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Note

On relationship between Hamiltonian path and holes in $L(3, 2, 1)$-coloring of minimum span

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ABSTRACT

The $L(2, 1)$-coloring and $L(3, 2, 1)$-coloring on a finite simple connected graph G originate from frequency assignment problem. Any $L(2, 1)$-coloring on G of minimum possible span is a $\lambda_{2,1}$-coloring of G and its span is the $\lambda_{2,1}$-number of G, denoted by $\lambda_{2,1}(G)$. Any integer i, $0 < i < \lambda_{2,1}(G)$, which is left unused as a color in a $\lambda_{2,1}$-coloring L, is a hole in L. The $\lambda_{3,2,1}$-number of G, $\lambda_{3,2,1}$-coloring of G and holes in a $\lambda_{3,2,1}$-coloring are defined in a similar manner. In their paper Georges et al. (1994) referred a hole i as a gap if $|L_{i-1}| = 1 = |L_{i+1}|$, L_j being the set of vertices colored j by L, and the vertices in $L_{i-1} \cup L_{i+1}$ are adjacent. They showed if G admits a $\lambda_{2,1}$-coloring without any gap, then the complement G^c has a Hamiltonian path. We first explore similar phenomenon in the perspective of $L(3, 2, 1)$-coloring, a natural generalization of $L(2, 1)$-coloring problem. Towards this, we coin a term 2-gap. Formally, a $\lambda_{3,2,1}$-coloring $\hat{L}$ on G is said to have a 2-gap $\{i + 1, i + 2\}$ if $\hat{L}$ has two consecutive holes $i + 1$, $i + 2$ and $|\hat{L}_i| = 1 = |\hat{L}_{i+3}|$, where $\hat{L}_j$ is the set of vertices colored j by $\hat{L}$. We prove if G admits a $\lambda_{3,2,1}$-coloring without any 2-gap, then G^c has a Hamiltonian path. But investigating the converse of this result poses an intriguing combinatorial problem. However, after exploring various interesting features of a 2-gap, we give a class of finite simple connected graphs for which the converse is true. For the same class of graphs, a characterization of the $L(3, 2, 1)$-coloring problem is given as an application.

1. Introduction

Several graph coloring problems find their motivations in the task of assigning frequencies to interfering radio and TV transmitters. The task is known as frequency assignment problem (FAP). Since radio signals attenuate over distance, same frequency may be used simultaneously by two transmitters without causing interference if they are at a sufficient distance called reuse distance [1]. In this perspective, the $L(3, 2, 1)$-coloring of graphs finds its relevance where the reuse distance is four. This is an assignment f of non-negative integers on the vertices of a simple finite connected graph $G = (V, E)$ such that $|f(u) - f(v)|$ is at least (i) 3, when $d(u, v) = 1$, (ii) 2, when $d(u, v) = 2$ and (iii) 1, when $d(u, v) = 3$, where $d(u, v)$ in G is the distance between u and v in G. Note that $f(u)$ is known as the color of $u \in V$. The span of f, denoted by *span(f)*, is $(max_{v \in V} f(v) - min_{v \in V} f(v))$. The $\lambda_{3,2,1}$-number of G is $min_f\{span(f) : f \text{ is an } L(3, 2, 1)\text{-coloring of } G\}$, denoted by $\lambda_{3,2,1}(G)$ or simply $\lambda_{3,2,1}$ when there is no confusion regarding the underlying graph. Without loss of generality, we will assume $min_{v \in V} f(v) = 0$ for any $L(3, 2, 1)$-coloring f on G. Any $L(3, 2, 1)$-coloring of G of span $\lambda_{3,2,1}$ is called a $\lambda_{3,2,1}$-coloring of G. A hole in a $\lambda_{3,2,1}$-coloring f is an integer h, $0 < h < \lambda_{3,2,1}$, left unassigned as a color by f.

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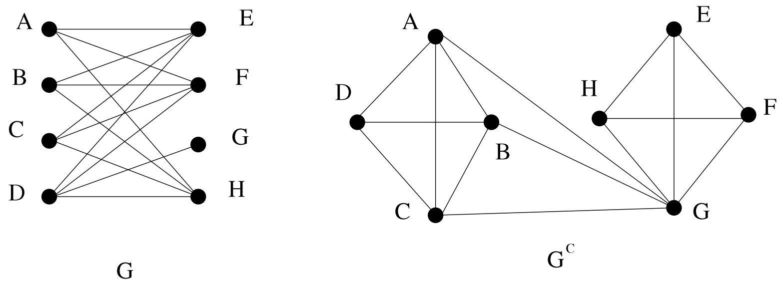

Fig. 1. Illustrating an example of $\hat{\mathfrak{G}}$.

Though some limited results on minimum span of $L(3, 2, 1)$-coloring and related issues on few classes of graphs are known so far (see [1,2,5,7,8] etc.), the study of holes in a $\lambda_{3,2,1}$-coloring and their relationship with different structural properties of a graph has remained an unexplored domain even for the most elementary graphs, to the best of our knowledge.

2. Our contribution

The concepts of holes and related issues have been studied widely for $L(2, 1)$-coloring problem (e.g. see [3,6]). In [4], Georges et al. termed a hole i in an $L(2, 1)$-coloring on a graph as a gap if each of the colors $i-1, i+1$ is assigned to exactly one vertex and these two vertices are adjacent. They showed if an optimal $L(2, 1)$-coloring on a graph G admits no gap, then the complement G^c has a Hamiltonian path. Note that a Hamiltonian path in a graph is a path through all the vertices. Taking a cue from the findings related to gaps in [4], we first probe the phenomenon in the perspective of the $L(3, 2, 1)$-coloring problem. Towards this, we introduce the notion of 2-gap from which a new parameter 2-gap index of a graph has been evolved. Formally, a $\lambda_{3,2,1}$-coloring L on G is said to have a *2-gap* $\{i+1, i+2\}$ if it has two consecutive holes $i+1$ and $i+2$, $0 \leq i \leq \lambda_{3,2,1} - 3$ and $|L_i| = |L_{i+3}| = 1$, where $L_i = \{v \in V : L(v) = i\}$. Further, the minimum number of 2-gaps in any $\lambda_{3,2,1}$-coloring of G is said to be the *2-gap index of* G, denoted by $\hat{\rho}_{3,2,1}(G)$ or simply by $\hat{\rho}_{3,2,1}$ if there is no confusion regarding the underlying graph.

We explore relationship between the 2-gap index of a graph and the existence of a Hamiltonian path in its complement. We state our main results whose proofs will be given in the subsequent sections.

Theorem 2.1. *Let G be a simple finite connected graph such that $\hat{\rho}_{3,2,1}(G) = 0$. Then G^c has a Hamiltonian path.*

In other words, if G admits a $\lambda_{3,2,1}$-coloring without any 2-gap, then G^c has a Hamiltonian path.

Our main focus lies on the converse of this theorem. In fact, proving the converse of Theorem 2.1 is a challenging combinatorial problem. However, we are able to provide a class $\hat{\mathfrak{G}}$ for which the converse of the theorem is true, where $\hat{\mathfrak{G}}$ is the class of C_3-free and C_5-free finite simple connected graphs, having at most $(n-5)$ 5-cycles in their complements. The graph G in Fig. 1 is a member of $\hat{\mathfrak{G}}$. Our main theorem is stated below.

Theorem 2.2. *Let $G \in \hat{\mathfrak{G}}$ be a graph with $n \geq 6$ vertices such that G^c has a Hamiltonian path. Then $\hat{\rho}_{3,2,1}(G) = 0$.*

Finally we conclude our paper by giving some applications of the above results.

3. Preliminaries

Henceforth, throughout the paper we shall assume G to be a simple finite connected graph with n number of vertices unless otherwise stated. A simple observation reveals that a $\lambda_{3,2,1}$-coloring has at most two consecutive holes. A $\lambda_{3,2,1}$-coloring L on G is said to have a *2-hole* $\{i+1, i+2\}$ if L has two consecutive holes $i+1$ and $i+2$, where $0 \leq i \leq span(L) - 3$. Clearly, a 2-gap is a 2-hole in a $\lambda_{3,2,1}$-coloring. Any $\lambda_{3,2,1}$-coloring on G with minimum number of 2-holes will be called a *minimum* $\lambda_{3,2,1}$*-coloring* on G. For any $\lambda_{3,2,1}$-coloring L on a graph G, a color $m \in \{0, 1, \ldots, \lambda_{3,2,1}\}$ is a multiple color if $l_m \geq 2$, where $l_m = |L_m|$.

If the vertices u and v are adjacent in G, we denote this by $u \sim v$ in G, otherwise we write $u \nsim v$ in G. The complete graph, path and cycle on n vertices are denoted by K_n, P_n and C_n respectively. For a $\lambda_{3,2,1}$-coloring L on G, the vertices in L_i are denoted as v_j^i, $1 \leq j \leq l_i$ and for $l_i = 1$, v_1^i is denoted simply as v^i. Let $\mathfrak{G}$ be the class of C_3-free and C_5-free finite simple connected graphs. Then $\hat{\mathfrak{G}}$ is a subclass of $\mathfrak{G}$.

Note that for any $\lambda_{3,2,1}$-coloring on G with a 2-gap $\{i+1, i+2\}$, $v^i \sim v^{i+3}$ in G, because otherwise, reducing the colors of vertices with color $\geq i+3$ by 1 and retaining the colors of other vertices, we get an $L(3, 2, 1)$-coloring of span $\lambda_{3,2,1} - 1$ on G, a contradiction.

We now prove some results which will be required throughout the paper.

Lemma 3.1. *Let L be any $\lambda_{3,2,1}$-coloring on any graph G and i and $i+1$ be two consecutive colors. Let $u \in L_i$ and $v \in L_{i+1}$. Then for any vertex $w \notin \{u, v\}$, $w \sim u$ or $w \sim v$ in G^c.*

Proof. Since $d(u, v) \neq 2$ in G, therefore the proof follows easily. □

Lemma 3.2. *Let L be any $\lambda_{3,2,1}$-coloring on any graph G and i and j be any two colors with $0 \leq |i-j| \leq 2$. Then for $u \in L_i$ and $v \in L_j$, $u \sim v$ in G^c.*

Proof. If not, then we would have $u \sim v$ in G implying $|i-j| \geq 3$, a contradiction. □

4. Proof of Theorem 2.1

Let L be a $\lambda_{3,2,1}$-coloring with no 2-gaps on G. Let $L_i \neq \emptyset$ where $0 \leq i \leq \lambda_{3,2,1}$. Then by Lemma 3.2, the vertices of L_i induce a clique in G^c and so we get a path in G^c of length $l_i - 1$ through all the vertices of L_i. Let this path be $P^{(i)}$ in G^c and the initial, end vertices of $P^{(i)}$ be u_i, v_i respectively.

Case I Let $L_{i+1} \neq \emptyset$. Then by the same reason as above, we get a path $P^{(i+1)}$ in G^c through all the vertices of L_{i+1} with u_{i+1} and v_{i+1} as its initial and end vertices respectively.

By Lemma 3.2, $v_i \sim u_{i+1}$ in G^c. Therefore concatenating the path $P^{(i)}$, the edge (v_i, u_{i+1}) and the path $P^{(i+1)}$, we obtain a new path, say, $Q_1^{(i)}$ in G^c through all the vertices of $L_i \cup L_{i+1}$, with u_i and v_{i+1} as its initial and end vertices respectively.

Case II Let $L_{i+1} = \emptyset$ and $L_{i+2} \neq \emptyset$. Then by repeating the previous argument, there is a path $P^{(i+2)}$ in G^c through the vertices of L_{i+2} with u_{i+2} and v_{i+2} as its initial and end vertices respectively.

By Lemma 3.2, $v_i \sim u_{i+2}$ in G^c. Therefore concatenating the path $P^{(i)}$, the edge (v_i, u_{i+2}) and the path $P^{(i+2)}$, we obtain a new path, say, $Q_2^{(i)}$ in G^c through all the vertices of $L_i \cup L_{i+2}$, with u_i and v_{i+2} as its initial end vertices respectively.

Case III Let $L_{i+1} = \emptyset = L_{i+2}$. Then $L_{i+3} \neq \emptyset$.

Since L has no 2-gap, therefore $l_i \neq 1$ or $l_{i+3} \neq 1$ and so one vertex in L_i must be adjacent to a vertex, say, $u_{i+3} \in L_{i+3}$ in G^c. Without loss of generality, we assume this vertex in L_i to be the end vertex v_i of $P^{(i)}$ in G^c, as described above. Therefore $v_i \sim u_{i+3}$ in G^c.

Using Lemma 3.2, starting from u_{i+3}, we can get a path $P^{(i+3)}$ in G^c through all the vertices of L_{i+3}, with v_{i+3} (say) as its end vertex. Concatenating the path $P^{(i)}$, the edge (v_i, u_{i+3}) and the path $P^{(i+3)}$, we obtain a new path, say, $Q_3^{(i)}$ in G^c through all the vertices of $L_i \cup L_{i+3}$, with u_i and v_{i+3} as its initial and end vertices respectively.

Starting from $i = 0$ as $L_0 \neq \emptyset$, using the above argument repeatedly, we obtain a path in G^c through all the vertices of G^c. Hence G^c has a Hamiltonian path.

5. Preparatory results for proving Theorem 2.2

We divide this section into two parts, first part focusing on $\mathfrak{G}$ and the later on $\hat{\mathfrak{G}}$.

5.1. Positive $\hat{\rho}_{3,2,1}$ inhibits multiple color in a minimum $\lambda_{3,2,1}$-coloring on $G \in \mathfrak{G}$

In this subsection we will focus on the class $\mathfrak{G}$ of C_3-free and C_5-free finite simple connected graphs. But first we need the following result.

Lemma 5.1. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on any graph G. Then $l_{g+4} \leq 1$ and $l_{g-1} \leq 1$.*

Proof. If possible, let $l_{g+4} \geq 2$. For each $u \in L_{g+4}$, $d(u, v^g) \geq 2$ in G, since otherwise we would have $d(u, v^{g+3}) = 2$ in G, a contradiction. Replacing the color of a vertex in L_{g+4} by $g+2$ would yield a new $\lambda_{3,2,1}$ coloring with fewer 2-holes, leading to a contradiction.

By similar argument, we have $l_{g-1} \leq 1$. □

Remark 5.1. Obviously for a minimum $\lambda_{3,2,1}$-coloring L with a 2-gap $\{g+1, g+2\}$ on G, if $l_{g+4} = 1$, then $v^{g+4} \nsim v^g$ and if $l_{g-1} = 1$, then $v^{g-1} \nsim v^{g+3}$ in G.

Now for any minimum $\lambda_{3,2,1}$-coloring L with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$, we consider all four possibilities arising from Lemma 5.1 : (i) $l_{g-1} = 0 = l_{g+4}$, (ii) $l_{g-1} = 1$, $l_{g+4} = 0$, (iii) $l_{g-1} = 0$, $l_{g+4} = 1$ and (iv) $l_{g-1} = 1 = l_{g+4}$. Finally we prove the main result (Theorem 5.1) of this subsection. We will proceed through the following results.

Lemma 5.2. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$ and m be a multiple color (if exists) in L. If $w \in L_m$ is not adjacent to any vertex in L_{g-1} (possibly empty) and L_{g+4} (possibly empty) in G, then w is adjacent to either v^g or v^{g+3} in G.*

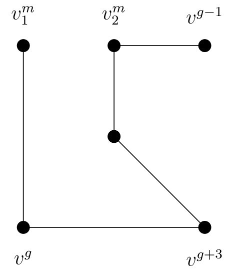

Fig. 2. Illustrating the situation when $v_1^m \sim v^g$ and $d(v_2^m, v^{g+3}) = 2$ in G.

Proof. Let $w \nsim v^g$ in G.

We claim that $d(w, v^g) = 2$ in G. If not, then either $d(w, v^g) = 3$ or $d(w, v^g) \geq 4$ in G.

If possible, let $d(w, v^g) = 3$ in G. Then $w \nsim v^{g+3}$ in G and the color of w can be replaced by $g + 1$ to get a $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction.

Again if possible, let $d(w, v^g) \geq 4$ in G. Then every shortest path between w and v^g in G avoids v^{g+3}, because otherwise we would have $d(w, v^{g+3}) \geq 3$ in G and so we can replace the color of w by $g + 2$ to obtain a $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction. But then again we can replace the color of w by $g + 1$ to obtain a new $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction.

Hence $d(w, v^g) = 2$ in G.

If possible let $w \nsim v^{g+3}$ in G. As G is C_3-free and C_5-free, so $d(w, v^{g+3}) \neq 2$ and therefore $d(w, v^{g+3}) = 3$ in G. Now the color of w can be replaced by $g + 2$ to produce a $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction. Hence $w \sim v^{g+3}$ in G.

This completes the proof. □

Corollary 5.3. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g + 1, g + 2\}$ on a graph $G \in \mathfrak{G}$ such that $l_{g-1} = 0 = l_{g+4}$. Then L has no multiple color.*

Proof. The proof immediately follows from the Lemma 5.2. □

Lemma 5.4. *Let $\{g + 1, g + 2\}$ be a 2-gap and $l_{g-1} = 1$ in a minimum $\lambda_{3,2,1}$-coloring L on any graph G. Then $l_{g-2} = 0$.*

Proof. If possible, let $l_{g-2} > 0$. Let $u = v^{g-1}$. Since $v^{g-1} \nsim v^{g+3}$ in G, replacing the color of u by $g + 1$, we have a new $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction. □

Proposition 5.5. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$ such that $l_{g-1} = 1,\ l_{g+4} = 0$. Then L has no multiple color.*

Proof. If possible, let L have a multiple color m. Then there is a $\hat{w} \in L_m$ such that $\hat{w} \sim v^{g-1}$ in G, by Lemma 5.2.

If possible, let $d(\hat{w}, v^{g+3}) \geq 3$ in G. Since $l_{g+4} = 0$ and $\hat{w} \nsim v^g$ in G, replacement of $L(\hat{w})$ by $g + 2$ to obtain a new $\lambda_{3,2,1}$-coloring leads to a contradiction because the new coloring has fewer 2-holes. Hence $d(\hat{w}, v^{g+3}) \leq 2$ in G.

Again, since m is a multiple color, $L_m \setminus \{\hat{w}\} \neq \emptyset$. Let $w \in L_m \setminus \{\hat{w}\}$. Then we should have $w \sim v^g$ or $w \sim v^{g+3}$ in G, by Lemma 5.2. Hence $|L_m \setminus \{\hat{w}\}| = 1$, i.e., $L_m = \{w,\ \hat{w}\}$. Now $w \sim v^{g+3}$ and $d(\hat{w}, v^{g+3}) \leq 2$ in G would together imply $d(\hat{w}, w) \leq 3$ in G, a contradiction. So $w \nsim v^{g+3}$ and hence $w \sim v^g$ in G. Also $d(\hat{w}, v^{g+3}) \neq 1$, i.e., $d(\hat{w}, v^{g+3}) = 2$, since otherwise $d(\hat{w}, w) = 3$ in G, a contradiction. (See Fig. 2 with $w = v_1^m$ and $\hat{w} = v_2^m$.)

Now, $v^{g-1} \nsim v^{g+3}$ and $v^{g-1} \sim \hat{w}$ in G. So, $2 \leq d(v^{g-1}, v^{g+3}) \leq 3$ in G. But $d(v^{g-1}, v^{g+3}) = 2$ would imply G has C_3 or C_5 as its subgraph, a contradiction. So $d(v^{g-1}, v^{g+3}) = 3$. Replacing $L(v^{g-1})$ by $g + 2$, we get a $\lambda_{3,2,1}$-coloring $\hat{L}$ with $\hat{l}_{g-1} = 0$. Also by Lemma 5.4, $\hat{l}_{g-2} = l_{g-2} = 0$. Therefore $\hat{l}_{g-3} \neq 0$. Thus $\hat{L}$ is a $\lambda_{3,2,1}$-coloring with a 2-hole $\{g - 2, g - 1\}$. Also the number of 2-holes in $\hat{L}$ is same as that of L, since $\hat{l}_{g+2} = 1$. So $\hat{L}$ is a minimum $\lambda_{3,2,1}$-coloring with a 2-hole $\{g - 2, g - 1\}$ and $\hat{l}_g = 1$.

If possible, let $\hat{l}_{g-3} \geq 2$. Then there is a vertex $u \in \hat{L}_{g-3}$ such that $u \nsim v^g$ in G, since $\hat{l}_g = 1$. Replacing the color of u by $g - 2$ gives a new $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction. Hence $\hat{l}_{g-3} = 1$, i.e., $\{g - 2, g - 1\}$ is a 2-gap in $\hat{L}$.

Also $\hat{l}_{g+1} = 0$, $\hat{l}_{g+2} = 1$. Since m is a multiple color, $\hat{l}_{g-4} = 1$, by Corollary 5.3. Now $\hat{l}_m = l_m = 2$ and repeating the argument as above, we have $v_i^m \sim v^{g-3}$, $v_i^m \nsim v^g$, $v_j^m \sim v^{g-4}$, $v_j^m \nsim v^{g-3}$, $d(v_j^m,\ v^g) = 2$ and $d(v^{g-4},\ v^g) = 3$ in G. (See Fig. 3.)

Therefore either $v_i^m = v_1^m$ requiring a C_3 or $v_i^m = v_2^m$ implying $d(v_1^m, v_2^m) = 3$ in G, a contradiction. Hence L has no multiple color. □

Corollary 5.6. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g + 1, g + 2\}$ on a graph $G \in \mathfrak{G}$. If $\lambda_{3,2,1} = g + 3$, then L has no multiple color.*

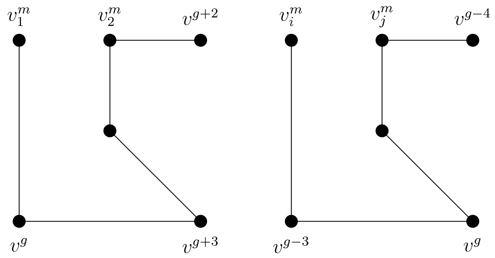

Fig. 3. Replacement of the color $g-1$ by $g+2$ together with the 2-gap $\{g-3,\ g\}$ in the new coloring.

Proof. Clearly, $l_{g+4}=0$. By Corollary 5.3 and Proposition 5.5, L has no multiple color. □

Lemma 5.7. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$. Then $\lambda_{3,2,1} \neq g+4$.*

Proof. If possible, let $\lambda_{3,2,1}=g+4$. By Lemma 5.1, $l_{g+4}=1$. As $v^{g+4} \nsim v^g$ in G, replacing $L(v^{g+4})$ by $g+2$ gives a new $L(3,2,1)$-coloring with span $\lambda_{3,2,1}-1$, a contradiction. □

Lemma 5.8. *Let $\{g+1, g+2\}$ be a 2-gap and $l_{g+4}=1$ in a minimum $\lambda_{3,2,1}$-coloring L on any graph G. Then $l_{g+5}=0$.*

Proof. If possible, let $l_{g+5}>0$. As $v^{g+4} \nsim v^g$ in G, replacing $L(v^{g+4})$ by $g+2$ gives a $\lambda_{3,2,1}$-coloring on G with fewer 2-holes, a contradiction. □

Proposition 5.9. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$ such that $l_{g-1}=0$, $l_{g+4}=1$. Then L has no multiple color.*

Proof. Using Lemma 5.8, argument similar to the proof of Proposition 5.5 is needed. □

The following lemma is required for considering the last possibility: $l_{g-1}=1=l_{g+4}$.

Lemma 5.10. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap $\{g+1, g+2\}$ on a graph $G \in \mathfrak{G}$. If $l_{g+4}=1$, then $\lambda_{3,2,1} \geq g+6$ and $l_{g+6}=1$.*

Proof. By Lemma 5.7, $\lambda_{3,2,1} \geq g+5$. But by Lemma 5.8, $l_{g+5}=0$. Therefore we should have $\lambda_{3,2,1} \geq g+6$.

We replace $L(v^{g+4})$ by $g+2$ to get a new $\lambda_{3,2,1}$-coloring $\hat{L}$ with holes $g+4$ and $g+5$. So we have $\hat{l}_{g+6}>0$. The number of 2-holes in $\hat{L}$ is same as that of L and hence $\hat{L}$ is also a minimum $\lambda_{3,2,1}$-coloring.

If possible, let $\hat{l}_{g+6} \geq 2$. Then there is a vertex $u \in \hat{L}_{g+6}$ such that $u \nsim v^{g+3}$ in G, as $\hat{l}_{g+3}=1$. Replacing the color of u by $g+5$ gives a $\lambda_{3,2,1}$-coloring with fewer 2-holes, a contradiction. Hence $\hat{l}_{g+6}=1$. So $l_{g+6}=1$ in L, as color of v^{g+6} in L remains unaltered in $\hat{L}$. □

Proposition 5.11. *Let L be a minimum $\lambda_{3,2,1}$-coloring on a graph $G \in \mathfrak{G}$ with a 2-gap $\{g+1,\ g+2\}$ such that $l_{g-1}=1=l_{g+4}$. Then L has no multiple color.*

Proof. By Lemma 5.8, $l_{g+5}=0$ and by Lemma 5.10, $l_{g+6}=1$. Since $v^{g+4} \nsim v^g$ in G, replacing $L(v^{g+4})$ by $g+2$ gives a new $\lambda_{3,2,1}$-coloring, say, $L^{(1)}$ of G with a new 2-gap $\{g+4,\ g+5\}$ and $l^{(1)}_{g+2}=1$. Thus the number of 2-gaps in $L^{(1)}$ is same as that of L and so $L^{(1)}$ is also a minimum $\lambda_{3,2,1}$-coloring on G.

If $l^{(1)}_{g+7}=0$, then $L^{(1)}$ and hence L has no multiple color, by Proposition 5.5.

If not, then $l^{(1)}_{g+7}=1$, by Lemma 5.1. By Lemma 5.8, $l^{(1)}_{g+8}=0$ and by Lemma 5.10, $l^{(1)}_{g+9}=1$. As $\{g+4,\ g+5\}$ is a 2-gap in $L^{(1)}$, so $v^{g+3} \sim v^{g+6}$ and hence $v^{g+7} \nsim v^{g+3}$ in G. Therefore, replacing $L^{(1)}(v^{g+7})$ by $g+5$ gives a new $\lambda_{3,2,1}$-coloring, say, $L^{(2)}$ of G with a new 2-gap $\{g+7,\ g+8\}$ and $l^{(2)}_{g+5}=1$. Arguing similarly as before, we get $L^{(2)}$ is also a minimum $\lambda_{3,2,1}$-coloring on G.

If $l^{(2)}_{g+10}=0$, then $L^{(2)}$ and hence L have no multiple color, by Proposition 5.5.

If not, then $l^{(2)}_{g+10}=1$, by Lemma 5.1. We repeat the argument as before and noting that G is a finite graph, after a finite number of repetitions, we get a minimum $\lambda_{3,2,1}$-coloring, say, $L^{(i)}$ of G with a new 2-gap $\{g+1+3i,\ g+2+3i\}$ and $l^{(i)}_{g-1+3i}=1$, $l^{(i)}_{g+4+3i}=0$, for some integer $i \geq 1$. By Proposition 5.5, $L^{(i)}$ and hence L have no multiple color. □

We now prove the main result of this subsection.

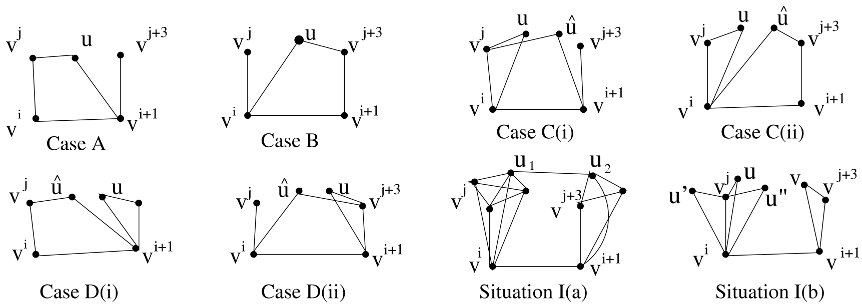

Fig. 4. Illustrating different cases in the proof of Proposition 5.12.

Theorem 5.1. *Let L be a minimum $\lambda_{3,2,1}$-coloring with a 2-gap on a graph $G \in \mathfrak{G}$. Then L has no multiple color.*

Proof. Proof directly follows from Corollary 5.3, Propositions 5.5, 5.9 and 5.11. □

5.2. Positive $\hat{\rho}_{3,2,1}$ inhibits consecutive colors in a minimum $\lambda_{3,2,1}$-coloring on $G \in \hat{\mathfrak{G}}$

After proving the following result for $\mathfrak{G}$, we will exploit it for $\hat{\mathfrak{G}}$ where $\hat{\mathfrak{G}} = \{G \in \mathfrak{G} : G^c$ has at most $(n-5)$ distinct 5-cycles$\}$.

Proposition 5.12. *Let L be a minimum $\lambda_{3,2,1}$-coloring on a graph $G \in \mathfrak{G}$ with $n \geq 6$ vertices and $\hat{\rho}_{3,2,1}(G) > 0$. If L has consecutive colors, then G^c has at least $(n-4)$ distinct 5-cycles.*

Proof. Let $i,\ i+1$ be two consecutive colors in L. Since $\hat{\rho}_{3,2,1}(G) > 0$, L has a 2-gap, say, $\{j+1,\ j+2\}$. By Theorem 5.1, L has no multiple color. Since $v^j \nsim v^{j+3}$ in G^c, every other vertex is adjacent to v^j or v^{j+3} in G^c, otherwise G would have a triangle, a contradiction. Again by Lemma 3.1, for every $w \in \{v^j,\ v^{j+3}\}$, $w \sim v^i$ or $w \sim v^{i+1}$ in G^c. Also, $v^i \sim v^{i+1}$ in G^c, by Lemma 3.2. Thus we get a P_4 in G^c with $v^j,\ v^{j+3}$ as end vertices and $v^i,\ v^{i+1}$ as internal vertices. Without loss of generality, let that path be $P:\ v^j,\ v^i,\ v^{i+1},\ v^{j+3}$. Let $A = \{v^j,\ v^{j+3},\ v^i,\ v^{i+1}\}$. Since $n \geq 6$, G^c has at least two more vertices.

Now, let u be any vertex in $V \setminus A$. If $u \sim v^j$ in G^c, we call u to be a type I vertex and if $u \sim v^{j+3}$ in G^c, we call u to be a type II vertex.

Again, a type I vertex $u \in V \setminus A$ is of two kinds: (i) $u \nsim v^i$ in G^c and we call it a type I(a) vertex; (ii) $u \sim v^i$ in G^c and we call it a type I(b) vertex. Note that for a type I(a) vertex u, we must have $u \sim v^{i+1}$ in G^c, by Lemma 3.1.

Similarly, a type II vertex $u \in V \setminus A$ is also of two kinds: (i) $u \nsim v^{i+1}$ in G^c and we call it a type II(a) vertex; (ii) $u \sim v^{i+1}$ in G^c and we call it a type II(b) vertex. Note that for a type II(a) vertex u, we must have $u \sim v^i$ in G^c, by Lemma 3.1.

We now consider the following cases.

Case A: Let $u \in V \setminus A$ be a type I(a) vertex. Since $\{u,\ v^i,\ v^{j+3}\}$ cannot be an independent set and $u \nsim v^i$ in G^c, so we have $u \sim v^{j+3}$ or $v^i \sim v^{j+3}$ in G^c. Hence u and vertices of A together form a C_5 in G^c. (See Fig. 4.)

Case B: Let $u \in V \setminus A$ be a type II(a) vertex. Using similar argument, it can be shown that u and vertices of A together form a C_5 in G^c. (See Fig. 4.)

Case C: Let $u \in V \setminus A$ be type I(b). Now G^c has at least one more vertex in $V \setminus A$.

Case C(i): Let $\exists\, \hat{u} \in V \setminus (A \cup \{u\})$ which is type I(a) (see Fig. 4). Then u, along with $\hat{u}$ and vertices of $A \setminus \{v^{j+3}\}$, forms a C_5 in G^c.

Case C(ii): Let $\exists\, \hat{u} \in V \setminus (A \cup \{u\})$ which is type II(a) (see Fig. 4). Since $\{u,\ \hat{u},\ v^{i+1}\}$ cannot be an independent set and $\hat{u} \nsim v^{i+1}$ in G^c, therefore $u \sim \hat{u}$ or $u \sim v^{i+1}$ in G^c. Hence u, along with $\hat{u}$ and vertices of $A \setminus \{v^j\}$, forms a C_5 in G^c.

Case C(iii): Let every vertex in $V \setminus (A \cup \{u\})$ is type I(b) or II(b). We will handle this case later.

Case D: Let $u \in V \setminus A$ be type II(b). Now G^c has at least one more vertex in $V \setminus A$.

Case D(i): Let $\exists\, \hat{u} \in V \setminus (A \cup \{u\})$ which is type I(a) (see Fig. 4). Since $\{u,\ \hat{u},\ v^i\}$ cannot be an independent set and $\hat{u} \nsim v^i$ in G^c, therefore $u \sim \hat{u}$ or $u \sim v^i$ in G^c. Hence u, along with $\hat{u}$ and vertices of $A \setminus \{v^{j+3}\}$, forms a C_5 in G^c.

Case D(ii): Let $\exists\, \hat{u} \in V \setminus (A \cup \{u\})$ which is type II(a) (see Fig. 4). Then u, along with $\hat{u}$ and vertices of $A \setminus \{v^j\}$, forms a C_5 in G^c.

Case D(iii): Let every vertex in $V \setminus (A \cup \{u\})$ is type I(b) or II(b). We shall consider Case *C(iii)* and *Case D(iii)* together. Hence we can assume that all the vertices in $V \setminus A$ is type I(b) or II(b). Now three situations may occur.

Situation I: Let there exist at least one type I(b) vertex and at least one type II(b) vertex.

Situation I(a): Let the type I(b) vertices together form a clique and the type II(b) vertices together form a clique in G^c. Then no type I(b) vertex can be simultaneously a type II(b) vertex, because otherwise G would have an isolated vertex, a contradiction. Let $X = \{v : v$ is a type I(b) vertex$\} \cup \{v^j\}$ and $Y = \{v : v$ is a type II(b) vertex$\} \cup \{v^{j+3}\}$.

If possible, let there be no edge in G^c between any vertex in X and any vertex in Y. Then any edge in G^c between $X \cup \{v^i\}$ and $Y \cup \{v^{i+1}\}$ is incident to v^i or v^{i+1}. But all the vertices of X cannot be adjacent to v^{i+1}, because otherwise v^{i+1} would be an isolated vertex in G. Similarly all the vertices of Y cannot be adjacent to v^i.

Now G is non-complete bipartite graph with $X \cup \{v^i\}$ and $Y \cup \{v^{i+1}\}$ as two partite sets. Also, the set $X \cup Y$ induces a complete bipartite subgraph in G. Again $d(x_1, x_2) = 2 = d(y_1, y_2)$ in G, for every $x_1,\ x_2$ in $X \cup \{v^i\}$ and for every $y_1,\ y_2$ in $Y \cup \{v^{i+1}\}$. Also, for any $p \in X \cup \{v^i\}$ with $p \nsim v^{i+1}$ in G, $d(p, v^{i+1}) = 3$ in G. Again, for any $q \in Y \cup \{v^{i+1}\}$ with $q \nsim v^i$ in G, $d(q, v^i) = 3$ in G. Hence diameter of G is 3 and so no $\lambda_{3,2,1}$-coloring of G can have multiple color. The structure of G indicates that any $\lambda_{3,2,1}$-coloring L of G satisfies the following properties: (i) $|L(x) - L(y)| \geq 3$, for any $x \in X, y \in Y$; (ii) $|L(x_1) - L(x_2)| \geq 2$, for any $x_1,\ x_2 \in X \cup \{v^i\}$; (iii) $|L(y_1) - L(y_2)| \geq 2$, for any $y_1,\ y_2 \in Y \cup \{v^{i+1}\}$; (iv) $L(p) \neq L(v^{i+1})$, for any $p \in X \cup \{v^i\}$ with $p \nsim v^{i+1}$ in G and (v) $L(q) \neq L(v^i)$, for any $q \in Y \cup \{v^{i+1}\}$ with $q \nsim v^i$ in G. Properties (iv) and (v) facilitate the existence of a $\lambda_{3,2,1}$-coloring of G without consecutive holes. Hence $\hat{\rho}_{3,2,1}(G) = 0$, a contradiction.

Hence we must have an edge in G^c between a vertex in X and a vertex in Y. Let v^j or a type I(b) vertex, say u_1, be adjacent to a type II(b) vertex, say u_2, or v^{j+3} in G^c (see Fig. 4). Note that $v^j \nsim v^{j+3}$ in G^c.

If $u_1 \sim u_2$ in G^c, then each of the remaining type I(b) and type II(b) vertices produces a C_5 in G^c along with $u_1,\ u_2,\ v^i$ and v^{i+1}. Also, u_1, along with u_2 and vertices in $A \setminus \{v^{j+3}\}$, yields a C_5 in G^c. Again u_2, along with u_1 and vertices in $A \setminus \{v^j\}$, yields a C_5 in G^c. Hence each vertex of $V \setminus A$ corresponds to a distinct C_5 in G^c.

Situation I(b): Let type I(b) or type II(b) vertices do not form a clique in G^c. Without loss of generality, we assume that $u,\ u'$ be two non-adjacent type I(b) vertices in G^c (see Fig. 4). Since $\{u,\ u',\ v^{i+1}\}$ cannot be an independent set and $u \nsim u'$ in G^c, therefore $v^{i+1} \sim u$ or $v^{i+1} \sim u'$ in G^c. Then we get a C_5 in G^c involving $u,\ u'$ and vertices of $A \setminus \{v^{j+3}\}$. Again since $\{u,\ u',\ v^{j+3}\}$ cannot be an independent set and $u \nsim u'$ in G^c, $v^{j+3} \sim u$ or $v^{j+3} \sim u'$ in G^c. Accordingly we get a C_5 in G^c involving A and u or u'. Thus we get two distinct C_5 in G^c (for every pair of non-adjacent type I(b) vertices).

If there is another type I(b) vertex u'' other than $u,\ u'$, then it must be adjacent to at least one of $u,\ u'$. Hence u'', along with $u,\ u',\ v^j,\ v^i$, yields a C_5 in G^c. Again, if v be any type II(b) vertex, then $v \sim u$ or $v \sim u'$, as $u \nsim u'$ in G^c. Accordingly, v, along with the vertices of $A \setminus \{v^j\}$ and u or u', forms a C_5 in G^c. Hence each vertex in $V \setminus A$ corresponds to a distinct C_5 in G^c.

Situation II: Let each vertex in $V \setminus A$ be type I(b). Proof is analogous to *Situation I*.

Situation III: Let each vertex in $V \setminus A$ be type II(b). Proof is analogous to *Situation I*.

Hence every vertex in $V \setminus A$ corresponds to a distinct C_5 in G^c. Therefore G^c has at least $(n - 4)$ distinct 5-cycles. □

Theorem 5.2. *Let L be a minimum $\lambda_{3,2,1}$-coloring on a graph $G \in \hat{\mathfrak{G}}$ with $n \geq 6$ vertices and $\hat{\rho}_{3,2,1}(G) > 0$. Then L has no consecutive colors.*

Proof. Proof directly follows from Proposition 5.12. □

6. Proof of Theorem 2.2

We first prove the following two results.

Lemma 6.1. *Let G^c have a Hamiltonian path. Then $\lambda_{3,2,1}(G) \leq 2n - 2$.*

Proof. Let $P_n :\ u_1,\ u_2, \ldots, u_n$ be a Hamiltonian path in G^c. Define a coloring L on G given by $L(u_i) = 2i - 2$, for $1 \leq i \leq n$. Clearly L is an $L(3,\ 2,\ 1)$-coloring on G with span $2n - 2$ and so $\lambda_{3,2,1}(G) \leq 2n - 2$. □

Lemma 6.2. *Let $\hat{\rho}_{3,2,1}(G) > 0$ for a graph $G \in \hat{\mathfrak{G}}$ with $n \geq 6$ vertices. Then $\lambda_{3,2,1}(G) \geq 2n - 1$.*

Proof. Let L be any minimum $\lambda_{3,2,1}$-coloring on G and ρ be the number of holes in L. Since $\hat{\rho}_{3,2,1} > 0$, therefore L has a 2-gap and so by Theorem 5.1, L has no multiple color. Hence the vertices of G can be identified with their respective colors. Since L has no consecutive colors by Theorem 5.2, any two colors, one of which is immediately following the other, must be separated by either a single hole or two consecutive holes. But L has n distinct colors and at least one 2-gap. Hence $n - 1 \leq \rho - 1$, i.e., $n \leq \rho$. As the minimum color assigned by L is 0, so $\lambda_{3,2,1} = span(L) = n + \rho - 1 \geq 2n - 1$. □

Proof of Theorem 2.2 follows directly from Lemmas 6.1 and 6.2.

7. Applications

The following two results together characterize the $L(3,\ 2,\ 1)$-coloring problem for $\hat{\mathfrak{G}}$.

Theorem 7.1. *For a graph $G \in \hat{\mathfrak{G}}$ with $n \geq 6$ vertices, $\hat{\rho}_{3,2,1}(G) = 0$ if and only if $\lambda_{3,2,1}(G) \leq 2n - 2$.*

Proof. Proof follows directly from Theorem 2.1, Lemmas 6.1 and 6.2. □

The following gives a closed formula for $\lambda_{3,2,1}$ when $\hat{\rho}_{3,2,1} > 0$ for a graph $G \in \hat{\mathfrak{G}}$.

Theorem 7.2. *For a graph $G \in \hat{\mathfrak{G}}$ with $n \geq 6$ vertices, let $\hat{\rho}_{3,2,1}(G) > 0$. Then $\lambda_{3,2,1}(G) = 2n - 2 + \hat{\rho}_{3,2,1}(G)$.*

Proof. Since $\hat{\rho}_{3,2,1}(G) > 0$, every minimum $\lambda_{3,2,1}$-coloring on G has no multiple color, by Theorem 5.1, and hence each 2-hole is a 2-gap in it. Therefore the minimum number of 2-holes in any $\lambda_{3,2,1}$-coloring on G is $\hat{\rho}_{3,2,1}(G)$.

Let L be a minimum $\lambda_{3,2,1}$-coloring on G and ρ be the number of holes in L. Then using the argument in the proof of Lemma 6.2, we have $\rho \geq n$.

Let $\rho = n - 1 + t$. By Theorem 5.2, L has no consecutive colors. Also L has at most two consecutive holes. Therefore t is equal to the number of 2-holes in $L = \hat{\rho}_{3,2,1}(G)$ and hence $\rho = n-1+\hat{\rho}_{3,2,1}(G)$. Therefore $\lambda_{3,2,1} = span(L) = n+\rho-1 = 2n - 2 + \hat{\rho}_{3,2,1}(G)$. □

Corollary 7.1. *For any two graphs $G_1, G_2 \in \hat{\mathfrak{G}}$ both with $n \geq 6$ vertices, let $\lambda_{3,2,1}(G_1) = \lambda_{3,2,1}(G_2)$. Then $\hat{\rho}_{3,2,1}(G_1) = \hat{\rho}_{3,2,1}(G_2)$.*

Proof. The proof immediately follows from Theorems 7.1 and 7.2. □

Corollary 7.2. *For any two graphs $G_1, G_2 \in \hat{\mathfrak{G}}$ both with $n \geq 6$ vertices, let $\hat{\rho}_{3,2,1}(G_1) = \hat{\rho}_{3,2,1}(G_2)$. Then either $\lambda_{3,2,1}(G_i) \leq 2n - 2$, for $i = 1, 2$, or $\lambda_{3,2,1}(G_1) = \lambda_{3,2,1}(G_2)$.*

Proof. The proof immediately follows from Theorems 7.1 and 7.2. □

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