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Multivariate Extensions of Univariate Life Distributions

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A general approach for the development of multivariate survival models, based on a set of given marginal survivals, is presented. Preservation of IFR and IFRA properties and the nature of dependence among the variables are examined, and a recursive relation is suggested to obtain the resultant density function. In particular, an absolutely continuous Weibull distribution is derived and a few of its properties are studied. © 1998 Academic Press

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1. INTRODUCTION

Various approaches for obtaining a multivariate probability distribution that would admit a given set of marginal distributions have been proposed. Characterization results provide an important approach, though it has met some problems while dealing with nonnormal distributions. As an illustration, let us consider different extensions of univariate exponential distributions based on characterization. Marshall and Olkin (1967) considered a bivariate version of the lack of memory property of the exponential distribution and obtained a bivariate exponential distribution with exponential marginals. The constancy of the hazard rate was generalized by Johnson and Kotz (1975) to obtain Gumbel's (1960) bivariate exponential distribution as a unique solution to local constancy of the hazard gradients. Many other multivariate exponential distributions were derived by Freund

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(1961), Downton (1970), Hawkes (1972), Paulson (1973), Block and Basu (1974), and Sarkar (1987), each having some useful properties. The problem becomes more involved when a univariate property of importance may have several bivariate/multivariate extensions. For example Basu (1971), Johnson and Kotz (1975), and Shanbhag and Kotz (1986) proposed different definitions for multivariate hazard rates.

Another approach is to use a functional equation for obtaining a multivariate distribution function starting from the marginal distribution functions. For the bivariate case, Morgenstern (1956) suggested one such equation. Farlie (1960) further generalized this model, and Gumbel (1960) used the Morgenstern model and suggested a bivariate exponential distribution.

Spherical generalization through Kelker's (1970) work is another approach that ensures retention of many univariate properties. This approach works with symmetric distributions only.

We present a different model for deriving multivariate extensions of continuous nonnegative random variables. A bivariate exponential distribution (of the third kind) due to Gumbel (1960) and multivariate Weibull distributions due to Hougaard (1986) and Crowder (1989) can be explicitly or implicitly obtained from the model proposed here. In Section 2 we present the general model along with a few important properties like retention of IFR and IFRA class properties. In Section 3 we examine a bivariate Weibull distribution derived as a particular case of our model.

2. A MODEL FOR MULTIVARIATE EXTENSION

Let $(X_1, X_2, ..., X_p)$ be a collection of p component lives of a system having the marginal distribution $F_i(x)$ for X_i, $i=1, 2, ..., p$. Writing $\bar{F}_i(x)$ as the survival function, $R_i(x)$ as the hazard function, and $r_i(x)$ as the hazard rate corresponding to $F_i(x)$, we have by definition $\bar{F}_i(x) = 1 - F_i(x)$, $R_i(x) = -\log \bar{F}_i(x)$, $r_i(x) = (d/dx)\, R_i(x)$, $i = 1, 2, ..., p$. By assuming the existence of hazard rates we restrict ourselves to absolutely continuous marginal distributions. We then introduce the following Multivariate Extension (ME) Model to describe the joint survival function $\bar{F}(x_1, x_2, ..., x_p)$ of $\mathbf{X} = (X_1, X_2, ..., X_p)$:

$$\text{Model: } \bar{F}(x_1, x_2, ..., x_p) = \exp\left[-\left\{ \sum_{i=1}^{p} (R_i(x_i))^{\nu} \right\}^{1/\nu} \right], \qquad (2.1)$$

where $\nu \geqslant 1$ is the dependency parameter.

RESULT 2.1. *It can be shown that the function $\bar{F}(x_1, x_2, ..., x_p)$ defined in ME (2.1) is a proper survival function with joint density function*

$$f(x_1, x_2, ..., x_p) = \left\{ \prod_{i=1}^{p} R_i^{v-1}(x_i)\, r_i(x_i) \right\} \exp\left[-\left\{ \sum_{i=1}^{p} (R_i(x_i))^v \right\}^{1/v} \right] \times \left\{ \sum_{i=1}^{p} {}_p a_i \left(\sum_{i=1}^{p} R_i^v(x_i) \right)^{i(1-v)/v-(p-i)} \right\} \tag{2.2}$$

where ${}_p a_i \geqslant 0$, $i = 1, 2, ..., p$, *are functions of* v *determined from the recursive relation*

$$\begin{aligned} {}_k a_i &= {}_{k-1}a_{i-1} + {}_{k-1}a_i((k-1)\,v - i), && i = 2, 3, ..., k-1 \\ {}_k a_k &= 1, \quad {}_k a_1 = (v-1)\cdots((k-1)\,v-1), && k = 2, ..., p. \end{aligned} \tag{2.3}$$

Remark 1. Strictly speaking the model under consideration does not require absolute continuity for the marginal distributions. $v = 1$ covers the case of independence. That this joint survival function admits the given marginal distributions can be easily verified.

It will be of interest to see whether the univariate class properties are retained through such a multivariate extension. Because the IFR and IFRA classes are of utmost importance in reliability analysis we examine preservation of those properties in the following results.

Result 2.2. *If* X_i *has an IFR distribution for each* $i = 1, 2, ..., p$ *then* $\mathbf{X}$, *following the ME model, has a multivariate IFR distribution.*

Proof. Following the multivariate definition of IFR class (Roy, 1994) we need to prove that $r_i(\mathbf{x}) \uparrow x_i$ for each choice of x_j; $j = 1, 2, ..., p$, $(\neq i)$. By definition (Johnson and Kotz, 1975), $r_i(\mathbf{x})$ under the given model works out as

$$r_i(\mathbf{x}) = \left[R_i^v(x_i) \Big/ \left\{ \sum_{j(\neq i)} R_j^v(x_j) + R_i^v(x_i) \right\} \right] r_i(x). \tag{2.4}$$

We observe that the first factor is increasing in x_i as $v \geqslant 1$ and as $R_i(x_i)$ is an increasing function of x_i, the second factor is increasing in x_i from the marginal IFR property of component life X_i. Hence $r_i(\mathbf{x})$ is increasing in x_i. But (2.4) is true for each $i = 1, 2, ..., p$, ensuring thereby the MIFR property of $\mathbf{X}$. ▌

Result 2.3. *If* X_i *has an IFRA distribution for each* $i = 1, 2, ..., p$ *then* $\mathbf{X}$, *following the ME model, has a multivariate IFRA distribution.*

Proof. Following the multivariate definition of IFRA class (Roy, 1994) and its equivalence relationship we need to show that for each $i = 1, 2, ..., p$

and for each $\mathbf{x}$, $r_i(\mathbf{x}) \geqslant A_i(\mathbf{x})$, where $A_i(\mathbf{x})$ is the multivariate version of failure rate average. Under the given model

$$A_i(\mathbf{x}) = \left[\left\{ \sum_{i=1}^{p} (R_i^v(x_i)) \right\}^{i/v} - \left\{ \sum_{j(\neq i)} R_i^v(x_j) \right\}^{1/v} \right] \Big/ x_i, \tag{2.5}$$

which we need to show is less than $r_i(\mathbf{x})$. Simplify

$$\begin{aligned} r_i(\mathbf{x}) - A_i(\mathbf{x}) = {} & \left\{ \sum_i R_i^v(x_i) \right\}^{1/v-1} R_i^{v-1}(x_i)\{r_i(x_i) - A_i(x_i)\} \\ & + \left\{ \sum R_i^v(x_i) \right\}^{1/v} \left[\left\{ \sum_{j(\neq i)} R_j^v(x_j) / \sum R_i^v(x_i) \right\}^{1/v} \right. \\ & \left. - \sum_{j(\neq i)} R_j^v(x_i) / \sum_i R_i^v(x_i) \right] \Big/ x_i, \end{aligned} \tag{2.6}$$

where $A_i(x_i) = R_i(x_i)/x_i$ is the marginal failure rate average of X_i. As X_i has an IFRA distribution we have $r_i(x_i) \geqslant A_i(x_i)$ and hence the first term of the right-hand side of (2.6) is nonnegative. Further, because v is greater than or equal to 1 we see that the second term is also nonnegative because

$$\left\{ \sum_{j(\neq i)} R_j^v(x_j) / \sum_i R_i^v(x_i) \right\} \leqslant 1$$

and hence

$$\left\{ \sum_{j(\neq i)} R_j^v(x_j) / \sum_i R_i^v(x_i) \right\}^{1/v} \geqslant \left\{ \sum_{j(\neq i)} R_j^v(x_j) / \sum_i R_i^v(x_i) \right\}.$$

Thus $r_i(\mathbf{x}) \geqslant A_i(\mathbf{x})$ for each i and hence $\mathbf{X}$ is IFRA. ▌

As the nature of dependence in the ME model may be of interest to users, we examine the underlying association following Lehmann (1966) and Shaked (1982). It is easy to observe that the ME model gives rise to positive quadrant dependence when $p = 2$, because for $v \geqslant 1$

$$\bar{F}(x_1, x_2, ..., x_p) \geqslant \bar{F}_1(x_1)\, \bar{F}(x_2), ..., \bar{F}_p(x_p), \tag{2.7}$$

which according to Shaked (1982) results in positive upper orthant dependence. When association is measured through a pairwise correlation coefficient we make an important observation through the following results.

RESULT 2.4. *Under the ME model the correlation coefficient between any two component lives is zero if and only if they are independently distributed.*

Proof. The "if" part follows from the standard result. To prove the "only if" part let us consider any two component lives X_i and X_j; $i \neq j$, $i, j \in (1, 2, ..., p)$ with

$$EX_i X_j = -EX_i EX_j = 0. \tag{2.8}$$

Since X_i and X_j are nonnegative, we have

$$\begin{aligned} EX_i X_j &\geqslant \int_0^\infty \int_0^\infty \bar{F}_i(u)\, \bar{F}_j(v)\, du\, dv \\ &= (EX_i)(EX_j). \end{aligned} \tag{2.9}$$

Following (2.8) we have a strict equality in (2.9), which implies thereby that X_i and X_j should necessarily be independent variables. ∎

RESULT 2.5. *Under the ME model, if any two variables are uncorrelated then all the variables are jointly independent.*

Proof. From result 2.4 we have independence of any two variables X_i and X_j when they are uncorrelated. This implies that $v = 1$. But for $v = 1$ the ME model reduces to

$$\bar{F}(x_1, x_2, ..., x_p) = \bar{F}_1(x_1)\, \bar{F}(x_2), ..., \bar{F}_p(x_p),$$

which implies thereby the joint independence of all the variables in **X**. ∎

In view of the above, the correlation matrix is such that either all the elements are positive or all the nondiagonal elements are zeros.

3. A PARTICULAR STUDY

We shall now examine a special case of the ME model where marginal distributions are of Weibull form. Writing $R_i(x_i) = \lambda_i x_i^\alpha$, $i = 1, 2, ..., p$, the resultant absolutely continuous multivariate Weibull distribution can be described in terms of the joint survival function as

$$\bar{F}(x_1, x_2, ..., x_p) = \exp\left[-\left\{ \sum_{i=1}^{p} (\lambda_i^v x_i^{\alpha v}) \right\}^{1/v} \right]. \tag{3.1}$$

We shall denote this distribution by MEWD (α, v; $\lambda_1, ..., \lambda_p$).

For $\alpha = 1$, it reduces to a multivariate exponential distribution proposed by Gumbel (1960). A slight variation of MEWD may be observed in Hougaard (1986) and Crowder (1989), who obtained their distributions through mixture, the mixing variate having a stable law with index δ $(0 < \delta \leqslant 1)$. As a result, from a distribution having the IFR property one may have a marginal distribution belonging to the DFR class. Under the ME model, both IFR and IFRA class properties remain unaffected.

We shall examine some additional properties of the MEWD $(\alpha, v, \lambda_1, ..., \lambda_p)$ with special reference to its univariate counterpart.

It is interesting to note that MEWD admits the Weibull minimum property as given below.

RESULT 3.1. *If* $\mathbf{X}$ *follows MEWD* $(\alpha, v, \lambda_1, ..., \lambda_p)$ *then* $Z = \mathrm{Min}_{1 \leqslant i \leqslant p} X_i$ *follows* $W(\alpha, \lambda)$, *where* $\lambda = (\sum_{i=1}^{p} \lambda_i^v)^{1/v}$.

Another interesting aspect of the MEWD is the Weibull form for the crude hazard rates. We note that for MEWD the ith crude hazard rate $h_i(x)$ is of the form $h_i(x) = r_i(x, x, ..., x) = \alpha\lambda^{1-v}\lambda_i^v h^{\alpha-1}$.

It was noted by Mukherjee and Roy (1987) that the class of IFR Weibull distributions is closed under failure rate transform. In the multivariate situation we generalize the concept of failure rate transform by crude hazard rate transform (CHR transform). We present below a related result in terms of CHR transform of the MEWD. The proof follows from a simple calculation.

RESULT 3.2. *If* $\mathbf{X}$ *follows* MEWD $(\alpha, v, \lambda_1, ..., \lambda_p)$ *with* $\alpha > 1$ *then* $(h_1(X_1), ..., h_p(X_p))$ *follows* MEWD $(\alpha', v; \beta_1, ..., \beta_p)$ *with* $\alpha' > 1$, *where* $1/\alpha + 1/\alpha' = 1$ *and* $\beta_i = \lambda^{\alpha'(v-1)}\lambda_i^{1-\alpha' v}\alpha^{-\alpha'}$.

It may be worth pointing out that the converse of Result 3.2 can be established under some mild assumptions and symmetry among the CHRs.

Roy and Mukherjee (1986) observed that

$$R(cx)\, R(1) = R(c)\, R(x) \qquad \forall x \geqslant 0, c \geqslant 0 \tag{3.2}$$

if and only if the underlying distribution is Weibull. The multivariate generalization of (3.2) may be

$$R(cx_1, ..., cx_p)\, R(1, ..., 1) = R(c, ..., c)\, R(x_1, ..., x_p) \qquad \forall \mathbf{x} \geqslant \mathbf{0} \quad \text{and} \quad c \geqslant 0. \tag{3.3}$$

RESULT 3.3. *If* $\mathbf{X}$ *follows MEWD, then condition* (3.3) *holds. Conversely, if* (3.3) *is true* $\mathbf{X}$ *follows the Weibull minimum property, each marginal distribution being Weibull.*

Proof. The first part is easy to show. To prove the second part let (3.3) be true. Then for a choice of $x_1 = x_2 = \cdots = x_p = x$ we have

$$R(cx, ..., cx)\, R(1, ..., 1) = R(c, ..., c)\, R(x, ..., x). \tag{3.4}$$

But $R(x, ..., x)$ is the hazard function of the variable $Z = \text{Min}_{1 \leqslant i \leqslant p} X_i$. Comparing the above observation with (3.2) we get via Roy and Mukherjee (1986) that Z follows the Weibull distribution. Thus (3.4) implies the Weibull minimum property. Similarly, $x_i = x$ and $x_j = 0$ in (3.3) ensures the Weibull distribution for the ith variable X_i under similar arguments. This is true for each $i = 1, 2, ..., p$. Hence the result follows. ▮

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