterpart of a portion of Theorem 4.11 in [**12**].

THEOREM 2.13. *For a zero-dimensional Hausdorff space X and a totally ordered field F, the following three statements are equivalent :*

1. *X is compact.*
2. *Each ideal in $C_c(X,F)$ is fixed.*
3. *Each maximal ideal in $C_c(X,F)$ is fixed.*

PROOF. (1) $\implies$ (2) and (2) $\implies$ (3) are trivial. We prove (3) $\implies$ (1) : Let (3) be true.
Suppose $\mathcal{B}$ is a subfamily of $Z_c(X,F)$ with finite intersection property. Since $Z_c(X,F)$ is a base for the closed sets in X(vide Theorem 2.9), it suffices to show that $\bigcap \mathcal{B} \neq \phi$.
Indeed $\mathcal{B}$ can be extended to a $Z_{F_c}-$ultrafilter $\mathcal{U}$ on X. In view of Theorem 2.4, we can write $\mathcal{U} = Z_{F,C}[M]$ for a maximal ideal M in $C_c(X,F)$. Hence $\bigcap \mathcal{B} \supset \bigcap \mathcal{U} \neq \phi$. □

Before proceeding further, we reproduce below the following basic facts about the structure space of a commutative ring with unity from $7M$, [**12**].
Let A be a commutative ring with unity and $\mathcal{M}(A)$, the set of all maximal ideals in A. For each $a \in A$, let $\mathcal{M}_a = \{M \in \mathcal{M}(A) : a \in M\}$. Then the family $\{\mathcal{M}_a : a \in A\}$ constitutes a base for the closed sets of some topology τ on $\mathcal{M}(A)$. The topological space $(\mathcal{M}(A), \tau)$ is known as the structure space of A and is a compact T_1 space. If A is a Gelfand ring, then it is established in Theorem 1.2, [**13**] that τ is a Hausdorff topology on $\mathcal{M}(A)$. The closure of a subset $\mathcal{M}_0$ of $\mathcal{M}(A)$ is given by : $\overline{\mathcal{M}_0} = \{M \in \mathcal{M}(A) : M \supset \bigcap \mathcal{M}_0\} \equiv$ the hull of the kernel of $\mathcal{M}_0$. [This is the reason why τ is also called the hull-kernel topology on $\mathcal{M}(A)$].
Let us denote the structure space of the ring $C_c(X,F)$ by the notation $\mathcal{M}_c(X,F)$. Since $C_c(X,F)$ is a Gelfand ring, already verified in Theorem 2.6, it follows that $\mathcal{M}_c(X,F)$ is a compact Hausdorff space. From now on, we assume that X is Hausdorff and zero-dimensional, and we will stick to this hypothesis throughout this article. It follows that the assignment $\psi : X \to \mathcal{M}_c(X,F)$ given by $\psi(p) = M^c_{p,F}$ is one-to-one.

Furthermore for any $f \in C_c(X,F)$,

$$\psi(Z_c(f)) = \{M^c_{p,F} : f \in M^c_{p,F}\} = \mathcal{M}_f \cap \psi(X)$$

where $\mathcal{M}_f = \{M \in \mathcal{M}_c(X,F) : f \in M\}$.
This shows that ψ exchanges the basic closed sets of the two spaces X and $\psi(X)$. Finally,

$$\begin{aligned}\overline{\psi(X)} &= \{M \in \mathcal{M}_c(X,F) : M \supset \bigcap \psi(X)\} \\ &= \{M \in \mathcal{M}_c(X,F) : M \supset \bigcap_{p \in X} \{M^c_{p,F}\} = \{0\}\} \\ &= \mathcal{M}_c(X,F)\end{aligned}$$

This leads to the following proposition :

THEOREM 2.14. *The map $\psi : X \to \mathcal{M}_c(X,F)$ given by $\psi(p) = M^c_{p,F}$ defines a topological embedding of X onto a dense subspace of $\mathcal{M}_c(X,F)$. In a more formal language, the pair $(\psi, \mathcal{M}_c(X,F))$ is a Hausdorff Compactification of X.*

The next result shows that the last-mentioned compactification enjoys a special extension property.

THEOREM 2.15. *The compactification $(\psi, \mathcal{M}_c(X,F))$ enjoys the $C-$extension property (see Definition* 2.5 *in* [**6**]*) in the following sense, given a compact Hausdorff zero-dimensional space Y and a continuous map $f : X \to Y$, there can be defined a continuous map $f^c : \mathcal{M}_c(X,F) \to Y$ with the following property : $f^c \circ \psi = f$.*

PROOF. This can be accomplished by closely adapting the arguments made in the second paragraph in the proof of the Theorem 2.7 in [**6**]. However, to make the paper self-contained, we sketch a brief outline of the main points of its proof.
Let $M \in \mathcal{M}_c(X,F)$. Define as in [**6**], $\widetilde{M} = \{g \in C_c(Y,F) : g \circ f \in M\}$. Then $\widetilde{M}$ is a prime ideal in $C_c(Y,F)$. Since $C_c(Y,F)$ is Gelfand ring and Y is compact and zero-dimensional, it follows from Theorem 2.13 that there exists a unique $y \in Y$ such that $\bigcap_{g \in \widetilde{M}} Z_c(g) = \{y\}$. Set $f^c(M) = y$. Then $f^c : \mathcal{M}_c(X,F) \to Y$ is the desired continuous map. $\square$

REMARK 2.16. If the structure space $\mathcal{M}_c(X,F)$ of $C_c(X,F)$ is zero-dimensional, then $(\psi, \mathcal{M}_c(X,F))$ is topologically equivalent to the Banaschewski Compactification $\beta_0 X$ of X. [see the comments after Definition 2.5 in [**6**]].

We shall now impose a condition on F; sufficient to make $\mathcal{M}_c(X,F)$ zero-dimensional.

THEOREM 2.17. *Suppose the totally ordered field F is either uncountable or a countable subfield of $\mathbb{R}$. Then given $f \in C_c(X,F)$, there* *exists an idempotent* $e \in C_c(X, F)$ *such that* e *is a multiple of* f *and* $(1 - e)$ *is a multiple of* $(1 - f)$

PROOF. We prove this theorem with the assumption that F is uncountable. The proof for the case when F is a countable subfield of $\mathbb{R}$ can be accomplished on using some analogous arguments. We first assert that the interval $[0, 1] = \{\alpha \in F : 0 \leq \alpha \leq 1\}$ is an uncountable set. This is immediate if F is Archimedean ordered because in that case $F^+ = \{\alpha \in F : \alpha \geq 0\} = \bigcup_{n \in \mathbb{N} \cup \{0\}} [n, n + 1]$ and for each $n \in \mathbb{N} \cup \{0\}$, $[n, n+1]$ is equipotent with $[0, 1]$ through the translation map : $\alpha \to (\alpha+n)$, $\alpha \in [0, 1]$. Now suppose that F is non-Archimedean ordered field. If possible let $[0, 1]$ be a countable set. Then the set $F^+ \backslash \bigcup_{n \in \mathbb{N} \cup \{0\}} [n, n + 1]$ becomes an uncountable set, which means that the set of all infinitely large members of F make an uncountable set. Consequently, the set $I = \{\alpha \in F^+ : 0 < \alpha < \frac{1}{n}\ for\ each\ n \in \mathbb{N}\}$ comprising of the infinitely small members of F is an uncountable set. But it is easy to see that $I \subset (0, 1)$ and therefore $(0, 1)$ turns out to be an uncountable set – a contradiction. Thus it is proved that $[0, 1]$ is an uncountable set [and consequently for any $\alpha > 0$ in F, $(0, \alpha)$ becomes an uncountable set]. So we can choose $r \in (0, 1)$ such that $r \notin f(X)$. Let, $W = \{x \in X : f(x) < r\} = \{x \in X : f(x) \leq r\}$ and so $X \backslash W = \{x \in X : f(x) > r\} = \{x \in X : f(x) \geq r\}$. It is clear that W and $X \backslash W$ are clopen sets in X and the function $e : X \to F$ defined by $e(W) = \{0\}$ and $e(X \backslash W) = \{1\}$ is an idempotent in $C_c(X, F)$. We see that $Z_c(f) \subset Z_c(e)$ and $Z_c(1 - f) \subset Z_c(1 - e)$ and we can say that $Z_c(e)$ is a neighbourhood of $Z_c(f)$ and $Z_c(1 - e)$ is a neighbourhood of $Z_c(1 - f)$ in the space X. Hence e is a multiple of f and $(1 - e)$ is a multiple of $(1 - f)$. [compare with the arguments made in Remark 3.6 in [**7**]]. □

THEOREM 2.18. *The structure space* $\mathcal{M}_c(X, F)$ *of* $C_c(X, F)$ *is zero-dimensional and hence* $\mathcal{M}_c(X, F) = \beta_0 X$.

[Here F is either uncountable or a countable subfield of $\mathbb{R}$]

PROOF. Recall the notation for $f \in C_c(X, F)$, $\mathcal{M}_f = \{M \in \mathcal{M}_c(X, F) : f \in M\}$. Suppose $M \in \mathcal{M}_c(X, F)$ is such that $M \notin \mathcal{M}_f$. It suffices to find out an idempotent e in $C_c(X, F)$ with the property : $\mathcal{M}_f \subset \mathcal{M}_e$ and $M \notin \mathcal{M}_e$. The simple reason is that $e.(1-e) = e - e^2 = e - e = 0$ and hence $\mathcal{M}_e = \mathcal{M}_c(X, F) \backslash \mathcal{M}_{(1-e)}$, consequently $\mathcal{M}_e$ is a clopen set in $\mathcal{M}_c(X, F)$. Now towards finding out such an idempotent let us observe that $M \notin \mathcal{M}_f$ implies that $f \notin M$, which further implies that $< f, M >= C_c(X, F)$. Hence we can write : $1 = f.h + g$, where $h \in C_c(X, F)$ and $g \in M$. By Theorem 2.17, there exists an idempotent e in $C_c(X, F)$ such that $e = g_1.g$ and $(1 - e) = g_2.(1 - g)$, where $g_1, g_2 \in C_c(X, F)$. Now let $N \in \mathcal{M}_f$, then $f \in N$ and so $f.h \in N$, which implies that $(1 - g) \in N$ consequently $(1 - e) \in N$. Therefore

$e \notin N$, which means that $N \notin \mathcal{M}_e$, i.e., $N \in \mathcal{M}_c(X,F)\backslash\mathcal{M}_e$. Again since $g \in M$, it follows that $e \in M$, thus $M \in \mathcal{M}_e$. □

REMARK 2.19. On choosing $F = \mathbb{R}$ and $X = \mathbb{Q}$ in the above Theorem 2.18, we get that $\beta_0\mathbb{Q}$ = structure space of $C(\mathbb{Q},\mathbb{R}) = \beta\mathbb{Q}$. Thus $\beta\mathbb{Q}$ becomes zero-dimensional, i.e., $\mathbb{Q}$ is strongly zero-dimensional. This is a standard result in General Topology – indeed a Lindelöf zero-dimensional space is strongly zero-dimensional. [Theorem 6.2.7, [**10**]].

One of the major achievements in the theory of $C(X)$ is that a complete description of the maximal ideals in this ring can be given. This is a remark made in the beginning of Chapter 6 in [**12**]. In order to give such a description, it becomes convenient to archive βX as the space of $Z-$ultrafilter on X equipped with the Stone- topology and formal construction of such a thing is dealt in rigorously in Chapter 6 in [**12**]. We follow the same technique in order to furnish an explicit description of maximal ideals in $C_c(X,F)$.
For each $p \in X$, let $A^c_{p,F} = \{Z \in Z_c(X,F) : p \in Z\} \equiv Z_{F,C}[M^c_{p,F}]$. Thus X is a readymade index set for the family of fixed $Z_{F_c}-$ultrafilters on X. As in Chapter 6, [**12**], we extend the set X to a set αX to serve as an index set for the family of all $Z_{F_c}-$ultrafilters on X. For $p \in \alpha X$, let the corresponding $Z_{F_c}-$ultrafilter be designated as $A^{p,F}_c$ with the understanding that if $p \in X$, then $A^{p,F}_c = A^c_{p,F}$.
For $Z \in Z_c(X,F)$, let $\overline{Z} = \{p \in \alpha X : Z \in A^{p,F}_c\}$. Then $\{\overline{Z} : z \in Z_c(X,F)\}$ makes a base for the closed sets of some topology on αX in which for $Z \in Z_c(X,F)$, $\overline{Z} = cl_{\alpha X}Z$. Furthermore, for $Z_1, Z_2 \in Z_c(X,F)$, $\overline{Z_1 \cap Z_2} = \overline{Z_1} \cap \overline{Z_2}$ and αX becomes a compact Hausdorff space containing X as a dense subset. Also given a point $p \in \alpha X$, $A^{p,F}_c$ is the unique $Z_{F_c}-$ultrafilter on X which converges to p and finally αX possesses the $C-$extension property meaning that if Y is a compact Hausdorff zero-dimensional space and $f : X \to Y$, a continuous map, then f can be extended to a continuous map $f^{\#} : \alpha X \to Y$. All these facts can be realized just by closely following the arguments in Chapter 6 in [**12**].

THEOREM 2.20. *The space αX is a zero-dimensional space.*
[Blanket assumption: F is either an uncountable field or a countable subfield of $\mathbb{R}$]

PROOF. Let $p \in \alpha X$ and $Z \in Z_c(X,F)$ be such that $p \notin \overline{Z}$.
It suffices to find out a clopen set K in αX such that $\overline{Z} \subset K$ and $p \notin K$.
Now $p \notin \overline{Z} \implies Z \notin A^{p,F}_c$. Since $A^{p,F}_c$ is a $Z_{F_c}-$ultrafilter, this implies that there exists $Z^* \in A^{p,F}_c$ such that $Z \cap Z^* = \phi$. Hence there exists $f \in C_c(X,F)$ such that $f : X \to [0,1]$ in F such that $f(Z^*) = \{0\}$ and $f(Z) = \{1\}$. Using the hypothesis that F is an uncountable field, and take note of the arguments in the proof of the Theorem 2.17, we can

find out an $r \in (0,1)$ in F such that $r \notin f(X)$ [analogous arguments can be made if F is a countable subfield of $\mathbb{R}$].
Let $K = \{x \in X : f(x) > r\} = \{x \in X : f(x) \geq r\}$. Then K is a clopen set in X containing Z and therefore $\overline{Z} \subset K$. Now $Z^* \subset X \backslash K$ implies $Z^* \cap K = \phi$ and hence $\overline{Z^*} \cap \overline{K} = \phi$, i.e., $\overline{Z^*} \cap K = \phi$. Since $Z^* \in A_c^{p,F}$ and therefore $p \in Z^*$, this further implies that $p \notin K$. □

REMARK 2.21. Since αX enjoys the $C-$extension property and is zero-dimensional, it follows from Definition 2.5 in [**6**] that αX is essentially the same as $\beta_0 X$, the Banaschewski Compactification of X and hence we can write for any $p \in \beta_0 X$ and $Z \in Z_c(X,F)$, $Z \in A_c^{p,F}$ if and only if $p \in cl_{\beta_0 X}$. If we now write $M_c^{p,F} = Z_{F,C}^{-1}[A_c^{p,F}]$, then this becomes a maximal ideal in $C_c(X,F)$. Since by Theorem 2.4(2), there is already realized a one-to-one correspondence between maximal ideals in $C_c(X,F)$ and $Z_{F_c}-$ultrafilters on X via the map $M \to Z_{F,C}[M]$, a complete description of the maximal ideals in $C_c(X,F)$ is given by the list $\{M_c^{p,F} : p \in \beta_0 X\}$ where $M_c^{p,F} = \{f \in c_c(X,F) : p \in cl_{\beta_0 X} Z_c(f)\}$

3. The ideals $O_c^{p,F}$ and a formula for $C_K^c(X,F)$

For each $p \in \beta_0 X$, set

$$O_c^{p,F} = \{f \in C_c(X,F) : cl_{\beta_0 X} Z_c(f) \textit{ is a neighbourhood of } p \textit{ in } \beta_0 X\}$$

Then the following facts come out as modified countable analogue of the relations between the ideals M^p and O^p in the classical scenario recorded in 7.12, 7.13, 7.15 in [**12**]. Also see Lemma 4.11 in [**7**] in this connection.

THEOREM 3.1. *Let the ordered field F be either uncountable or a countable subfield of $\mathbb{R}$. Then for a zero-dimensional Hausdorff space X, the following statements are true :*

(1) $O_c^{p,F}$ *is a $Z_{F_c}-$ideal in $C_c(X,F)$ contained in $M_c^{p,F}$.*
(2) $O_c^{p,F} = \{f \in C_c(X,F) :$ *there exists an open neighbourhood V of p in $\beta_0 X$ such that* $Z_c(f) \supset V \cap X\}$.
(3) *For $p \in \beta_0 X$ and $f \in C_c(X,F)$, $f \in O_c^{p,F}$ if and only if there exists $g \in C_c(X,F) \backslash M_c^{p,F}$ such that $f.g = 0$, hence each non-zero element in $O_c^{p,F}$ is a divisor of zero in $C_c(X,F)$. Indeed $O_c^{p,F}$ is a z^o-ideal in $C_c(X,F)$.*
(4) *An ideal I in $C_c(X,F)$ is extendable to a unique maximal ideal if and only if there exists $p \in \beta_0 X$ such that $O_c^{p,F} \subset I$.*
(5) *For $p \in \beta_0 X$, $O_c^{p,F}$ is a fixed ideal if and only if $p \in X$.*

PROOF. The statements (1), (2) and (4) can be proved by making arguments parallel to those adopted to prove the corresponding results in the classical situation with $F = \mathbb{R}$ in Sections 7.12, 7.13, 7.15 in [**12**]. We prove only the statements (3) and (5).
To prove (3), let $f \in O_c^{p,F}$. Then by (2), there exists an open neighbourhood V of p in $\beta_0 X$ such that $Z_c(f) \supset V \cap X$. Since $\beta_0 X$ is zero

dimensional, there exists a clopen set K in $\beta_0 X$ such that $\beta_0 X\backslash V \subset K$ and $p \notin K$. The function $h : \beta_0 X \to F$, defined by $h(K) = \{0\}$ and $h(\beta_0 X\backslash K) = \{1\}$ belongs to $C_c(\beta_0 X, F)$. Take $g = h|_X$. Then $g \in C_c(X, F)$, $f.g = 0$ and $p \notin cl_{\beta_0 X} Z_c(g)$, hence $g \notin M_c^{p,F}$.
Conversely let there exist $g \in C_c(X, F)\backslash M_c^{p,F}$ such that $f.g = 0$. Then $p \notin cl_{\beta_0 X} Z_c(g)$. Therefore there exists an open neighbourhood V of p in $\beta_0 X$ such that $V \cap Z_c(g) = \phi$. Since $Z_c(f) \cup Z_c(g) = X$, it follows that $X \cap V \subset Z_c(f)$. Hence from (2), we get that $f \in O_c^{p,F}$.
To prove the last part of (3), we recall that an ideal I in a commutative ring A with unity is called a z^o-ideal if for each $a \in I$, $P_a \subset I$, where P_a is the intersection of all minimal prime ideals in A containing a. We reproduce the following useful formula from Proposition 1.5 in [**8**], which is also recorded in Theorem 3.10 in [**6**] : if A is a reduced ring meaning that 0 is the only nilpotent member of A, then $P_a = \{b \in A : Ann(a) \subset Ann(b)\}$, where $Ann(a) = \{c \in A : a.c = 0\}$ is the annihilator of a in A. Hence for any $f \in C_c(X, F)$, $P_f \equiv$ the intersection of all minimal prime ideals in $C_c(X, F)$ which contain $f = \{g \in C_c(X, F) : Ann(f) \subset Ann(g)\}$.
Now to show that $O_c^{p,F}$ is a z^o-ideal in $C_c(X, F)$, for any $p \in \beta_0 X$, choose $f \in O_c^{p,F}$ and $g \in P_f$. Therefore $Ann(f) \subset Ann(g)$. But from the result (3), we see that there exists $h \in C_c(X, F)\backslash M_c^{p,F}$ such that $f.h = 0$ and hence $h \in Ann(f)$. Consequently, $h \in Ann(g)$, i.e., $g.h = 0$. Thus $P_f \subset O_c^{p,F}$ and hence $O_c^{p,F}$ is a z^o-ideal in $C_c(X, F)$.
Proof of (5) : If $p \in X$, then $M_c^{p,F} = M_{p,F}^c$, a fixed ideal, hence $O_c^{p,F}$ is also fixed.
Now let $p \in \beta_0 X\backslash X$. Choose $x \in X$ and a closed neighbourhood W of p in $\beta_0 X$ such that $x \notin W$. Since $\beta_0 X$ is zero-dimensional, there exists a clopen set K in $\beta_0 X$ such that $W \subset K$ and $x \notin K$. Let $g : \beta_0 X \to F$ be defined by $g(K) = \{0\}$ and $g(\beta_0 X\backslash K) = \{1\}$. Then $g \in C_c(\beta_0 X, F)$ and hence $h = g|_X \in C_c(X, F)$. We observe that $h(x) = 1$ and $Z_c(h) \supset K \cap X$. It follows from the result (2) that $h \in O_c^{p,F}$. This proves that $O_c^{p,F}$ is a free ideal in $C_c(X, F)$ □

The following properties of $C_K^c(X, F) = \{f \in C_c(X, F) : f$ *has compact support i.e.* $cl_X(X\backslash Z_c(f))$ *is compact*$\}$ can be established as parallel to the analogous properties of the ring $C_K(X) = \{f \in C(X) :$ *f has compact support*$\}$ given in $4D$, [**12**].

THEOREM 3.2. *Let X be Hausdorff and zero-dimensional. Then :*

(1) $C_K^c(X, F) \subset C_c(X, F) \cap C^*(X, F)$, *where* $C^*(X, F) = \{f \in C(X, F) : cl_F f(X)$ *is compact*$\}$ *and equality holds if and only if X is compact.*
(2) *If X is non-compact, then* $C_K^c(X, F)$ *is an ideal(proper) of* $C_c(X, F)$.

(3) *$C^c_K(X,F)$ is contained in every free ideal of $C_c(X,F)$. $C^c_K(X,F)$ itself is a free ideal of $C_c(X,F)$ if and only if X is non-compact and locally compact.*
(4) *X is nowhere locally compact if and only if $C^c_K(X,F)=\{0\}$ and this is the case when and only when $\beta_0X\backslash X$ is dense in β_0X. [Compare with 7F4, [**12**]]*

REMARK 3.3. $C^c_K(X,F)\subset\bigcap\{O^{p,F}_c : p\in\beta_0X\backslash X\}$.
This follows from Theorem 3.1(5) and Theorem 3.2(3).

To show that equality holds in the last inclusion relation, we need the following subsidiary result.

THEOREM 3.4. *Let $f\in C_c(X,F)$ be such that $cl_{\beta_0X}Z_c(f)$ is a neighbourhood of $\beta_0X\backslash X$. Then $f\in C^c_K(X,F)$.*

PROOF. It suffices to show that $supp(f)\equiv cl_X(X\backslash Z_c(f))$ is closed in β_0X and hence compact. As $Z_c(f)$ is closed in X, it follows that $cl_{\beta_0X}Z_c(f)\cap(X\backslash Z_c(f))=\phi$. The hypothesis tells that there exists an open set W in β_0X such that $\beta_0X\backslash X\subset W\subset cl_{\beta_0X}Z_c(f)$. Hence $W\cap(X\backslash Z_c(f))=\phi$, which further implies because W is open in β_0X that $W\cap cl_{\beta_0X}(X\backslash Z_c(f))=\phi$. Consequently $W\cap cl_X(X\backslash Z_c(f))=\phi$. Since $\beta_0X\backslash X\subset W$, it follows therefore that no point of $\beta_0X\backslash X$ is a limit point of $cl_X(X\backslash Z_c(f))$ in the space β_0X. Thus there does not exist any limiting point of $cl_X(X\backslash Z_c(f))$ outside it in the entire space β_0X. Hence $cl_X(X\backslash Z_c(f))$ is closed in β_0X. □

THEOREM 3.5. *Let X be zero-dimension and Hausdorff. Then $C^c_K(X,F)=\bigcap\{O^{p,F}_c : p\in\beta_0X\backslash X\}$*

PROOF. Let $f\in O^{p,F}_c$ for each $p\in\beta_0X\backslash X$. Then $cl_{\beta_0X}Z_c(f)$ is a neighbourhood of each point of $\beta_0X\backslash X$ in the space β_0X. It follows from Theorem 3.4 that $f\in C^c_K(X,F)$. Thus $\bigcap\{O^{p,F}_c : p\in\beta_0X\backslash X\}\subset C^c_K(X,F)$. The reversed implication relation is already realized in Remark 3.3. Hence $C^c_K(X,F)=\bigcap\{O^{p,F}_c : p\in\beta_0X\backslash X\}$. □

4. Von Neumann regularity of $C_c(X,F)$ versus $P-$space X

We recall from [**3**] that X is called P_F-space if $C(X,F)$ is a Von-Neumann regular ring. By borrowing the terminology from [**11**], we call a zero-dimensional space X, a countably P_F-space or CP_F-space if $C_c(X,F)$ is Von-Neumann regular ring. Thus in this terminology, $CP_{\mathbb{R}}-$spaces are precisely $CP-$spaces introduced in [**11**], Definition 5.1. It is still undecided whether there exist an ordered field F and a zero-dimensional space X for which X is a P_F-space without being a $P-$space (see the comments preceding Definition 3.3 in [**3**]). However, we shall prove that subject to the restrictions imposed on the field F, already used several times in this paper, CP_F-spaces and $P-$spaces are one and the same. We want to mention in this context that the zero set of a function f in $C(X, F)$ may not be a $G_\delta-$set [see Theorem 2.2 in [**3**]]. In contrast, we shall show that the zero set of a function lying in $C_c(X, F)$ is necessarily a $G_\delta-$set. Before proceeding further, we make the assumption throughout the rest of this article that the ordered field F is either uncountable or a countable subfield of $\mathbb{R}$.

THEOREM 4.1. *A zero set $Z \in Z_c(X, F)$ is a $G_\delta-$set.*

PROOF. We can write $Z = Z_c(f)$ for some $f \geq 0$ in $C_c(X, F)$. Since $f(X)$ is a countable subset of F, we can write, $f(X)\setminus\{0\} = \{r_1, r_2, ..., r_n, ...\}$; a countable set in F^+. It follows that $Z_c(f) = \bigcap_{n=1}^{\infty} f^{-1}(-r_n, r_n) =$ a $G_\delta-$set in X. □

The following results are generalized versions of Proposition 4.3, Theorem 5.5 and Corollary 5.7 in [**11**].

THEOREM 4.2. *If A and B are disjoint closed sets in X with A, compact, then there exists $f \in C_c(X, F)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$*

THEOREM 4.3. *For $f \in C_c(X, F)$, $Z_c(f)$ is a countable intersection of clopen sets in X*

PROOF. As in the proof of Theorem 4.1, we can assume $f \geq 0$ and $f(X)\setminus\{0\} = \{r_1, r_2, ..., r_n, ...\}$. Now if F is an uncountable ordered field, then for each $n \in \mathbb{N}$, we can choose $s_n \in F$ such that $0 < s_n < r_n$ and $s_n \notin \{r_1, r_2, ..., r_n, ...\}$. On the other hand, if F is a countable subfield of $\mathbb{R}$, then we can pick up for each $n \in \mathbb{N}$ an irrational point denoted by the same symbol s_n with the above-mentioned condition, i.e., $0 < s_n < r_n$ and $s_n \notin \{r_1, r_2, ..., r_n, ...\}$. It follows that $Z_c(f) = \bigcap_{n=1}^{\infty} f^{-1}(-s_n, s_n) = \bigcap_{n=1}^{\infty} f^{-1}[-s_n, s_n] =$ a countable intersection of clopen sets in X. □

THEOREM 4.4. *A countable intersection of clopen sets in X is a zero set in $Z_c(X, F)$ (equivalently, a countable union of clopen sets in X is a co-zero set, i.e., the complement in X of a zero set in $Z_c(X, F)$).*

PROOF. Since in any topological space, a countable union of clopen sets can be expressed as a countable union of pairwise disjoint clopen sets, we can start with a countable family $\{L_i\}_{i=1}^{\infty}$ of pairwise disjoint clopen sets in X. For each $n \in \mathbb{N}$, define a function $e_n : X \to F$ as follows : $e_n(L_n) = \{1\}$ and $e_n(X\setminus L_n) = \{0\}$. Then $e_n \in C_c(X, F)$ and is an idempotent in this ring. Furthermore, it is easy to see that if $m \neq n$, then $e_m.e_n = 0$. Let $h(x) = \sum_{n=1}^{\infty} \frac{e_n(x)}{3^n}$, $x \in X$. Then $h : X \to F$ is a continuous function and $h(X) \subset \{0, \frac{1}{3}, \frac{1}{3^2}, ...\}$. Thus $h \in C_c(X, F)$. It is clear that $\bigcup_{n=1}^{\infty} L_n = X\setminus Z_c(h)$. □

THEOREM 4.5. *$Z_c(X, F)$ is closed under countable intersection.*

PROOF. Follows from Theorem 4.3 and Theorem 4.4. □

THEOREM 4.6. *Suppose a compact set K in X is contained in a $G_\delta-$set G. Then there exists a zero set Z in $Z_c(X,F)$ such that $K \subset Z \subset G$.*

PROOF. We can write $G = \bigcap_{n=1}^{\infty} W_n$ where each W_n is open in X. For each $n \in \mathbb{N}$, K and $X \backslash W_n$ are disjoint closed sets in X with K compact. Hence by Theorem 4.2, there exists $g_n \in C_c(X,F)$ such that $g_n(K) = \{0\}$ and $g_n(X\backslash W_n) = \{1\}$. It follows that $K \subset Z_c(g_n) \subset W_n$ for each $n \in \mathbb{N}$. Consequently, $K \subset \bigcap_{n=1}^{\infty} Z_c(g_n) \subset G$. But by Theorem 4.5, we can write $\bigcap_{n=1}^{\infty} Z_c(g_n) = Z_c(g)$ for some $g \in C_c(X,F)$. Hence $K \subset Z_c(g) \subset G$. □

Before giving several equivalent descriptions of the defining property of CP_F-space in the manner $4J$ of [**12**] and the theorem 5.8 in [**11**]. We like to introduce a suitable modified countable version of $m-$topology on $C(X)$ as dealt with in $2N$, [**12**].
For each $g \in C_c(X,F)$ and a positive unit u in this ring, set $M_F(g,u) = \{f \in C_c(X,F) : |f(x) - g(x)| < u(x)\ for\ each\ x \in X\}$. Then it can be proved by routine computation that $\{M_F(g,u) : g \in C_c(X,F),\ u,\ a$ *positive unit in* $C_c(X,F)\}$ is an open set for some topology on $C_c(X,F)$, which we call m_c^F-topology on $C_c(X,F)$. A special case of this topology with $F = \mathbb{R}$ is already considered in [**6**], Section 3. The following two can be established by making straight forward modifications in the arguments adopted to prove Theorem 3.1 and Theorem 3.7 in [**6**].

THEOREM 4.7. *Each maximal ideal in $C_c(X,F)$ is closed in the m_c^F-topology.*

THEOREM 4.8. *For any ideal I in $C_c(X,F)$, its closure in m_c^F- topology is given by : $\overline{I} = \bigcap\{M_c^{p,F} : p \in \beta_0 X$ and $M_c^{p,F} \supset I\} \equiv$ the intersection of all the maximal ideals in $C_c(X,F)$ which contains I.*

[compare with $7Q2$, [**12**]]

THEOREM 4.9. *An ideal I in $C_c(X,F)$ is closed in m_c^F-topology if and only if it is the intersection of all the maximal ideals in this ring which contains I.*

[This follows immediately from Theorem 4.8]
We are now ready to offer a bunch of statements, each equivalent to the requirement that X is a CP_F-space.

THEOREM 4.10. *Let X be a zero-dimensional Hausdorff space and F, a totally ordered field with the property mentioned in the beginning of this section. Then the following statements are equivalent :*

(1) *X is a CP_F-space.*
(2) *Each zero set in $Z_c(X,F)$ is open.*
(3) *Each ideal in $C_c(X,F)$ is a $Z_{F_c}-$ideal.*
(4) *For all f,g in $C_c(X,F)$, $< f,g > = < f^2 + g^2 >$.*
(5) *Each prime ideal in $C_c(X,F)$ is maximal.*

(6) *For each* $p \in X$, $M^c_{p,F} = O^c_{p,F}$.
(7) *For each* $p \in \beta_0 X$, $M^{p,F}_c = O^{p,F}_c$.
(8) *Each ideal in* $C_c(X,F)$ *is the intersection of all the maximal ideals containing it.*
(9) *Each* $G_\delta-$*set in* X *is open (which eventually tells that* X *is a* $P-$*space)*
(10) *Every ideal in* $C_c(X,F)$ *is closed in the* m^F_c-*topology.*

Proof. Equivalence of the first eight statements can be proved by making an almost repetition of the arguments to prove the equivalence of the analogous statements in $4J$, [**12**] [Also see the Theorem 5.8 in [**11**]]. We prove the equivalence of the statements $(2),(9),(10)$.
$(9) \implies (2)$ is immediate because of Theorem 4.1.
$(2) \implies (9)$: Let (2) be true and G be a non-empty $G_\delta-$set in X. Then by Theorem 4.6, for each point $x \in G$, there exists a zero set $Z_x \in Z_c(X,F)$ such that $x \in Z_x \subset G$. Since Z_x is open in X by (2), it follows that x is an interior point of G. In other words, G is open in X.
Equivalence of (8) and (10) follows from Theorem 4.9. □

Remark 4.11. On choosing $F = \mathbb{Q}$ in Theorem 4.10, we get that a zero dimensional space X is a $P-$space if and only if $C(X,\mathbb{Q})$ is a Von Neumann regular ring, i.e., X is a $P_{\mathbb{Q}}-$space. Thus each $P_{\mathbb{Q}}-$space is a $P-$space. But we note that, though the cofinality character of $\mathbb{Q}$ is ω_0, it is not Cauchy complete. This improves the conclusion of the Theorem 3.5 in [**3**], which says that if F is a Cauchy complete totally ordered field with cofinality character ω_0, then every P_F-space is a $P-$space.

Open question : If $p \in \beta_0 X$, then does the set of prime ideals in $C_c(X,F)$ that lie between $O^{p,F}_c$ and $M^{p,F}_c$ make a chain?

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