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On extensions of topological spaces in terms of ideals

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Abstract

In the present paper, a kind of extension, termed ideal extension of a given topological space is considered via the concept of ideals. A general method of construction of such an extension of a T_0—space is worked out and it is finally shown that under certain condition imposed on the ideals involved, the said extension space turns out to be the compactification of a given space.

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1. Introduction

There have been great many attempts, so far, by topologists to use the concept of ideals for manoeuvring investigations of different problems of topology. In this connection one may refer to the works in [1,2,4,5].

In this paper, our endeavour is to study a certain extension problem via ideals. In the next section, we shall introduce the notion of strength system of an extension of a topological space and show that it can equivalently be arrived at in terms of a type of ideals. It is shown that two compactifications of a Tychonoff space, with identical strength systems, are equivalent.

Our principle aim of this paper is to define a kind of extension, termed ideal extension, of a T_0-space. We show that each such extension is topologically equivalent to a suitable extension constructed in terms of a class of ideals. The latter extension, apart from giving a general method of construction of an ideal extension, becomes a compactification of the given space under certain condition involving ideals.

In what follows, by a space X or simply by X we mean a topological space. We know that in any topological space X, the mapping $c:\mathcal{P}(X)\to\mathcal{P}(X)$ ($\mathcal{P}(X)$ denoting the power set of X) given by $c(A)=\overline{A}$ (the closure of A) for any $A\subseteq X$, is a Kuratowski closure operator on $\mathcal{P}(X)$. In our discussions, we shall, in general, denote a topological space X by the pair (X,c), where c is the Kuratowski closure operator inducing the topology of the space.

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2. Strength system of an extension

We start by recalling that a subcollection I of the power $\mathcal{P}(X)$ of a set X is called an ideal [3] on X if

(i) for $A, B \subseteq X$, $A \subseteq B \in I \Rightarrow A \in I$ and
(ii) $A, B \in I \Rightarrow A \cup B \in I$.

Definition 2.1. For a space (X, c) and $x \in X$, $I_c(x) = \{A \subseteq X\colon\ x \notin c(A)\}$ is an ideal on X, called an adherence free or simply a free ideal on (X, c).

Proposition 2.2. *A space (X, c) is T_0 iff for any two distinct points x, y of X, $I_c(x) \neq I_c(y)$.*

Proof. Let x, y be distinct points in a T_0-space (X, c). Then we may assume that $x \notin c(\{y\})$, which gives $\{y\} \in I_c(x)$, but $\{y\} \notin I_c(y)$. Thus $I_c(x) \neq I_c(y)$.

Conversely, let $x, y \in X$ with $x \neq y$. Then $I_c(x) \neq I_c(y)$. Suppose $A \subseteq X$ is such that $A \in I_c(x)$ but $A \notin I_c(y)$. Then $x \notin c(A)$ and $y \in c(A)$ for some $A \subseteq X$. Thus $x \in X \setminus c(A)$ but $y \notin X \setminus c(A)$, proving (X, c) to be T_0. □

Definition 2.3. An ideal I on a space (X, c) is called a c-ideal if for all $A \subseteq X$, $A \in I \Rightarrow c(A) \in I$.

Remark 2.4. Every free ideal is a c-ideal.

By an extension of a space (X, c) we mean, as usual, a pair $E \equiv (\psi, (Y, k))$ (with k the closure operator on Y), where ψ is a homeomorphism of (X, c) onto a subspace of (Y, k) such that $k(\psi(X)) = Y$. Two extensions $E_1 \equiv (\psi_1, (Y_1, k_1))$, and $E_2 \equiv (\psi_2, (Y_2, k_2))$ of a topological space X are said to be (topologically) equivalent if there exists a homeomorphism h of (Y_1, k_1) onto (Y_2, k_2) with $h \circ \psi_1 = \psi_2$.

We now introduce the following concept.

Definition 2.5. Let $E \equiv (\psi, (Y, k))$ be an extension of a space (X, c). Then the strength $S(y, E)$ of an arbitrary point y of Y on (X, c) is defined by

$$S(y, E) = \bigl\{A \subseteq X\colon\ y \notin k\bigl(\psi(A)\bigr)\bigr\} = \bigl\{A \subseteq X\colon\ \psi(A) \in I_k(y)\bigr\}.$$

Clearly $S(y, E)$ is a c-ideal on (X, c).

We shall call the collection $\widehat{X} = \{S(y, E)\colon\ y \in Y\}$ the strength system of E.

Proposition 2.6. *Let $E \equiv (\psi, (Y, k))$ be an extension of a space (X, c). Then for each $x \in X$, $S(\psi(x), E) = I_c(x)$.*

Proof. Indeed, $S(\psi(x), E) = \{A \subseteq X\colon\ \psi(x) \notin k\psi(A)\} = \{A \subseteq X\colon\ \psi(x) \notin k\psi(A) \cap \psi(X)\} = \{A \subseteq X\colon\ \psi(x) \notin \psi(c(A))\} = \{A \subseteq X\colon\ x \notin c(A)\} = I_c(x)$. □

Theorem 2.7. *Two equivalent extensions of a topological space have identical strength systems.*

Proof. Let $E_1 \equiv (\psi_1, (Y_1, k_1))$ and $E_2 \equiv (\psi_2, (Y_2, k_2))$ be two equivalent extensions of a space (X, c). Then there exists a homeomorphism h of (Y_1, k_1) onto (Y_2, k_2) with $h \circ \psi_1 = \psi_2$. We show that for each point $y \in Y_1$, $S(y, E_1) = S(h(y), E_2)$. Indeed, $A \in S(y, E_1) \Leftrightarrow y \notin k_1\psi_1(A) \Leftrightarrow h(y) \notin hk_1\psi_1(A) = k_2h\psi_1(A) = k_2\psi_2(A) \Leftrightarrow A \in S(h(y), E_2)$. □

Henceforth by a compactification αX of a Tychonoff space X, we shall mean an extension $(\alpha, (Y, k))$, where $Y = \alpha X$ is a compact space, $\alpha : X \to \alpha(X)$ is a homeomorphism and $k(\alpha(X)) = Y$. In this connection, we shall denote $k(B)$ (for $B \subseteq Y$) by $\overline{B}$.

Theorem 2.8. *The strengths of different points of any compactification αX of a Tychonoff space X are different.*

Proof. Let y_1, y_2 be two distinct points of αX. Then there are disjoint open sets G and H in αX containing y_1 and y_2 respectively. Clearly, $y_2 \notin \overline{G \cap \alpha(X)}$. Take $A = \alpha^{-1}(G)$. Then $\alpha(A) = G \cap \alpha(X)$, so that $y_2 \notin \overline{\alpha(A)}$ and hence $A \in S(y_2, \alpha X)$. But $y_1 \in \overline{\alpha(A)} \Rightarrow A \notin S(y_1, \alpha X)$. Thus $S(y_1, \alpha X) \neq S(y_2, \alpha X)$. □

Remark 2.9. It follows from the proof of the above theorem that the strengths of distinct points of any Hausdorff extension of a Hausdorff space are different.

As to the converse of Theorem 2.7 we have:

Theorem 2.10. *Let αX and γX be two compactifications of a Tychonoff space X with identical strength systems. Then αX and γX are topologically equivalent.*

Proof. According to the given condition we have, $\{S(y, \alpha X): y \in Y\} = \{S(z, \gamma X): z \in \gamma X\}$. Then Theorem 2.8 assures that none of the elements in either set is repeated. Therefore for each point $y \in \alpha X$, we can associate a unique point $z \in \gamma X$ such that $S(y, \alpha X) = S(z, \gamma X)$. This defines a mapping $f : \alpha X \to \gamma X$ given by $f(y) = z$. Thus f is a bijective map of αX onto γX and for each $y \in \alpha X$, $S(y, \alpha X) = S(f(y), \gamma X) \ldots$ (i). In particular for any point $x \in X$, $S(\alpha(x), \alpha X) = S(f(\alpha(x)), \gamma X)$, i.e., $I(x)$ ($\equiv$ the free ideal in X corresponding to the point x) $= S(f(\alpha(x)), \gamma X)$. But $I(x) = S(\gamma(x), \gamma X)$. Therefore, $S(f(\alpha(x)), \gamma X) = S(\gamma(x), \gamma X) \Rightarrow f(\alpha(x)) = \gamma(x)$, for any $x \in X$, i.e., $f \circ \alpha = \gamma$.

To complete the proof we need only to show that f is a homeomorphism of αX onto γX. In fact, from (i) we have for any $A \subseteq X$, $y \notin \overline{\alpha(A)} \Leftrightarrow f(y) \notin \overline{\gamma(A)}$, i.e., $f(\overline{\alpha(A)}) = \overline{\gamma(A)}$. Now $\{\overline{\alpha(A)}: A \subseteq X\}$ and $\{\overline{\gamma(A)}: A \subseteq X\}$ constitute closed bases for the compact spaces αX and γX respectively. Therefore the basic closed subsets of the topological spaces αX and γX correspond to each other in a one–one manner via the bijective map. This shows that f is a homeomorphism of αX onto γX. □

3. Ideal extensions

Definition 3.1. An extension $E \equiv (\psi, (Y, k))$ of a topological space (X, c) is said to be an ideal extension if

(i) any two distinct points of Y have different strengths, and
(ii) $\{k\psi(A): A \subseteq X\}$ is a base for the closed sets in (Y, k).

Remark 3.2. (a) It follows from the definition that a topological space (X, c) which admits an ideal extension $E \equiv (\psi, (Y, k))$ is necessarily T_0. In fact, for any $x_1, x_2 \in X$, $I_c(x_1) = I_c(x_2) \Rightarrow S(\psi(x_1), E) = S(\psi(x_2), E)$ (by Proposition 2.6) $\Rightarrow \psi(x_1) = \psi(x_2)$. The rest follows from Proposition 2.2.

(b) Any ideal extension $(\psi, (Y, k))$ of a T_0 topological space (X, c) is also a T_0 space. In fact, for any $y_1, y_2 \in Y$, $I_k(y_1) = I_k(y_2) \Rightarrow S(y_1, E) = \{A \subseteq X: y_1 \notin k\psi(A)\} = \{A \subseteq X: \psi(A) \in I_k(y_1)\} = \{A \subseteq X : \psi(A) \in I_k(y_2)\} = \{A \subseteq X: y_2 \notin k\psi(A)\} = S(y_2, E) \Rightarrow y_1 = y_2$, proving (by Proposition 2.2) that (Y, k) is T_0.

(c) As Hausdorff extensions satisfy condition (i) of the above definition and semiregular extensions satisfy condition (ii) of the definition, it follows that every semiregular Hausdorff extension is an ideal extension.

Theorem 3.3. *Let $E \equiv (\psi, (Y, k))$ be an ideal extension of a T_0 topological space (X, c). Then E is equivalent to an extension $E^* \equiv (\Phi, (X^*, d))$ of (X, c), for some suitable collection X^* of c-ideals on (X, c) containing all the free ideals of X.*

Proof. Let $X^* = \widehat{X} = \{S(y, E): y \in Y\}$, the strength system of E on (X, c). Then X^* is clearly a set of c-ideals on (X, c) containing every free ideal on the same. We shall first introduce a Kuratowski closure operator on X^*, together with a one–one map $\Phi : (X, c) \to X^*$ by the rule $\Phi(x) = I_c(x)$, for all $x \in X$. The T_0—property of X ensures (by Proposition 2.2) that Φ is a one–one map. For any $A \subseteq X$, set $A^c = \{I \in X^*: A \notin I\}$. We first show that $\mathcal{B} = \{A^c: A \subseteq X\}$ is a base for the closed sets with respect to some topology on X^*. Clearly $\phi^c = \phi$. Next let $A^c, B^c \in \mathcal{B}$. Now, $(A \cup B)^c = \{I \in X^*: A \cup B \notin I\} = \{I \in X^*: A \notin I \text{ or } B \notin I\} = \{I \in X^*: A \notin I\} \cup \{I \in X^*: B \notin I\}$ $= A^c \cup B^c \Rightarrow A^c \cup B^c \in \mathcal{B}$.

Let d be the Kuratowski closure operator associated with this topology on X^*. Then for all $\alpha \subseteq X^*$, $d(\alpha) =$ the intersection of all the basic closed subsets of (X^*, d) containing $\alpha = \bigcap\{A^c\colon \alpha \subseteq A^c \text{ and } A \subseteq X\}$.

Let us first observe some of the relations involving d and $\Phi : X \to X^*$.

(i) For all $A \subseteq X$, $\Phi(c(A)) = A^c \cap \Phi(X)$. In fact, $x \in c(A) \Leftrightarrow A \notin I_c(x) = \Phi(x) \in A^c \cap \Phi(X)$.
(ii) For all $A \subseteq X$, $\Phi(A) \subseteq A^c$. For, $\Phi(A) \subseteq \Phi(c(A)) \subseteq A^c$ (by (i)).
(iii) For all $A \subseteq X$, $d(\Phi(A)) = A^c$.

In view of (ii) it is sufficient to show that whenever $\Phi(A) \subseteq B^c$ for some $B \subseteq X$, then $A^c \subseteq B^c$.

Assume therefore that $\Phi(A) \subseteq B^c$ for some $B \subseteq X$. Now, $x \in A \Rightarrow \Phi(x) \in B^c \Rightarrow I_c(x) \in B^c \Rightarrow B \notin I_c(x) \Rightarrow x \in c(B)$. Thus $A \subseteq c(B)$.

We now note the following:

(a) $A \subseteq B \Rightarrow A^c \subseteq B^c$; in fact, $I \in A^c \Rightarrow A \notin I \Rightarrow B \notin I \Rightarrow I \in B^c$.
(b) For all $B \subseteq X$, $(c(B))^c = B^c$. In fact, $I \in (c(B))^c \Leftrightarrow c(B) \notin I \Leftrightarrow B \notin I$ (since I is a c-ideal)$\Leftrightarrow I \in B^c$. Thus $(c(B))^c = B^c$.

Now, $A \subseteq c(B) \Rightarrow A^c \subseteq (c(B))^c = B^c$. Thus $d(\Phi(A)) \supseteq A^c$ and (iii) follows from the definition of d.

Now, from (i) and (iii) we have $\Phi(c(A)) = d(\Phi(A)) \cap \Phi(X)$ and $d(\Phi(X)) = X^c$. We also have $X^c = X^*$. In fact, $X^c = \{I \in X^*\colon X \notin I\}$. Obviously $X^c \subseteq X^*$. Now for $S(y, E) \in X^*$ for any $y \in Y$, where $S(y, E) = \{A \subseteq X\colon y \notin k\psi(A)\}$, we have $X \notin S(y, E) \Rightarrow S(y, E) \in X^c$. Hence $X^c = X^*$.

Thus (X^*, d) is an extension of (X, c).

Let us now define a map $f : (Y, k) \to (X^*, d)$, given by $f(y) = S(y, E)$. Now, for all $x \in X$, $f(\psi(x)) = S(\psi(x), E) = I_c(x) = \Phi(x)$, so that $f \circ \psi = \Phi$.

Again, let $A \subseteq X$. Then for any $y \in Y$, $y \in k\psi(A) \Leftrightarrow A \notin S(y, E) = f(y) \Leftrightarrow f(y) \in A^c$. Therefore, the bijective map $f : (Y, k) \to (X^*, d)$ establishes a one to one correspondence between the basic closed sets of those two spaces and hence f is a homeomorphism of (Y, k) onto (X^*, d). Therefore E is equivalent to the extension $E^* \equiv (\Phi, (X^*, d))$. □

Having furnished a method of construction of an ideal extension of a T_0—space in the above theorem, we now look for the condition under which such an ideal extension becomes compact, i.e., it becomes a compactification of the underlying space. To that end we first prove the following result which incidentally gives a characterization of the compactness of a space in terms of ideals.

Theorem 3.4. *A space (X, c) is compact iff for every ideal I on it satisfying the condition*:

(C) *If a finite intersection of closed sets is empty then one of the closed sets is a member of the ideal I,*

there exists $x \in X$ such that $I_c(x) \subseteq I$.

Proof. Let (X, c) be compact and I be any ideal on X satisfying the condition (C). Then $\{c(A)\colon A \in \mathcal{P}(X) \setminus I\}$ is a collection of closed subsets of X with the finite intersection property. Hence by compactness of (X, c), there exists $x \in X$ such that $x \in \bigcap\{c(A)\colon A \in \mathcal{P}(X) \setminus I\}$, i.e., $x \in c(A)$, for all $A \in \mathcal{P}(X) \setminus I$, i.e., $A \notin I_c(x)$ whenever $A \notin I$. Hence $I_c(x) \subseteq I$.

Conversely, let $\mathcal{U}$ be an ultrafilter on (X, c) and let $I = \{A \subseteq X\colon X \setminus A \in \mathcal{U}\}$. Then it is easy to see that I is an ideal on (X, c). We shall now show that I satisfies the condition (C). For this, let $A_1, A_2, \ldots, A_n \notin I$, where each A_i is a closed set. We claim that $\bigcap_{i=1}^n A_i \notin I$ and hence $\bigcap_{i=1}^n A_i \neq \phi$. In fact,

$$\bigcap_{i=1}^{n} A_i \in I \Rightarrow X \setminus \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X \setminus A_i) \in \mathcal{U} \Rightarrow X \setminus A_i \in \mathcal{U}$$

for some $i \Rightarrow A_i \in I$, a contradiction.

Thus by the given condition, there is $x \in X$ such that $I_c(x) \subseteq I$, that is, $A \in I_c(x) \Rightarrow A \in I$, i.e., $x \notin c(A) \Rightarrow A \in I \Rightarrow X \setminus A \in \mathcal{U} \ldots$ (i). Let U be any open neighbourhood of x. Then $x \notin c(X \setminus U) = X \setminus U \Rightarrow X \setminus (X \setminus U) \in \mathcal{U}$ (by (i)), i.e., $U \in \mathcal{U}$. This shows that $\mathcal{U}$ converges to x, and hence X becomes compact. □

We now define a class of ideals, bigger than that of ideals satisfying the condition (C).

Definition 3.5. An ideal I on a space (X, c) is said to be a c-joined ideal if for any two elements A, B of $\mathcal{P}(X) \setminus I$, $c(A) \cap c(B) \neq \phi$. The space (X, c) is called c-joined compact if for each c-joined ideal I on (X, c), $I_c(x) \subseteq I$ for some $x \in X$.

Remark 3.6. The class of c-joined ideals is strictly larger than the class of ideals satisfying condition (C). For example, let $A = \{2, 4, 6, \ldots, 28\}$, $B = \{3, 6, 9, \ldots, 27\}$, and $C = \{5, 10, 15, 20, 25\}$. Let $X = A \cup B \cup C$ have closed sets $\tau^c = \{\phi, X, A, B, C, A \cap B, A \cap C, B \cap C, A \cup B, A \cup C, B \cup C\}$. Define an ideal on X by $I = \{\phi, \{15\}, \{10, 20\}, \{10\}, \{20\}, \{10, 15, 20\}, \{15, 20\}, \{10, 15\}, \{6, 12, 18, 24\}, \{6\}, \{12\}, \{18\}, \{24\}, \{6, 12\}, \{6, 18\}, \{6, 24\}, \{12, 18\}, \{12, 24\}, \{18, 24\}, \{6, 12, 18\}, \{6, 12, 24\}, \{6, 18, 24\}, \{12, 18, 24\}\}$. $A, B, C \notin I \Rightarrow I$ does not satisfy condition (C) since $A \cap B \cap C = \phi$. So, if $\phi \neq H, K \subseteq X$ have $c(H) \cap c(K) = \phi$, then $c(H), c(K) \in \{X, A, B, C, A \cap B, A \cap C, B \cap C\}$ and in each case, either $c(H) \in I$ or $c(K) \in I$ and hence either $H \in I$ or $K \in I$. For instance, if $c(H) = A$ then $c(K) = B \cap C = \{15\} \in I$. In particular, $A \cap B, A \cap C, B \cap C \in I$. Thus, I is a c-joined ideal.

It thus follows from Theorem 3.4 that

Corollary 3.7. *Every c-joined compact space is compact.*

It is still an open problem whether the converse of Corollary 3.7 is true. However, the converse holds in a regular space, that is

Theorem 3.8. *In the family of regular spaces, the notions of compactness and c-joined compactness are identical.*

Proof. In view of Theorem 3.4 we need only to show that an arbitrary regular compact space (X, c) is c-joined compact. Let I be a c-joined ideal on (X, c). If $X \in I$, then $I = \mathcal{P}(X)$ and hence for each $x \in X$, $I_c(x) \subseteq I$. Thus suppose $X \notin I$. If possible, let $I_c(x) \nsubseteq I$ for any $x \in X$. Then for each $x \in X$, there exists $A_x \in I_c(x)$, i.e., $x \notin c(A_x)$ such that $A_x \notin I$. Now by using regularity of (X, c), we can find an open neighbourhood V_x of x such that $c(V_x) \cap c(A_x) = \phi$. Now the open cover $\{V_x\colon x \in X\}$ has a finite subcover $\{V_{x_i}\colon i = 1, 2, \ldots, n\}$ (say). Since for each $i = 1, 2, \ldots, n$, $A_{x_i} \notin I$, $c(A_{x_i}) \cap c(V_{x_i}) = \phi$ and I is c-joined, we have $V_{x_i} \in I$ for each i and hence $X \in I$, a contradiction. Hence $I_c(x) \subseteq I$ for some $x \in X$, and (X, c) becomes c-joined compact. □

Definition 3.9. A family $\mathcal{F} = \{I_\alpha\colon \alpha \in \Lambda\}$ of ideals on a space (X, c) is said to satisfy 'Condition $(*)$' if whenever any two subsets A, B of X are not members of an ideal I on X, there exists $\alpha \in \Lambda$ with $A, B \notin I_\alpha$, then for some $\beta \in \Lambda$, $I_\beta \subseteq I$.

We are now in a position to prove the desired result as follows.

Theorem 3.10. *The ideal extension $E \equiv (\Phi, (X^*, d))$ (the symbols have their meanings as given in Theorem* 3.3) *of a space (X, c) is c-joined compact iff X^* is a collection of ideals on (X, c) satisfying condition $(*)$.*

Proof. Let (X^*, d) be c-joined compact and I be an ideal on X such that any two subsets of X which are not in I, are not contained in some member of X^*. We need to show that $I_0 \subseteq I$ for some $I_0 \in X^*$. We first observe that for any $A, B \notin I$, $A^c \cap B^c \neq \phi \ldots$ (1). Indeed, $A, B \notin I \Rightarrow$ there exists $I^* \in X^*$ such that $A, B \notin I^* \Rightarrow I^* \in A^c \cap B^c$.

Define $\mathcal{A} = \{\alpha \subseteq X^*\colon \Phi(A) \cap (X^* \setminus \alpha) \neq \phi, \forall A \notin I\}$.

We first show that $\mathcal{A}$ is a c-joined ideal on X^*. For this, let $\alpha, \beta \subseteq X^*$ be such that $\alpha \subseteq \beta$ with $\beta \in \mathcal{A}$. Then $\Phi(A) \cap (X^* \setminus \beta) \neq \phi$, for all $A \notin I \Rightarrow \Phi(A) \cap (X^* \setminus \alpha) \neq \phi, \forall A \notin I \Rightarrow \alpha \in \mathcal{A}$.

Next let $\alpha, \beta \in \mathcal{A}$. Then $\Phi(A) \cap (X^* \setminus \alpha) \neq \phi, \forall A \notin I$ and $\Phi(B) \cap (X^* \setminus \beta) \neq \phi, \forall B \notin I$. We have to show that $\alpha \cup \beta \in \mathcal{A}$. If possible, let $\alpha \cup \beta \notin \mathcal{A}$. Then there is a $C \notin I$ such that $\Phi(C) \cap (X^* \setminus (\alpha \cup \beta)) = \phi$, i.e., $\Phi(C) \subseteq \alpha \cup \beta$. Define $C_1 = \{x \in C\colon \Phi(x) \in \alpha\}$ and $C_2 = \{x \in C\colon \Phi(x) \in \beta\}$. Clearly $C = C_1 \cup C_2$. Since $C \notin I$, we have $C_1 \notin I$ or $C_2 \notin I$. But $\Phi(C_1) \cap (X^* \setminus \alpha) \neq \phi$ when $C_1 \notin I$, a contradiction to the definition of C_1; similar is the contradiction when we consider $C_2 \notin I$. Hence $\alpha \cup \beta \in \mathcal{A}$. Thus $\mathcal{A}$ is an ideal on X^*. Next let $\alpha, \beta \notin \mathcal{A}$. Then $\Phi(A) \subseteq \alpha$ and $\Phi(B) \subseteq \beta$ for some $A, B \notin I \Rightarrow A^c = d(\Phi(A)) \subseteq d(\alpha)$, $d(\beta) \supseteq d(\Phi(B)) = B^c$ and $A, B \notin I$. Thus $d(\alpha) \cap d(\beta) \supseteq A^c \cap B^c \neq \phi$ (see (ii) of Theorem 3.3). Since (X^*, d) is assumed to be c-joined compact, there exists $I_0 \in X^*$ such that $\mathcal{A} \supseteq I_d(I_0)$.We show that $I_0 \subseteq I$. In fact, $A \notin I \Rightarrow \Phi(A) \notin \mathcal{A} \Rightarrow \Phi(A) \notin I_d(I_0) \Rightarrow I_0 \in d(\Phi(A)) = A^c \Rightarrow A \notin I_0$. Hence $I_0 \subseteq I$ and thus X^* satisfies Condition $(*)$.

Conversely assume that X^* satisfies Condition $(*)$. We shall prove that X^* is c-joined compact. Let $\mathcal{A}$ be any c-joined ideal on (X^*, d). We have to find out an $I_0 \in X^*$ such that $I_d(I_0) \subseteq \mathcal{A}$. Define $\mathcal{A}_X = \{A \subseteq X\colon \Phi(A) \in \mathcal{A}\}$ and $\mathcal{A}_Y = \{A \subseteq X\colon A^c \setminus \Phi(X) \in \mathcal{A}\}$.

We first prove that $\mathcal{A}_X, \mathcal{A}_Y$ are ideals. Let $A, B \subseteq X$ be such that $A \subseteq B \in \mathcal{A}_X$. Then $\Phi(A) \subseteq \Phi(B) \in \mathcal{A} \Rightarrow \Phi(A) \in \mathcal{A}$ (as $\mathcal{A}$ is an ideal) $\Rightarrow A \in \mathcal{A}_X$. Next let $A, B \in \mathcal{A}_X$. Then $\Phi(A), \Phi(B) \in \mathcal{A}$. Now $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cup B)$. Next, $x \in A \cup B \Rightarrow x \in A$ or $x \in B \Rightarrow \Phi(x) \in \Phi(A)$ or $\Phi(x) \in \Phi(B) \Rightarrow \Phi(x) \in \Phi(A) \cup \Phi(B)$. Hence $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$. Since $\Phi(A), \Phi(B) \in \mathcal{A}$ and $\mathcal{A}$ is an ideal, $\Phi(A \cup B) = \Phi(A) \cup \Phi(B) \in \mathcal{A}$ so that $A \cup B \in \mathcal{A}_X$. Thus $\mathcal{A}_X$ is an ideal.

Next let $A, B \subseteq X$ be such that $A \subseteq B \in \mathcal{A}_Y$. Then $B^c \setminus \Phi(X) \in \mathcal{A}$ and since $\mathcal{A}$ is an ideal $A^c \setminus \Phi(X) \in \mathcal{A} \Rightarrow A \in \mathcal{A}_Y$. Now let $A, B \in \mathcal{A}_Y$. Then $A^c \setminus \Phi(X)$, $B^c \setminus \Phi(X) \in \mathcal{A}$. Since $\mathcal{A}$ is an ideal, $(A^c \setminus \Phi(X)) \cup (B^c \setminus \Phi(X)) = (A^c \cup B^c) \setminus \Phi(X) = (A \cup B)^c \setminus \Phi(X) \in \mathcal{A}$. Thus $A \cup B \in \mathcal{A}_Y$ and hence $\mathcal{A}_Y$ is an ideal and hence $\mathcal{A}_X \cap \mathcal{A}_Y$ is an ideal.

We now show that for two arbitrary subsets A, B of X with $A, B \notin \mathcal{A}_X \cap \mathcal{A}_Y$, neither A nor B is contained in some ideal of X^*. Consider the case when $A \notin \mathcal{A}_X$ and $B \notin \mathcal{A}_Y$. Then $\Phi(A) \notin \mathcal{A}$ and $B^c \setminus \Phi(X) \notin \mathcal{A}$. Since $\mathcal{A}$ is a c-joined ideal on (X^*, d), we get, $d(\Phi(A)) \cap d(B^c \setminus \Phi(X)) \neq \phi \Rightarrow d(\Phi(A)) \cap d(B^c) \neq \phi \Rightarrow A^c \cap B^c \neq \phi$. The case for $B \notin \mathcal{A}_X$ and $A \notin \mathcal{A}_Y$ is similar. In case $A, B \notin \mathcal{A}_X$ or $A, B \notin \mathcal{A}_Y$, one obtains almost immediately that $A^c \cap B^c \neq \phi$. Now, $A^c \cap B^c \neq \phi \Rightarrow$ there exists $I \in X^*$ such that $I \in A^c \cap B^c \Rightarrow A, B \notin I$. Since X^* is assumed to satisfy Condition $(*)$, we must have $\mathcal{A}_X \cap \mathcal{A}_Y \supseteq I_0 \in X^*$. We now show that $I_d(I_0) \subseteq \mathcal{A}$. Choose any $\alpha \notin \mathcal{A}$, where $\alpha \subseteq X^*$. Now, $\alpha = (\alpha \cap \Phi(X)) \cup (\alpha \setminus \Phi(X))$. Since $\alpha \notin \mathcal{A}$, then either $\alpha \cap \Phi(X) \notin \mathcal{A}$ or $\alpha \setminus \Phi(X) \notin \mathcal{A}$. First assume that $\alpha \cap \Phi(X) \notin \mathcal{A}$. Then $\Phi^{-1}(\alpha) \cap X \notin \mathcal{A}_X \Rightarrow \Phi^{-1}(\alpha) \notin I_0 \Rightarrow I_0 \in (\Phi^{-1}(\alpha))^c = d(\Phi\Phi^{-1}(\alpha)) \subseteq d(\alpha) \Rightarrow \alpha \notin I_d(I_o)$.

Next assume that $\alpha \setminus \Phi(X) \notin \mathcal{A}$. Let A^c be any basic closed set in the space (X^*, d) containing α. Then $A^c \setminus \Phi(X) \supseteq \alpha \setminus \Phi(X)$. Since $\alpha \setminus \Phi(X) \notin \mathcal{A}$, we have $A^c \setminus \Phi(X) \notin \mathcal{A} \Rightarrow A \notin \mathcal{A}_Y \Rightarrow A \notin I_0 \Rightarrow I_0 \in A^c$. Hence $I_0 \in \bigcap\{A^c\colon \alpha \subseteq A^c\} = d(\alpha) \Rightarrow \alpha \notin I_d(I_0)$.

Thus in any case, $\mathcal{A} \supseteq I_d(I_0)$. Hence (X^*, d) is c-joined compact. □

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Further reading

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