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Optimum covariate designs in a binary proper equi-replicate block design set-up

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ABSTRACT

The choice of covariates values for a given block design attaining minimum variance for estimation of each of the regression parameters of the model has attracted attention in recent times. In this article, we consider the problem of finding the optimum covariate design (OCD) for the estimation of covariate parameters in a binary proper equi-replicate block (BPEB) design model with covariates, which cover a large class of designs in common use. The construction of optimum designs is based mainly on Hadamard matrices.

1. Introduction

Consider the following model with non-stochastic controllable covariates in a one-way classification set-up:

$$(\mathbf{Y}, \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}) \tag{1.1}$$

where $\mathbf{Y}^{n\times 1}$ is the response vector, $\mathbf{X}$ is the design matrix corresponding to the treatment effect vector $\boldsymbol{\tau}^{v\times 1}$ and $\mathbf{Z}$ is the design matrix corresponding to the covariate effect (also called the regression coefficient) vector $\boldsymbol{\gamma}^{c\times 1}$. When the covariates are not under the control of the experimenter, the choice of design has been considered by Harville [15,16], Haggstrom [14] and Wu [31]. However, in many situations, the covariates can be controlled by the experimenter. The problem of choice of covariate values for the estimation of the parameters of the model was considered by Lopes Troya [20,21] using a D-optimality criterion. Subsequently, many authors contributed in this area, namely [19,28–30,5,3,4,8]. We are considering the situation where the covariates are controllable.

Consider the block design set-up which can be written as

$$(\mathbf{Y}, \mu\mathbf{1} + \mathbf{X}_1\boldsymbol{\beta} + \mathbf{X}_2\boldsymbol{\tau} + \mathbf{Z}\boldsymbol{\gamma}, \sigma^2\mathbf{I}) \tag{1.2}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\tau}$ represent the vectors of block and treatment effects respectively and $\mathbf{X}_1^{n\times b}$ and $\mathbf{X}_2^{n\times v}$ are the corresponding incidence matrices. We consider the problem of estimation of the covariate parameters in $\boldsymbol{\gamma}$ optimally for the given block

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design set-up. For the covariates, we assume without loss of generality, the (location-scale-)transformed version, where $|z_{ij}| \le 1$, z_{ij} the (i,j)th element of $\mathbf{Z}$. By appropriate selection of the values of the covariates z_{ij}, one can optimise the estimation of the parameters in $\boldsymbol{\gamma}$ while maintaining the properties of the design with regard to the treatment and block. We consider how to select the values of z_{ij} optimally. Optimality here refers to attaining the least possible value $\frac{\sigma^2}{n}$ of individual variances simultaneously for all the estimators of the parameters in $\boldsymbol{\gamma}$. Such a design is termed in the literature *globally optimal* (see e.g. [27, page 143]). From now on, unless otherwise stated, by optimality we mean *global optimality*. Das et al. [7] studied the construction of *optimum covariate designs* (OCDs) extensively for the first time in the above set-up. They exploited mutually orthogonal Latin squares (MOLS) and Hadamard matrices to construct such designs and obtained an upper bound for the maximum number of covariates that could be accommodated in the set-up of randomised block design (RBD) and some series of balanced incomplete block design (BIBD). Rao et al. [26] revisited the problem in completely randomized design (CRD) and RBD set-ups and identified the solution as a mixed orthogonal array (MOA), thereby providing some new solutions. Moreover, for OCDs in BIBD and partially balanced incomplete design (PBIBD) set-ups, references can be made to [7,9,10,12]. Dutta et al. [11] also considered the problem of finding OCDs in the correlated set-up of split-plot and strip-plot designs. The optimum choice of covariate values in the above set-up for the estimation of covariate parameters using the D-optimality criterion has also been considered by Dutta et al. [13].

With reference to the model (1.2), it is evident that for the estimation of the covariate effects orthogonal to the treatment and block effect contrasts, we must have (see [23])

$$\mathbf{Z}'\mathbf{X}_1 = \mathbf{0}, \qquad \mathbf{Z}'\mathbf{X}_2 = \mathbf{0} \tag{1.3}$$

and for most efficient estimation of the regression parameters, we must have

$$\mathbf{Z}'\mathbf{Z} = n\mathbf{I}_c. \tag{1.4}$$

Therefore a most efficient estimation of each of the regression parameters is possible if and only if $\mathbf{Z}$ satisfies the conditions (1.3) and (1.4) simultaneously. In this article, our aim is to find $\mathbf{Z}$ which is globally optimal for given $\mathbf{X}_1$ and $\mathbf{X}_2$.

Let $z_{rs}^{(i)}$ be the element of $\mathbf{Z}_i$ corresponding to the observation on the rth treatment in the sth block, where $\mathbf{Z}_i$ is the ith column of $\mathbf{Z}$, $i = 1, 2, \ldots, c$, $r = 1, 2, \ldots, v$, $s = 1, 2, \ldots, b$. There are different ways of finding the OCDs. In the present set-up, it is very difficult to visualise the $\mathbf{Z}$-matrix satisfying the conditions (1.3) and (1.4). Following Das et al. [7], we recast each column of the $\mathbf{Z}$-matrix to a $\mathbf{W}^{v\times b}$-matrix. We construct a matrix $\mathbf{W}_i$ corresponding to $\mathbf{Z}_i$, putting the element $z_{rs}^{(i)}$ in the rth row and sth column of $\mathbf{W}_i$. Therefore a covariate design $\mathbf{Z}$ for c covariates is equivalent to c $\mathbf{W}$-matrices. At this point, it may be mentioned that one can use a mixed orthogonal array (MOA) to construct such OCDs and this has been successfully done by [26,11]. However, in the set-up of BPEB design, we find OCDs using $\mathbf{W}$-matrices which are convenient to work with.

The conditions for global optimality for the estimation of $\boldsymbol{\gamma}$ in (1.3) and (1.4) reduce, in terms of the elements of the $\mathbf{W}$-matrices, to:

($\mathbf{C}_1$) each $\mathbf{W}$-matrix has all column-sums and all row-sums equal to zero;
($\mathbf{C}_2$) the grand total of all the entries in the Hadamard product (see [25, page 30]) of any two distinct $\mathbf{W}$-matrices is zero.

Henceforth, the conditions $\mathbf{C}_1$–$\mathbf{C}_2$ will be referred to as the single condition $\mathbf{C}$.

Definition 1.1. With respect to the model (1.2), the $c\mathbf{W}$-matrices corresponding to the c covariates are said to be optimum (in the sense of global optimality) if they satisfy the condition $\mathbf{C}$.

Remark 1.1. It is to be noted that if $c = 1$, only condition $\mathbf{C}_1$ is to be satisfied by the $\mathbf{W}$-matrix to be optimum.

In this context, it may be mentioned that the estimation of $\boldsymbol{\gamma}$ comes from the error functions of the model $(\mathbf{Y}, \mu\mathbf{1} + \mathbf{X}_1\boldsymbol{\beta} + \mathbf{X}_2\boldsymbol{\tau}, \sigma^2\,\mathbf{I})$ and hence the maximum number of covariates cannot exceed the error degrees of freedom of a given set-up.

In block design set-up, our aim is to construct as many $\mathbf{W}$-matrices as possible satisfying the condition $\mathbf{C}$. There are two reasons behind this. Firstly, it is known that the experimental error decreases with increase in the number of covariates. Secondly, the experimenter should have an idea about the maximum number of covariates, say c^*, that can be used satisfying the condition $\mathbf{C}$. Otherwise, if the experimenter uses more than the c^* covariates available to the experimenter, he or she will fail to achieve maximum accuracy in the estimation of the regression coefficients attached to the covariates in the model.

In our approach, optimum $\mathbf{W}$-matrices are being constructed from the incidence matrices corresponding of the incomplete block design set-ups, by placing ± 1's judiciously in the k non-zero positions of every column and in r non-zero positions of every row such that the $\mathbf{W}$-matrices satisfy the condition $\mathbf{C}$ mentioned above.

The paper is organised as follows. In Section 2 of this article, we have considered the construction of OCDs in the class of *binary proper equi-replicate block* (BPEB) designs where the number of blocks is a multiple of the number of treatments. This covers large classes of BIB, PBIB and cyclic designs among a host of others. In Section 3, a general method of construction of OCDs for the larger class of cyclic designs has been proposed. Finally, in the appendix, lists of BPEB designs covering BIBD and PBIBD where OCDs can be constructed are given.

2. BPEB designs with $b = mv$

The construction of OCDs for any arbitrary block design is rather difficult. The procedures depend heavily on the method of construction of the corresponding block designs and often optimum **W**-matrices are obtained for designs which are mainly constructed through the method of differences (cf. [2]). But now we shall describe a technique which does not depend on the method of construction and hence can be widely applied to a large class of commonly used block designs. The following lemma and theorem will help us in the construction of OCDs.

Lemma 2.1. *Let $\mathbf{C}^*$ be a $k \times b$ matrix with v elements $t_1, t_2, \ldots, t_v$ where $b = mv$, $m =$ a positive integer such that each element occurs at most once in each column and an equal number of times in the whole matrix $\mathbf{C}^*$. Then from $\mathbf{C}^*$ we can construct a $v \times b$ matrix $\boldsymbol{A}$ with $(k+1)$ symbols $a_1, a_2, \ldots, a_k$ and 0 such that each of the non-null symbols occurs once and only once in each of the b columns and m times in each of the v rows of $\boldsymbol{A}$.*

Proof. From the properties of the matrix $\mathbf{C}^*$ it can be easily seen that the columns can be identified with the b blocks of a BPEB design d with constant block size k and with v treatments $t_1, t_2, \ldots, t_v$. We know from [1] that for a BPEB design with $b = mv$, the k treatments in the b blocks of d can always be arranged such that each treatment occurs m times in each of the k positions in the blocks. We denote such an arrangement by a $k \times b$ matrix $\mathbf{B}$. From the above matrix $\mathbf{B}$, we can construct a $v \times b$ matrix $\mathbf{A}$ by putting the element a_l in its (i, j)th cell if t_i occurs in the lth row and jth column of $\mathbf{B}$, $l = 1, 2, \ldots, k$, $j = 1, 2, \ldots, b$, $i = 1, 2, \ldots, v$. Other positions are filled in with zeros. Obviously it follows from the property of $\mathbf{B}$ that each of $a_1, a_2, \ldots, a_k$ occurs once and only once in each of the b columns of $\mathbf{A}$. As every treatment occurs m times in each of the k rows of $\mathbf{B}$, it is evident that each of the symbols $a_1, a_2, \ldots, a_k$ occurs m times in each row of $\mathbf{A}$. Thus the lemma is proved. □

Remark 2.1. It may sometimes be challenging to construct such a $\mathbf{B}$. But if a BPEB design with $b = mv$ has a cyclic solution, it is very straightforward to construct the $\mathbf{B}$-matrix. When the block design with $b = mv$ does not have a cyclic solution, the construction of $\mathbf{B}$ seems to be difficult and we have used a trial and error method to get the desired configuration, whose existence is guaranteed by Lemma 2.1.

Now we prove the main theorem.

Theorem 2.1. *For any BPEB design $d(v, b, r, k)$ with $b = mv$, m (≥ 1) a positive integer, $(k-1)$ optimum $\mathbf{W}$-matrices can be constructed provided $\mathbf{H}_k$, a Hadamard matrix of order k, exists.*

Proof. We write the matrix $\mathbf{H}_k$ as

$$\mathbf{H}_k = \left(\mathbf{1}, \quad \mathbf{h}_1, \quad \mathbf{h}_2, \quad \ldots \quad , \mathbf{h}_{k-1}\right). \tag{2.1}$$

From a BPEB design $d(v, b, r, k)$, we can always, by Lemma 2.1, construct a $v \times b$ matrix $\mathbf{A}$ where each of $a_1, a_2, \ldots, a_k$ occurs m times in each row and once in each column. We identify the k elements of $\mathbf{h}_i$ with the symbols $a_1, a_2, \ldots, a_k$ and replace these symbols in $\mathbf{A}$ with their identified elements of $\mathbf{h}_i$, $i = 1, 2, \ldots, (k-1)$. Thus we get $(k-1)$ matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{k-1}$ corresponding to $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}$ respectively. From the properties of the matrix $\mathbf{A}$ and those of $\mathbf{H}_k$, it easily follows that the $\mathbf{W}_i$'s satisfy the optimality condition $\mathbf{C}$. □

Example 2.1. Let us consider the symmetric BIBD with parameters $v = b = 7$, $r = k = 4$, $\lambda = 2$ constructed heuristically by Nandi [22]. The blocks are: (1, 2, 3, 4), (1, 2, 5, 6), (1, 3, 6, 7), (1, 4, 5, 7), (2, 3, 5, 7), (2, 4, 6, 7), (3, 4, 5, 6).

The **B**- and **A**-matrices of Lemma 2.1 can respectively be written as

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 7 & 6 \\ 2 & 1 & 6 & 5 & 7 & 4 & 3 \\ 3 & 5 & 1 & 7 & 2 & 6 & 4 \\ 4 & 6 & 7 & 1 & 3 & 2 & 5 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 & a_3 & a_4 & 0 \\ a_3 & 0 & a_1 & 0 & a_4 & 0 & a_2 \\ a_4 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_3 & 0 & a_2 & a_1 & 0 & a_4 \\ 0 & a_4 & a_2 & 0 & 0 & a_3 & a_1 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & 0 \end{pmatrix}.$$

Consider

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = (\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3). \tag{2.2}$$

Using the identification $\mathbf{a} = \mathbf{h}_1$ where $\mathbf{a}' = (a_1, a_2, a_3, a_4)$, we construct $\mathbf{W}_1$ from $\mathbf{A}$ as given below:

$$\mathbf{W}_1 = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Similarly, using the identifications $\mathbf{a} = \mathbf{h}_2$ and $\mathbf{a} = \mathbf{h}_3$ respectively, we can construct $\mathbf{W}_2$ and $\mathbf{W}_3$. It is easy to verify that the $\mathbf{W}_i$'s satisfy the condition $\mathbf{C}$.

Remark 2.2. If k is even, then it follows from Theorem 2.1 that at least one optimum $\mathbf{W}$-matrix can always be constructed by identifying the $\mathbf{a}$'s with the elements of $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$, where $\mathbf{1}'_p = (1, 1, \ldots, 1)^{1\times p}$.

In the following theorem, we shall see that the number of optimum $\mathbf{W}$-matrices can be increased substantially if the BPEB design obeys an additional condition of k-resolvability ([24], page 59). This requires that the $b = mv$ blocks can be partitioned into m sets $S_1, S_2, \ldots, S_m$ each of which contains v blocks such that each of the v treatments occurs k times in each S_i, $i = 1, 2, \ldots, m$.

Theorem 2.2. *For a k-resolvable BPEB design with $b = mv$, it is possible to construct $m(k-1)$ optimum $\mathbf{W}$-matrices, provided $\mathbf{H}_k$ and $\mathbf{H}_m$ exist.*

Proof. As the design is k-resolvable, then Lemma 2.1 is applicable to the blocks of each S_i and from these v blocks a matrix $\mathbf{A}_i^{v\times v}$ can be constructed where $\mathbf{A}_i$ contains each of the symbols $a_1, a_2, \ldots, a_k$ once and only once in each row and in each column, $i = 1, 2, \ldots, m$. It is also to be noted that

$$\mathbf{A} = \left(\mathbf{A}_1, \quad \mathbf{A}_2, \quad \ldots \quad , \mathbf{A}_m\right) \tag{2.3}$$

is the $\mathbf{A}$-matrix of Lemma 2.1 corresponding to the $b = mv$ blocks of BPEB design where each of the symbols $a_1, a_2, \ldots, a_k$ occurs m times in each row and just once in each column of $\mathbf{A}$.

According to the method described in Theorem 2.1, we can construct a matrix $\mathbf{W}_{ji}$ from $\mathbf{A}_j$ by identifying a's with the elements of $\mathbf{h}_i$, the ith column of $\mathbf{H}_k$ in (2.1). By juxtaposing $\mathbf{W}_{ji}$, $j = 1, 2, \ldots, m$, for fixed i, we obtain a matrix $\mathbf{W}_i$, where

$$\mathbf{W}_i = \left(\mathbf{W}_{1i}, \quad \mathbf{W}_{2i}, \quad \ldots \quad , \mathbf{W}_{mi}\right), \; i = 1, 2, \ldots, (k-1). \tag{2.4}$$

Thus we get $(k-1)$ matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{k-1}$ which are optimum $\mathbf{W}$-matrices for the BPEB design. As the BPEB design is k-resolvable and $\mathbf{H}_m$ exists, we can increase the number of optimal $\mathbf{W}$-matrices. By taking the Khatri–Rao product (termed a 'new product' in [25, page 30]), denoted by $\odot$, of $\mathbf{h}_j^* = \left(h_{j1}^*, h_{j2}^*, \ldots, h_{jm}^*\right)'$, the jth column of $\mathbf{H}_m$ with $\mathbf{W}_i$ of (2.4), $m(k-1)$ matrices $\mathbf{W}_{ji}^*$ can be constructed, where

$$\mathbf{W}_{ji}^* = \mathbf{h}_j^{*\prime} \odot \mathbf{W}_i = \left(h_{j1}^*\mathbf{W}_{1i}, h_{j2}^*\mathbf{W}_{2i}, \ldots, h_{jm}^*\mathbf{W}_{mi}\right), \quad \forall i = 1, 2, \ldots, (k-1);\; j = 1, 2, \ldots, m. \tag{2.5}$$

It is easy to verify that the $\mathbf{W}_{ji}^*$'s satisfy the condition $\mathbf{C}$ and hence give $m(k-1)$ optimum $\mathbf{W}$-matrices for the k-resolvable BPEB design. $\square$

Example 2.2. Let us consider the following 2-resolvable BIBD with parameters $v = 5, b = 10, r = 4, k = 2, \lambda = 1$ where the blocks can be represented in the form of a $\mathbf{B}$-matrix of order 2×10 as

$$\mathbf{B} = \left(\begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 & 1 & 2 \end{array}\right) = (\mathbf{B}_1, \; \mathbf{B}_2).$$

Now the $\mathbf{A}$-matrix of order 5×10 can be constructed as

$$\begin{array}{ccc} \textit{Blocks} \longrightarrow & & \begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \textit{Treatments} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \\ \downarrow & & \end{array}$$

$$\mathbf{A} = \left(\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 \\ a_2 & a_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & a_2 \\ 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & a_2 & 0 & a_1 \end{array}\right) = \left(\mathbf{A}_1, \; \mathbf{A}_2\right).$$

Considering the column $(1, -1)'$ of $\mathbf{H}_2$ and identifying 1 with a_1 and -1 with a_2, the following $\mathbf{W}$-matrix can be constructed:

$$\mathbf{W}_1 = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right) = \left(\mathbf{W}_{11}, \quad \mathbf{W}_{21}\right).$$

Since this is a resolvable design and $m = 2$, where $\mathbf{H}_2$ exists, two optimum $\mathbf{W}$-matrices can be constructed via Theorem 2.2 as

$$\mathbf{W}_{11}^* = \left(1, \quad 1\right) \odot \left(\mathbf{W}_{11}, \quad \mathbf{W}_{21}\right) = \left(\mathbf{W}_{11}, \quad \mathbf{W}_{21}\right);$$
$$\mathbf{W}_{21}^* = \left(1, \quad -1\right) \odot \left(\mathbf{W}_{11}, \quad \mathbf{W}_{21}\right) = \left(\mathbf{W}_{11}, \quad -\mathbf{W}_{21}\right).$$

In the following remarks we consider some methods of construction of optimum $\mathbf{W}$-matrices where at least one of $\mathbf{H}_k$ and $\mathbf{H}_m$ does not exist, so Theorem 2.2 cannot be applied.

Remark 2.3. Let $\mathbf{H}_k$ exist and $m\ (> 2)$ be even but let $\mathbf{H}_m$ not exist. Then $2(k-1)$ optimum $\mathbf{W}$-matrices can be obtained for a resolvable BPEB design by using $(k-1)$ columns (except the column of all 1's) of $\mathbf{H}_k$ and $\mathbf{1}_m$ and $(\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})'$ as two choices of the vector $\mathbf{h}^*$ in the Khatri–Rao product in Theorem 2.2.

Remark 2.4. If both of $k\ (> 2)$, $m\ (> 2)$ are even but neither of $\mathbf{H}_k$ and $\mathbf{H}_m$ exists, then two optimum $\mathbf{W}$-matrices can be constructed for a resolvable BPEB design by using the two pairs of vectors $((\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})', \mathbf{1}_m)$ and $((\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})', (\mathbf{1}'_{\frac{m}{2}}, -\mathbf{1}'_{\frac{m}{2}})')$ as two choices of $(\mathbf{h}, \mathbf{h}^*)$ in Theorem 2.2.

Remark 2.5. Let $\mathbf{H}_m$ exist, $\mathbf{H}_k$ not exist and $k\ (> 2)$ be even. Then m optimum $\mathbf{W}$-matrices can be constructed for a resolvable BPEB design by using m columns of $\mathbf{H}_m$ and $(\mathbf{1}'_{\frac{k}{2}}, -\mathbf{1}'_{\frac{k}{2}})'$ for $\mathbf{h}$ in Theorem 2.2.

Remark 2.6. If t optimum $\mathbf{W}$-matrices exist for any BPEB design, then the same number of $\mathbf{W}$-matrices exist for the dual of that design where the blocks of the original design play the role of the treatments. This is because $\mathbf{W}'$ is the optimum $\mathbf{W}$-matrix for the dual design when $\mathbf{W}$ is the optimum $\mathbf{W}$-matrix for the original design.

3. Cyclic designs

Cyclic designs are BPEB designs obtained by developing $m\ (\geq 1)$ initial blocks. All cyclic designs belong to the class of PBIBDs with at most $\frac{v}{2}$ associate classes. Many incomplete block designs may be set out as cyclic designs. If there are v treatments denoted by $0, 1, \ldots, (v-1)$ which are elements of a module M, and are arranged in blocks of size k so that each treatment is replicated r times, then the cyclic design with these parameters is denoted by $C(v, k, r)$. Given any initial block, another block is generated by adding $\alpha \pmod{v}$ to each treatment of the initial block where $\alpha \in M$. If all the v blocks thus obtained from the given initial block are all distinct then this set of blocks is said to form a full set. If v and k are relatively prime to each other then the v blocks generated from an initial block always give a full set with parameters $(v, k = r)$. On the other hand if v and k have a common divisor d, then for every value of d, there always exists at least one initial block where all the v blocks generated from an initial block are not distinct; only $\frac{v}{d}$ of them are distinct. This set of $\frac{v}{d}$ blocks forms a partial set with parameters $(v, k, r = \frac{k}{d})$. Full or partial sets can be used singly or in combination to construct cyclic designs. For a detailed study, one is referred to [18,17].

Example 3.1. Let $v = 8$, $k = 4$. The eight blocks (1, 2, 3, 4), (2, 3, 4, 5), (3, 4, 5, 6), (4, 5, 6, 7), (5, 6, 7, 0), (6, 7, 0, 1), (7, 0, 1, 2), (0, 1, 2, 3) generated from the initial block (1, 2, 3, 4) form a full set. On the other hand, with $d = 2$, the $\frac{v}{d} = 4$ blocks (0, 1, 4, 5), (1, 2, 5, 6), (2, 3, 6, 7) and (3, 4, 7, 0) generated from (0, 1, 4, 5) form a partial set of size $(8, 4, \frac{k}{d} = 2)$. For $d = 4$, the two blocks (0, 2, 4, 6) and (1, 3, 5, 7) generated from (0, 2, 4, 6) form a partial set of size $(8, 4, \frac{k}{d} = 1)$.

If there exists a partial set consisting of $\frac{v}{d}$ blocks in a cyclic design, then we see that each treatment is replicated $\frac{k}{d}$ times in these blocks. So the number of covariates to be accommodated in a cyclic design depends on whether the sets are full sets or partial sets and also on the number of sets. When cyclic designs consist of full sets only, then a systematic way for assigning values to the covariates can be developed. However, when a design contains partial sets, it is difficult to specify the number of covariates to be accommodated beforehand. Some examples of cyclic designs containing the partial sets are considered where we have provided a solution for OCDs through an ad hoc method.

It is to be noted that [7,9] gave solutions for the series of BIBD's which belonged to the class of cyclic designs. Moreover, all irreducible BIBDs can also be obtained by cyclically developing the set of some initial blocks. So we can cover all these designs and a lot of other designs under the general technique described in this section.

3.1. Cyclic designs containing full sets only

It is proved in Theorem 2.2 that for resolvable BPEB designs the number of covariates can be increased over the number of designs for ordinary BPEB design with the same parameters. It can be easily noted that cyclic designs with m initial blocks giving m full sets of blocks always give resolvable BPEB designs. So for such cyclic designs a larger number of covariates can be accommodated in an optimum way following Theorem 2.2. But for the particular case when the resolvable designs are of cyclic nature, construction can be done more easily by exploiting the circular nature of the blocks. The precise statement follows.

Theorem 3.1. *Let a cyclic design with parameters $v, b = mv, r = mk, k$ be obtained by developing m initial blocks each of size k and also let $\mathbf{H}_k$ and $\mathbf{H}_m$ exist. Then $m(k-1)$ optimum $\mathbf{W}$-matrices can be constructed through cyclic development.*

Proof. Let $\mathbf{H}_k$ and $\mathbf{H}_m$ be written respectively in the forms

$$\mathbf{H}_k = \left(\mathbf{1}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{k-1}\right) \quad \text{and} \quad \mathbf{H}_m = \left(\mathbf{h}_1^*, \mathbf{h}_2^*, \ldots, \mathbf{h}_m^*\right)$$

where

$$\mathbf{h}_j^{*\prime} = \left(h_{j1}^*, h_{j2}^*, \ldots, h_{jm}^*\right); \quad i = 1, 2, \ldots, m.$$

Also let the m initial blocks of the specified design be displayed in the form of the column vectors in the treatment–block incidence matrix of that design. The k non-zero elements of the jth initial block are replaced by the k elements of $\mathbf{h}_i$ in that order and are cyclically permuted to get a $v \times v$ matrix $\mathbf{W}_{ji}$. After obtaining the $\mathbf{W}_{ji}$'s the construction of $m(k-1)$ optimum $\mathbf{W}$-matrices, $\mathbf{W}_{ji}^*$ $(i = 1, 2, \ldots, (k-1), j = 1, 2, \ldots, m)$ follows routinely from (2.4) and (2.5). □

Example 3.2. Let the cyclic design with parameters $v = 13, b = 26, r = 8, k = 4$ with the initial blocks (1, 4, 12, 13), (1, 4, 10, 13) mod 13 be considered.

Let $\mathbf{H}_2$ be written as

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (\mathbf{h}_1^*, \mathbf{h}_2^*).$$

Then by Theorem 3.1, six optimum $\mathbf{W}$-matrices for this design can be constructed by using $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ of (2.2) and the two columns of $\mathbf{H}_2^*$. For instance, by using $\mathbf{h}_1$ of (2.2) in two full cycles of the incidence matrix corresponding to the initial blocks (1, 4, 12, 13) and (1, 4, 10, 13), the matrices $\mathbf{W}_{11}$ and $\mathbf{W}_{21}$ are obtained respectively as

$$\begin{array}{c} \longrightarrow \textit{Treatments} \\ \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \end{array}$$

$$\mathbf{W}_{11}' = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \multicolumn{13}{c}{\textit{and cyclic permutations}} \end{pmatrix};$$

$$\begin{array}{c} \longrightarrow \textit{Treatments} \\ \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} \end{array}$$

$$\mathbf{W}_{21}' = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ \multicolumn{13}{c}{\textit{and cyclic permutations}} \end{pmatrix}.$$

After obtaining $\mathbf{W}_{11}$ and $\mathbf{W}_{21}$, the two $\mathbf{W}$-matrices $\mathbf{W}_{11}^*, \mathbf{W}_{21}^*$ are obtained by using $\mathbf{h}_1^*$ as

$$\mathbf{W}_{11}^* = \mathbf{h}_1^{*\prime} \odot \mathbf{W}_1 = \begin{pmatrix}1, & 1\end{pmatrix} \odot \begin{pmatrix}\mathbf{W}_{11}, & \mathbf{W}_{21}\end{pmatrix} = \begin{pmatrix}\mathbf{W}_{11}, & \mathbf{W}_{21}\end{pmatrix}$$
$$\mathbf{W}_{21}^* = \mathbf{h}_2^{*\prime} \odot \mathbf{W}_1 = \begin{pmatrix}1, & -1\end{pmatrix} \odot \begin{pmatrix}\mathbf{W}_{11}, & \mathbf{W}_{21}\end{pmatrix} = \begin{pmatrix}\mathbf{W}_{11}, & -\mathbf{W}_{21}\end{pmatrix}.$$

Incidentally, it is seen that this is a PBIBD with parameters $v = 13, b = 26, r = 8, k = 4, \lambda_1 = 1, \lambda_2 = 3, n_1 = n_2 = 6$. The first associates of the treatment i are $(i+2, i+5, i+6, i+7, i+8, i+11)$ (mod 13).

In the next two sections we consider cyclic designs which are not exactly of the type considered in this section. We illustrate through examples how OCDs can be constructed through ad hoc methods.

3.2. Cyclic designs containing some partial sets

It was mentioned earlier that it is difficult to propose a systematic method for finding OCDs for cyclic designs containing partial sets. It should be noted that the number of optimum $\mathbf{W}$-matrices depends on the properties of the partial sets and consequently on the nature of the columns of $\mathbf{H}_k$ whose elements are used to replace the non-zero elements in the blocks of the incidence matrix. We consider the following example illustrating an ad hoc method which depends on the nature of the partial set.

Example 3.3. Consider the irreducible BIBD with parameters $v = 6, b = \binom{6}{4} = 15, r = \binom{5}{3} = 10, k = 4, \lambda = \binom{4}{2} = 6$. The design can be obtained from the three initial blocks: $[(0, 1, 2, 3), (0, 2, 3, 4), (0, 2, 3, 5)]$ mod 6, where the first two give full sets containing six distinct blocks each and the last one gives a partial set containing only three distinct blocks. We consider $\mathbf{H}_4$ of (2.2). It is to be noted that, like for Theorem 3.1, all the three columns $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ of (2.2) cannot be used in each of the three subsets of blocks obtained by developing cyclically three initial blocks. The last three blocks obtained from the third initial block are partially cyclic; only $\mathbf{h}_1, \mathbf{h}_2$ can be used to construct $\mathbf{W}_j$ matrices but $\mathbf{h}_3$ cannot be used as it will not lead to zero row-sums. Using $\mathbf{h}_1$ and $\mathbf{h}_2$, we get two $\mathbf{W}$-matrices, namely $\mathbf{W}_{(1)}$ and $\mathbf{W}_{(2)}$ respectively:

$$\mathbf{W}_{(1)} = \begin{array}{ll} & \quad Bl. \longrightarrow \\ & \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \begin{array}{cc} Tr. & 1 \\ \downarrow & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \end{array} & \left(\begin{array}{rrrrrr|rrrrrr|rrr} 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \end{array}$$

$$\mathbf{W}_{(2)} = \begin{array}{ll} & \quad Bl. \longrightarrow \\ & \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \begin{array}{cc} Tr. & 1 \\ \downarrow & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \end{array} & \left(\begin{array}{rrrrrr|rrrrrr|rrr} 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \end{array}.$$

3.3. Cyclic designs where each element corresponds to a number of symbols

Here any treatment is denoted by α_j where α is any element of the module $M = (0, 1, \ldots, m-1)$ and j is one of the n symbols $1, 2, \ldots, n$. The following is an example of a design which is obtained by the classical method of difference (cf. [2]) where each symbol of the module $M = (0, 1, 2, 3, 4)$ corresponds to two symbols 1 and 2. If the blocks can be grouped into sets which have cycles, then optimum $\mathbf{W}$-matrices can be constructed by exploiting the properties of the blocks of the design. The method is illustrated through the following example.

Example 3.4. Consider the GD design with parameters $v = 10, b = 20, r = 8, k = 4, \lambda_1 = 0, \lambda_2 = 3, m = 5, n = 2$ with the initial group $(0_1, 0_2)$ mod 5. The initial blocks are $(0_1, 1_2, 2_2, 4_2)$, $(0_2, 1_1, 2_1, 4_1)$, $(0_1, 2_2, 3_2, 4_2)$, $(0_2, 2_1, 3_1, 4_1)$ mod 5. We divide the four initial blocks into two sets $S_1 = \{(0_1, 1_2, 2_2, 4_2), (0_1, 2_2, 3_2, 4_2)\}$ and $S_2 = \{(0_2, 1_1, 2_1, 4_1), (0_2, 2_1, 3_1, 4_1)\}$. In the first five rows of the incidence matrix corresponding to the initial block $(0_1, 1_2, 2_2, 4_2)$ of S_1, the non-zero elements are replaced by the elements of $\mathbf{h}_1$ of (2.2) and in the last five rows corresponding to $(0_1, 1_2, 2_2, 4_2)$ of S_1, the non-zero elements are replaced by those of $-\mathbf{h}_1$. We call this matrix, of order 10×10, $\mathbf{U}_1^{(1)}$. In the same way using $\mathbf{h}_1$ and $-\mathbf{h}_1$, we can get the matrix $\mathbf{U}_1^{(2)}$ from the blocks corresponding to the two initial blocks of S_2. Therefore $\mathbf{U}_1^{(1)}$ and $\mathbf{U}_1^{(2)}$ are given by

$$\mathbf{U}_1^{(1)} = \begin{array}{c} \longrightarrow \; Treatments \\ \begin{array}{cccccccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \end{array} \\ \left(\begin{array}{rrrrr|rrrrr} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \end{array},$$

$$\begin{array}{cc} & \longrightarrow \textit{Treatments} \\ & \begin{array}{cccccccccc} 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \end{array} \\ \mathbf{U}_1^{(2)} = & \left(\begin{array}{rrrrr|rrrrr} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \end{array}$$

From $\mathbf{U}_1^{(1)}, \mathbf{U}_1^{(2)}$, we construct two optimum $\mathbf{W}$-matrices as

$$\mathbf{W}^{(1,1)} = \left(\mathbf{U}_1^{(1)\prime}, \quad \mathbf{U}_1^{(2)\prime}\right), \qquad \mathbf{W}^{(1,2)} = \left(\mathbf{U}_1^{(1)\prime}, \quad -\mathbf{U}_1^{(2)\prime}\right).$$

Similarly we can get four more optimum $\mathbf{W}$-matrices, namely $\mathbf{W}^{(2,1)}$, $\mathbf{W}^{(2,2)}$, $\mathbf{W}^{(3,1)}$ and $\mathbf{W}^{(3,2)}$ if we use $\mathbf{h}_2$ and $\mathbf{h}_3$ of (2.2).

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Appendix

A.1. List of OCDs in BPEB designs and cyclic design set-ups

As mentioned earlier, BIBD and PBIBDs form an important subclass of BPEB designs and the lists of BIBD and PBIBDs are readily available. So for using OCDs in a BPEB design set-up we have considered BIBDs with $b = mv$ where $v, b \leq 100$, $r, k \leq 15$ from the list given in [24, page 91–94] and the PBIBDs with $b = mv$ and $r \leq 10$, $k \leq 10$ from [6, pages 107–311]. A number of BIBDs and PBIBDs with $b = mv$ are cyclic designs. These designs have not been considered separately. Only a separate list for the cyclic BIBDs having partial cycles has been considered. Here c^* stands for the number of covariates which are accommodated optimally.

A.1.1. BIBD with $b = mv$

See Table A.1.

A.1.2. PBIBDs with two associate classes and $b = mv$

PBIBDs with two associate classes are classified into the following types:

(a) Group divisible (GD), which again are classified as:
Singular (S), Semi-Regular (SR), Regular (R),
(b) Triangular (T),
(c) Latin Square types (LS),
(d) Cyclic (C),
(e) Partial Geometry (PG),
(f) Miscellaneous (M).

See Tables A.2–A.9.

A.2. Cyclic designs

We have considered only the BIBDs listed in [24] where the designs have partial cycles, so that $b \neq mv$. See Table A.10.

Table A.1
BIBD with $b = mv$

Sl. No.	Design No.	v	b	r	k	λ	c^*	Method of construction
1	3	5	10	4	2	1	2	Theorem 2.2
2	4	5	5	4	4	3	3	Theorem 2.1
3	11	7	7	4	4	2	3	Theorem 2.1
4	12	7	21	6	2	1	1	Theorem 2.1
5	13	7	7	6	6	5	1	Remark 2.2

Table A.1 (*continued*)

Sl. No.	Design No.	v	b	r	k	λ	c^*	Method of construction
6	18	9	36	8	2	1	4	Theorem 2.2
7	19	9	18	8	4	3	6	Theorem 2.2
8	21	9	9	8	8	7	7	Theorem 2.1
9	30	11	11	6	6	3	1	Remark 2.2
10	31	11	55	10	2	1	1	Theorem 2.1
11	32	11	11	10	10	9	1	Remark 2.2
12	37	13	13	4	4	1	3	Theorem 2.1
13	40	13	26	12	6	5	2	Remark 2.5
14	44	15	15	8	8	4	7	Theorem 2.1
15	47	16	16	6	6	2	1	Remark 2.2
16	49	16	16	10	10	6	1	Remark 2.2
17	56	19	19	10	10	5	1	Remark 2.2
18	57	19	57	12	4	2	3	Theorem 2.1
19	61	21	42	12	6	3	2	Remark 2.5
20	Dual of 64	23	23	12	12	6	11	Theorem 2.1
21	66	25	50	8	4	1	6	Theorem 2.2
22	Dual of 71	27	27	14	14	7	1	Remark 2.2
23	75	31	31	6	6	1	1	Remark 2.2
24	76	31	31	10	10	3	1	Remark 2.2
25	85	45	45	12	12	3	11	Theorem 2.1
26	87	57	57	8	8	1	7	Theorem 2.1
27	91	91	91	10	10	1	1	Remark 2.2

Table A.2
(a) Group divisible designs: Singular (S).

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	m	n	c^*	Method of construction
1	S1	6	3	2	4	2	1	3	2	2	Remark 2.6
2	S2	6	6	4	4	4	2	3	2	3	Theorem 2.1
3	S4	6	12	8	4	8	4	3	2	6	Theorem 2.2
4	S9	10	10	4	4	4	1	5	2	3	Theorem 2.1
5	S10	10	20	8	4	8	2	5	2	6	Theorem 2.2
6	S15	18	36	8	4	8	1	9	2	6	Theorem 2.2
7	S19	8	8	6	6	6	4	4	2	1	Remark 2.2
8	S21	9	3	2	6	2	1	3	3	1	Remark 2.6
9	S23	9	9	6	6	6	3	3	3	1	Remark 2.2
10	S26	10	10	6	6	6	3	5	2	1	Remark 2.2
11	S29	12	12	6	6	6	2	4	3	1	Remark 2.2
12	S33	14	14	6	6	6	2	7	2	1	Remark 2.2
13	S42	21	21	6	6	6	1	7	3	1	Remark 2.2
14	S44	26	26	6	6	6	1	13	2	1	Remark 2.2
15	S51	10	5	4	8	4	3	5	2	6	Remark 2.6
16	S52	10	10	8	8	8	6	5	2	7	Theorem 2.1
17	S53	12	3	2	8	2	1	3	4	4	Remark 2.6
18	S54	12	6	4	8	4	2	3	4	6	Remark 2.6
19	S56	12	12	8	8	8	4	3	4	7	Theorem 2.1
20	S59	14	7	4	8	4	2	7	2	6	Remark 2.6
21	S60	14	14	8	8	8	4	7	2	7	Theorem 2.1
22	S65	18	18	8	8	8	3	9	2	7	Theorem 2.1
23	S66	20	10	4	8	4	1	5	4	6	Remark 2.6
24	S68	20	20	8	8	8	2	5	4	7	Theorem 2.1
25	S71	26	13	4	8	4	1	13	2	6	Remark 2.6
26	S72	26	26	8	8	8	2	13	2	7	Theorem 2.1
27	S77	36	36	8	8	8	1	9	4	7	Theorem 2.1
28	S80	50	50	8	8	8	1	25	2	7	Theorem 2.1
29	S99	12	12	10	10	10	8	6	2	1	Remark 2.2
30	S100	15	3	2	10	2	1	3	5	1	Remark 2.6
31	S104	15	15	10	10	10	5	3	5	1	Remark 2.2
32	S105	18	18	10	10	10	5	9	2	1	Remark 2.2
33	S111	22	22	10	10	10	4	11	2	1	Remark 2.2
34	S115	30	30	10	10	10	2	6	5	1	Remark 2.2
35	S119	42	42	10	10	10	2	21	2	1	Remark 2.2
36	S123	55	55	10	10	10	1	11	5	1	Remark 2.2
37	S124	82	82	10	10	10	1	41	2	1	Remark 2.2

Table A.3
(a) Group divisible designs: Semi-Regular (SR).

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	m	n	c^*	Method of construction
1	SR1	4	4	2	2	0	1	2	2	1	Theorem 2.1
2	SR2	4	8	4	2	0	2	2	2	2	Theorem 2.2
3	SR3	4	12	6	2	0	3	2	2	1	Theorem 2.1
4	SR4	4	16	8	2	0	4	2	2	4	Theorem 2.2
5	SR5	4	20	10	2	0	5	2	2	1	Theorem 2.1
6	SR7	6	18	6	2	0	2	2	3	1	Theorem 2.1
7	SR9	8	16	4	2	0	1	2	4	2	Theorem 2.2
8	SR10	8	32	8	2	0	2	2	4	4	Theorem 2.2
9	SR12	10	50	10	2	0	2	2	5	1	Theorem 2.1
10	SR13	12	36	6	2	0	1	2	6	1	Theorem 2.1
11	SR15	16	64	8	2	0	1	2	8	4	Theorem 2.2
12	SR17	20	100	10	2	0	1	2	10	1	Theorem 2.1
13	SR36	8	8	4	4	0	2	4	2	3	Theorem 2.1
14	SR39	8	16	8	4	0	4	4	2	6	Theorem 2.2
15	SR44	16	16	4	4	0	1	4	4	3	Theorem 2.1
16	SR45	16	32	8	4	0	2	4	4	6	Theorem 2.2
17	SR49	32	64	8	4	0	1	4	8	6	Theorem 2.2
18	SR67	12	12	6	6	0	3	6	2	1	Remark 2.2
19	SR68	12	12	6	6	2	3	3	4	1	Remark 2.2
20	SR72	18	18	6	6	0	2	6	3	1	Remark 2.2
21	SR92	16	16	8	8	0	4	8	2	7	Theorem 2.1
22	SR95	32	32	8	8	0	2	8	4	7	Theorem 2.1
23	SR97	64	64	8	8	0	1	8	8	7	Theorem 2.1
24	SR108	20	20	10	10	0	5	10	2	1	Remark 2.2

Table A.4
(a) Group divisible designs: Regular (R).

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	m	n	c^*	Method of construction
1	R1	4	8	4	2	2	1	2	2	2	Theorem 2.2
2	R4	4	12	6	2	4	1	2	2	1	Theorem 2.1
3	R8	4	16	8	2	6	1	2	2	4	Theorem 2.2
4	R9	4	16	8	2	4	2	2	2	4	Theorem 2.2
5	R10	4	16	8	2	2	3	2	2	4	Theorem 2.2
6	R14	4	20	10	2	8	1	2	2	1	Theorem 2.1
7	R15	4	20	10	2	6	2	2	2	1	Theorem 2.1
8	R16	4	20	10	2	4	3	2	2	1	Theorem 2.1
9	R17	4	20	10	2	2	4	2	2	1	Theorem 2.1
10	R18	6	12	4	2	0	1	3	2	2	Theorem 2.2
11	R19	6	18	6	2	2	1	3	2	1	Theorem 2.1
12	R22	6	24	8	2	4	1	3	2	4	Theorem 2.2
13	R23	6	24	8	2	0	2	3	2	4	Theorem 2.2
14	R24	6	24	8	2	1	2	2	3	4	Theorem 2.2
15	R28	6	30	10	2	6	1	3	2	1	Theorem 2.1
16	R29	8	24	6	2	0	1	4	2	1	Theorem 2.1
17	R30	8	32	8	2	2	1	4	2	4	Theorem 2.2
18	R32	8	40	10	2	2	1	2	4	1	Theorem 2.1
19	R33	8	40	10	2	4	1	4	2	1	Theorem 2.1
20	R34	9	27	6	2	0	1	3	3	1	Theorem 2.1
21	R35	9	45	10	2	2	1	3	3	1	Theorem 2.1
22	R36	10	40	8	2	0	1	5	2	4	Theorem 2.2
23	R37	10	50	10	2	2	1	5	2	1	Theorem 2.1
24	R38	12	48	8	2	0	1	3	4	4	Theorem 2.2
25	R40	12	60	10	2	0	1	6	2	1	Theorem 2.1
26	R41	15	75	10	2	0	1	3	5	1	Theorem 2.1
27	R94	6	6	4	4	3	2	2	3	3	Theorem 2.1
28	R95	6	12	8	4	6	4	2	3	6	Theorem 2.2
29	R96	6	12	8	4	4	5	3	2	6	Theorem 2.2
30	R98	8	16	8	4	4	3	2	4	6	Theorem 2.2
31	R99	8	16	8	4	6	3	4	2	6	Theorem 2.2
32	R104	9	9	4	4	3	1	3	3	3	Theorem 2.1
33	R105	9	18	8	4	6	2	3	3	6	Theorem 2.2
34	R106	10	20	8	4	0	3	5	2	6	Theorem 2.2
35	R109	12	12	4	4	2	1	6	2	3	Theorem 2.1
36	R110	12	24	8	4	4	2	6	2	6	Theorem 2.2
37	R112	14	14	4	4	0	1	7	2	3	Theorem 2.1
38	R113	14	28	8	4	0	2	7	2	6	Theorem 2.2

Table A.4 (*continued*)

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	m	n	c^*	Method of construction
39	R114	15	15	4	4	0	1	5	3	3	Theorem 2.1
40	R115	15	30	8	4	6	1	5	3	6	Theorem 2.2
41	R116	15	30	8	4	0	2	5	3	6	Theorem 2.2
42	R117	15	30	8	4	1	2	3	5	6	Theorem 2.2
43	R120	16	32	8	4	4	1	4	4	6	Theorem 2.2
44	R128	26	52	8	4	0	1	13	2	6	Theorem 2.2
45	R129	27	54	8	4	0	1	9	3	6	Theorem 2.2
46	R130	28	56	8	4	0	1	7	4	6	Theorem 2.2
47	R166	10	10	6	6	5	2	2	5	1	Remark 2.2
48	R168	15	15	6	6	5	1	3	5	1	Remark 2.2
49	R170	27	27	6	6	3	1	9	3	1	Remark 2.2
50	R171	28	28	6	6	2	1	7	4	1	Remark 2.2
51	R186	12	12	8	8	6	5	6	2	7	Theorem 2.1
52	R187	14	14	8	8	7	2	2	7	7	Theorem 2.1
53	R188	21	21	8	8	7	1	3	7	7	Theorem 2.1
54	R189	24	24	8	8	4	2	4	6	7	Theorem 2.1
55	R190	48	48	8	8	4	1	12	4	7	Theorem 2.1
56	R191	63	63	8	8	0	1	9	7	7	Theorem 2.1
57	R203	12	12	10	10	9	8	4	3	1	Remark 2.2
58	R204	14	14	10	10	8	6	2	7	1	Remark 2.2
59	R205	14	14	10	10	6	7	7	2	1	Remark 2.2
60	R206	18	18	10	10	9	2	2	9	1	Remark 2.2
61	R207	27	27	10	10	9	1	3	9	1	Remark 2.2
62	R208	32	32	10	10	6	2	4	8	1	Remark 2.2
63	R209	75	75	10	10	5	1	15	5	1	Remark 2.2

Table A.5
(b) Triangular PBIBDs.

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
1	T1	10	30	6	2	1	0	6	3	1	Theorem 2.1
2	T3	10	30	6	2	0	2	6	3	1	Theorem 2.1
3	T5	15	60	8	2	1	0	8	6	1	Theorem 2.1
4	T6	15	45	6	2	0	1	8	6	1	Theorem 2.1
5	T7	21	105	10	2	1	0	10	10	1	Theorem 2.1
6	T8	21	105	10	2	0	1	10	10	1	Theorem 2.1
7	T28	10	5	2	4	1	0	6	3	2	Remark 2.6
8	T29	10	10	4	4	2	0	6	3	3	Theorem 2.1
9	T31	10	20	8	4	4	0	6	3	6	Theorem 2.2
10	T33	10	10	4	4	1	2	6	3	3	Theorem 2.1
11	T34	10	20	8	4	3	2	6	3	6	Theorem 2.2
12	T36	10	20	8	4	2	4	6	3	6	Theorem 2.2
13	T38	15	30	8	4	3	0	8	6	3	Theorem 2.1
14	T39	28	56	8	4	2	0	12	15	3	Theorem 2.1
15	T42	91	182	8	4	1	0	24	66	3	Theorem 2.1
16	T58	10	10	6	6	4	2	6	3	1	Remark 2.2
17	T60	10	10	6	6	3	4	6	3	1	Remark 2.2
18	T61	15	15	6	6	3	1	8	6	1	Remark 2.2
19	T65	21	7	2	6	1	0	10	10	1	Remark 2.6
20	T67	21	21	6	6	3	0	10	10	1	Remark 2.2
21	T78	36	9	2	8	1	0	14	21	2	Remark 2.6
22	T79	36	18	4	8	2	0	14	21	6	Remark 2.6
23	T81	36	36	8	8	4	0	14	21	7	Theorem 2.1
24	T94	21	21	10	10	6	3	10	10	1	Remark 2.2
25	T95	21	21	10	10	5	4	10	10	1	Remark 2.2
26	T96	55	11	2	10	1	0	18	36	1	Remark 2.6
27	T100	55	55	10	10	5	0	18	36	1	Remark 2.2

Table A.6
(c) Latin square type PBIBDs.

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
1	LS1	9	18	4	2	1	0	4	4	2	Theorem 2.2
2	LS2	9	36	8	2	2	0	4	4	4	Theorem 2.2
3	LS3	16	48	6	2	1	0	6	9	1	Theorem 2.1
4	LS5	25	100	8	2	1	0	8	16	4	Theorem 2.2
5	LS6	36	180	10	2	1	0	10	25	1	Theorem 2.1
6	LS26	9	9	4	4	1	2	4	4	3	Theorem 2.1

(*continued on next page*)

Table A.6 (*continued*)

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
7	LS27	9	18	8	4	2	4	4	4	6	Theorem 2.2
8	LS28	16	8	2	4	1	0	6	9	2	Remark 2.6
9	LS30	16	16	4	4	2	0	6	9	3	Theorem 2.1
10	LS34	16	32	8	4	4	0	6	9	6	Theorem 2.2
11	LS38	16	32	8	4	2	1	9	6	3	Theorem 2.1
12	LS42	16	32	8	4	1	2	6	9	6	Theorem 2.2
13	LS45	25	50	8	4	3	0	8	16	6	Theorem 2.2
14	LS46	49	98	8	4	2	0	12	36	6	Theorem 2.2
15	LS47	169	338	8	4	1	0	24	144	6	Theorem 2.2
16	LS74	36	12	2	6	1	0	10	25	1	Remark 2.6
17	LS77	36	36	6	6	2	0	15	20	1	Remark 2.2
18	LS78	36	36	6	6	3	0	10	25	1	Remark 2.2
19	LS82	49	49	6	6	0	1	18	30	1	Remark 2.2
20	LS101	25	25	8	8	3	2	8	16	7	Theorem 2.1
21	LS102	64	16	2	8	1	0	14	49	4	Remark 2.6
22	LS104	64	32	4	8	1	0	28	35	6	Remark 2.6
23	LS108	64	32	4	8	2	0	14	49	3	Remark 2.6
24	LS110	64	64	8	8	2	0	28	35	7	Theorem 2.1
25	LS114	64	64	8	8	4	0	14	49	7	Theorem 2.1
26	LS136	36	36	10	10	4	2	10	25	1	Remark 2.2
27	LS137	100	20	2	10	1	0	18	81	1	Remark 2.6
28	LS146	100	100	10	10	5	0	18	81	1	Remark 2.2

Table A.7
(d) Cyclic PBIBDs.

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
1	C1	5	5	2	2	1	0	2	2	1	Theorem 2.1
2	C2	5	10	4	2	2	0	2	2	2	Theorem 2.2
3	C3	5	15	6	2	3	0	2	2	1	Theorem 2.1
4	C4	5	20	8	2	4	0	2	2	4	Theorem 2.2
5	C5	5	25	10	2	5	0	2	2	1	Theorem 2.1
6	C6	5	15	6	2	2	1	2	2	1	Theorem 2.1
7	C7	5	20	8	2	3	1	2	2	4	Theorem 2.2
8	C8	5	25	10	2	4	1	2	2	1	Theorem 2.1
9	C9	5	25	10	2	3	2	2	2	1	Theorem 2.1
10	C10	13	39	6	2	1	0	6	6	1	Theorem 2.1
11	C11	17	68	8	2	1	0	8	8	4	Theorem 2.2
12	C21	13	26	8	4	1	3	6	6	6	Theorem 2.2
13	C22	17	34	8	4	1	2	8	8	6	Theorem 2.2
14	C23	13	13	6	6	3	2	6	6	1	Remark 2.2
15	C26	17	17	8	8	4	3	8	8	7	Theorem 2.1
16	C27	29	29	8	8	3	1	14	14	7	Theorem 2.1
17	C29	13	13	10	10	8	7	6	6	1	Remark 2.2

Table A.8
(e) PBIBDs based on partial geometries.

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
1	PG3	40	40	4	4	1	0	12	27	3	Theorem 2.1
2	PG8	156	156	6	6	1	0	30	125	1	Remark 2.2
3	PG11	50	25	4	8	1	0	28	21	6	Remark 2.6

Table A.9
(f) Miscellaneous PBIBDs.

Sl. No.	Design No.	v	b	r	k	λ_1	λ_2	n_1	n_2	c^*	Method of construction
1	M2	16	48	6	2	1	0	6	9	1	Theorem 2.1
2	M3	16	80	10	2	1	0	10	5	1	Theorem 2.1
3	M4	16	80	10	2	2	0	5	10	1	Theorem 2.1
4	M6	26	130	10	2	1	0	10	15	1	Theorem 2.1
5	M7	27	135	10	2	1	0	10	16	1	Theorem 2.1
6	M19	40	80	8	4	2	0	12	27	6	Theorem 2.2
7	M26	26	26	6	6	2	0	15	10	1	Remark 2.2
8	M33	50	50	8	8	2	0	28	21	7	Theorem 2.1
9	M38	26	26	10	10	4	3	15	10	1	Remark 2.2
10	M39	27	27	10	10	1	5	10	16	1	Remark 2.2

Table A.10
BIBD.

Sl. No.	Design No.	v	b	r	k	λ	Solution	c^*	Method of construction
1	9	6	15	10	4	6	Two full sets of 5 blocks each and the initial blocks: [(0, 1, 2, 3), (0, 2, 3, 4)] mod 6; One partial set of 3 blocks and the initial block (0, 2, 3, 5) mod 6	2	Analogous to Example 3.3
2	63	22	77	14	4	2	Difference set:$(x_1^0, x_1^3, x_2^\alpha, x_2^{\alpha+3})$; $(x_1^1, x_1^4, x_2^{\alpha+1}, x_2^{\alpha+4})$;$(x_1^2, x_1^5, x_2^{\alpha+2}, x_2^{\alpha+5})$; $(x_2^0, x_2^3, x_3^\alpha, x_3^{\alpha+3})$;$(x_2^1, x_2^4, x_3^{\alpha+1}, x_3^{\alpha+4})$; $(x_2^2, x_2^5, x_3^{\alpha+2}, x_3^{\alpha+5})$;$(x_3^0, x_3^3, x_1^\alpha, x_1^{\alpha+3})$; $(x_3^1, x_3^4, x_1^{\alpha+1}, x_1^{\alpha+4})$; $(x_3^2, x_3^5, x_1^{\alpha+2}, x_1^{\alpha+5})$;$(\infty, 0_1, 0_2, 0_3)$;$(\infty, 0_1, 0_2, 0_3)$; $x =$ Primitive root of GF(7)	2	Analogous to Examples 3.3 and 3.4

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