

# Multidimensional persistence behaviour in an Ising system

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We consider a periodic Ising chain with nearest-neighbour and  $r$ -th neighbour interaction and quench it from infinite temperature to zero temperature. The persistence probability  $P(t)$ , measured as the probability that a spin remains unflipped upto time  $t$ , is studied by computer simulation for suitable values of  $r$ . We observe that as time progresses,  $P(t)$  first decays as  $t^{-0.22}$  (-the *first* regime), then the  $P(t) - t$  curve has a small slope (in log-log scale) for some time (-the *second* regime) and at last it decays nearly as  $t^{-3/8}$  (-the *third* regime). We argue that in the first regime, the persistence behaviour is the usual one for a two-dimensional system, in the second regime it is like that of a non-interacting ('zero-dimensional') system and in the third regime the persistence behaviour is like that of a one dimensional Ising model. We also provide explanations for such behaviour.

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## I. INTRODUCTION

The tendency of a spin in a spin- $\frac{1}{2}$  Ising system to remain in its original state following a quench from infinite temperature to zero temperature has been extensively studied over the last decade and is an example of the phenomenon called persistence in dynamical systems [1]-[5]. The probability  $P(t)$  that a spin does not flip upto time  $t$ , exhibits a power law behaviour

$$P(t) \sim t^{-\theta}, \quad (1)$$

where  $\theta$  is a non-trivial exponent, as it is not related to any other static or dynamic exponent. In one dimension with nearest-neighbour interaction, it has been proved exactly that this exponent is  $\theta = 3/8$  [3]. In two dimensions, again with nearest-neighbour interaction, this exponent has been numerically evaluated [6] as  $\theta = 0.209$ . For Glauber dynamics, the one-dimensional Ising system is equivalent to a one-dimensional  $A + A \rightarrow 0$  diffusion system (see below). Hence, the "zero-dimensional" persistent behaviour should correspond to that of a system of non-interacting particles initially spread randomly over a chain with density  $\rho$  and then diffusing independent of each other. (The rule of diffusion is to take a step to the right or to the left with probability  $1/2$ .) The persistence probability for this system has been shown [7] to decay stretched-exponentially,

$$P(t) = \exp[-(2\sqrt{2/\pi\rho})\sqrt{t}] \quad (2)$$

The objective of this communication is to report the observation that an Ising system with nearest-neighbour interaction on a rectangular helical lattice, when quenched

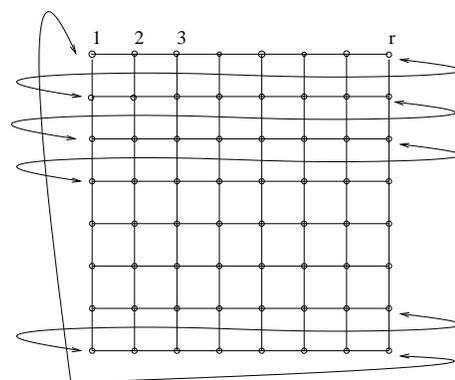


FIG. 1:  
The lattice with helical boundary condition.

to zero-temperature from an infinite temperature shows zero-, one- and two-dimensional persistence behaviour in different regions of its temporal evolution. In the next section, we shall describe the details of the system simulated and the algorithm followed, alongwith the results. An explanation of the simulation results will also be presented in Sec. III. In the last section we shall discuss some subtle issues.

## II. THE ALGORITHM AND THE RESULTS

In this section we shall first describe the details of our system and our algorithm, and then present the results. Consider Ising spins  $s_1, s_2, \dots, s_L$  placed on a chain of  $L$  sites with the periodic boundary condition  $i + L \equiv i$ . Initially, each spin is  $+1$  or  $-1$  with equal probability. One iteration of the system consists of the following 4 steps:

(i) choose one (say,  $k$ -th) spin randomly,

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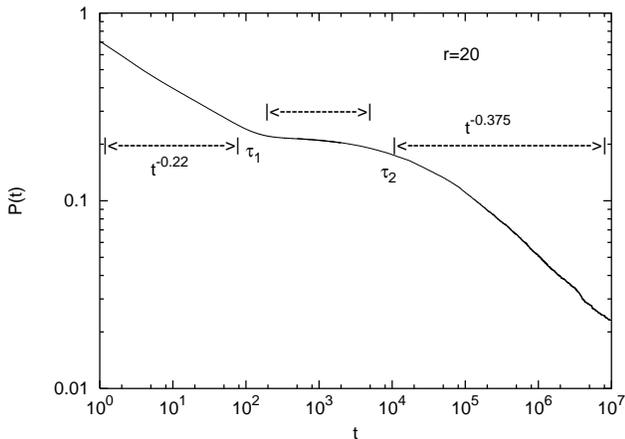


FIG. 2:

Plot of  $P(t)$  as a function of  $t$  for  $r = 20$ . The curve is for  $L = 10000$  and averaged over 100 configurations. The first, second and third regime are marked in the figure. The curves for  $L = 5000$  and  $L = 20000$  fall on the curve shown here.

(ii) calculate its energy

$$E_k \equiv s_k(s_{k-1} + s_{k+1} + s_{k-r} + s_{k+r}), \quad (3)$$

(iii) flip  $s_k$  with probability 1 if  $E_k > 0$ , and with probability  $\frac{1}{2}$  if  $E_k = 0$  (do not flip at all if  $E_k < 0$ ),  
 (iv) repeat the steps (i) to (iii)  $(L - 1)$  times more (random updating).

Here,  $r$  is a parameter of the model and must lie between 2 and  $L$ . Clearly, we have nearest and  $r$ -th neighbour interaction and our system is equivalent to a nearest neighbour rectangular Ising system of size  $r \times (L/r)$ . The lattice is *not* periodic in the two axial directions, rather the chain is wound as a helix of periodicity  $r$  with the ends (first and  $L$ -th sites) put side by side (Fig. 1). When  $r \ll \sqrt{L}$ , the system is effectively a one-dimensional one, while for  $r \sim \sqrt{L}$  it is a two-dimensional one with aspect ratio  $a = r^2/L$ . In this communication we study the case of  $a \sim 0.01$ , so that the system is effectively one-dimensional, or at least a narrow strip. We do not consider the situation when the condition  $r \ll \sqrt{L}$  is not satisfied.

After every iteration we compute the fraction of spins that has not been flipped till now. This fraction, averaged over many realisations of the system gives us the probability of persistence  $P(t)$ . We present in Figs. 2 and 3 the simulation results for this quantity. At first  $P(t)$  decreases as  $t^{-\theta}$  with  $\theta = 0.22 \pm 0.01$  showing two-dimensional behaviour. This behaviour continues upto a certain time, say,  $\tau_1$ . For a given  $L$ , the value of  $\tau_1$  has been observed to increase with  $r$  as  $r^2$  (Fig. 4). We call the region  $1 < t < \tau_1$  the *first regime*. Next follows the *second regime* extending upto some  $\tau_2$  iteration where the  $P(t) - t$  curve has a small slope in the log-log scale. We

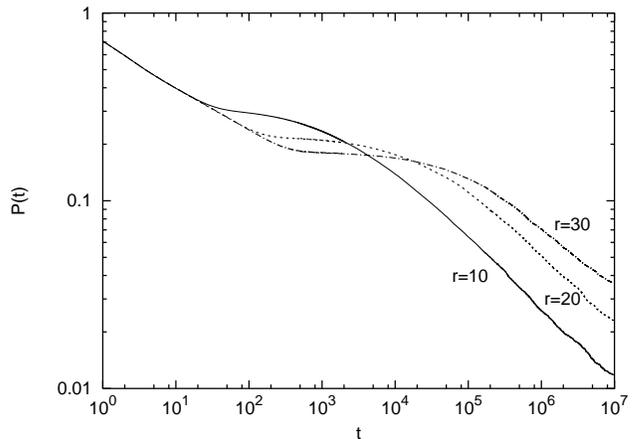


FIG. 3:

Plot of  $P(t)$  as a function of  $t$  for  $r = 10, 20$  and  $30$ . The curve is for  $L = 10000$  and averaged over 100 configurations.

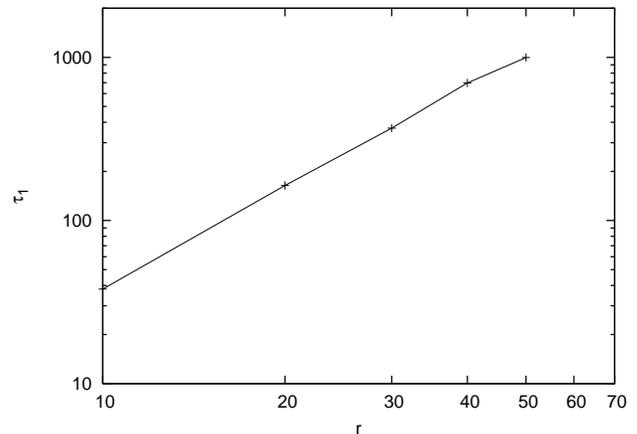


FIG. 4:

Variation of  $\tau_1$  vs  $r$ . The curve is for  $L = 10000$  and averaged over 100 configurations.

shall see below that in this regime the system behaves as a zero-dimensional one. For limitation of computational resources, we could not achieve precise evaluation of  $\tau_2$ , but could observe that, like  $\tau_1$ , this quantity increases with  $r$ . At last comes the *third regime* (for  $\tau_2 < t < \infty$ ) where  $P(t)$  decays nearly as  $t^{-\theta}$  with  $\theta = 0.375 \pm 0.01$  showing the one-dimensional behaviour.

### III. EXPLANATION OF THE RESULTS

We now explain the observations in the three regimes one by one, by approximate analytic arguments but an exact analytic calculation of the persistence behaviour spanning over the three regimes is yet to be done.

*The First Regime*: Here the system shows normal two-dimensional behaviour. The persistence curve  $P(t)$  vs.  $t$

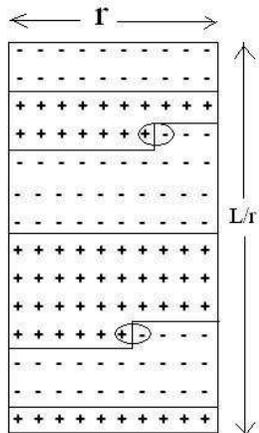


FIG. 5:

Presence of only a small number of kinks in the domain wall ( $A$  particles) in the second regime. Dynamics occurs only at the kinks.

saturates at a time  $\tau_1 \sim r^z$  to a value that varies as  $r^{-z\theta}$  where  $z$  is the dynamical critical exponent ( $\approx 2$ ) [1, 2]. After the system reaches the saturation stage, the domain walls (lines separating unlike spins) are mostly parallel to the  $X$  axis, with only a small number of kinks (Fig. 5). The dynamics occurs only at these kinks, which we call ‘ $A$  particles’. For the updating rule stated above, each  $A$  particle jumps to the left or right with probabilities  $1/2, 1/2$ , assuming that the walls are sparsely distributed over the system (an  $A$  particle does not have another at a distance  $\leq r$ ). The density  $\rho_A$  of  $A$  particles, measured as the number of vertical domain walls per site is shown in Fig. 6. This quantity also shows a plateau region in the second regime, like persistence. It is important to note that, for a fully periodic system shown in Fig. 7, there can be only an even number of  $A$  particles in a row. Presence of one particle in a row is hence ruled out and two or more particles get annihilated within time  $\tau_1$ . The dynamics therefore *stops completely* at  $t = \tau_1$  in the case of a fully periodic system, and the second and the third regimes do not appear.

*The Second Regime* : What happens to our (helical) system for  $t > \tau_1$  ? For an interval of time  $\tau_1 < t < \tau_2$ , the  $A$  particles (kinks in the domain walls) are so far away from each other that they cannot “see” each other and diffuse independently. For a system of independent random walkers of density  $\rho$ , the persistence has been shown [7] to obey Eq. 2. To compare our persistence data with Eq. 2, we assign each site to be persistent at  $t = \tau_1$ . This makes  $P(\tau_1) = 1$  and obliterates the distribution of persistent sites created in the first regime. Then we note down the (almost constant) density  $\rho_A$  in the second

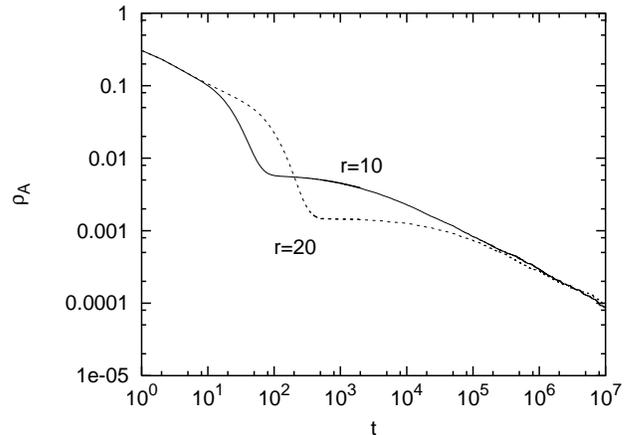


FIG. 6:

Plot of density  $\rho_A$  of  $A$  particles as a function of  $t$  for  $r = 10$  and  $20$ ,  $L = 10000$ . The data was averaged over 1000 configurations. In the nearly horizontal region (the second regime)  $\rho_A = 5.50 \times 10^{-3}$  for  $r = 10$  and  $\rho_A = 1.44 \times 10^{-3}$  for  $r = 20$ .

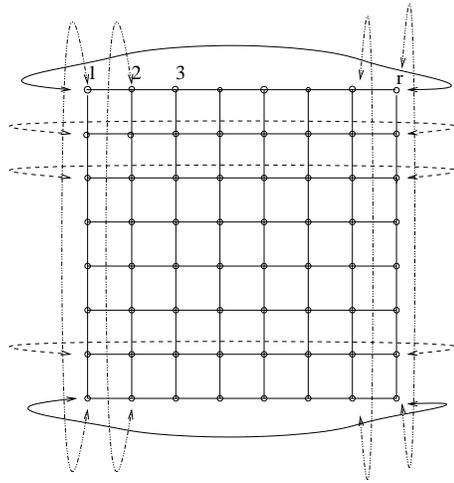


FIG. 7:

The lattice with periodic boundary condition.

regime and observe that the  $P(t)$  data here obeys the relation

$$P(t) = 1 - \alpha \rho_A \sqrt{t - \tau_1} \quad (4)$$

with  $\alpha \approx 1.6$  (Fig. 8). Since the value of the slope  $\alpha$  is close to  $2\sqrt{2/\pi} = 1.596$ , and since the value of  $\rho_A$  is small, Eq. 2 is obeyed and we conclude that in the second regime, the system behaves as one of zero-dimension.

*The Third Regime* : When the system evolves further ( $t > \tau_2$ ), the  $A$  particles start meeting (and annihilating) each other and the usual one-dimensional dynamics leads to  $t^{-\theta}$  behaviour with  $\theta \approx 3/8$ . Since the density  $\rho_A$  is too small, one needs to average over a large number of realisations and simulation of the third regime is a

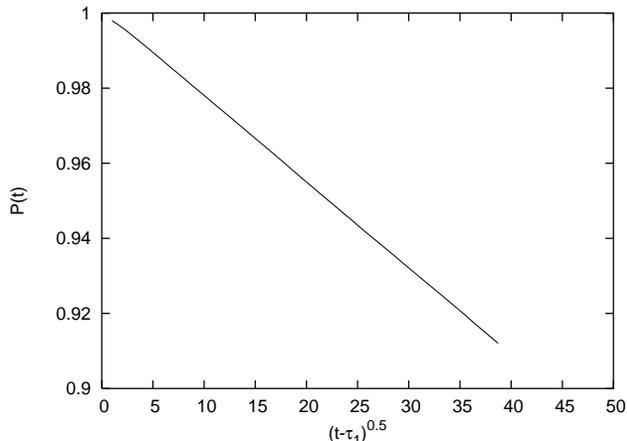


FIG. 8:

Plot of  $P(t)$  as a function of  $\sqrt{t - \tau_1}$  in the second regime of the persistence curve for  $r = 20$  and  $L = 10000$ .  $P(t)$  was set to be 1 at  $\tau_1 = 500$ . The curve fits to  $1 - 1.58\rho_A\sqrt{t - \tau_1}$  with  $\rho_A = 1.44 \times 10^{-3}$ . The data were averaged over 1000 configurations.

computationally intensive job. The  $P(t) - t$  curves remain the same for different values of  $L$  for a given  $r$ , but get shifted (maintaining  $t^{-3/8}$  behaviour) as one varies  $r$  at a given  $L$  (Figs. 2 and 3). That the dynamics in the second and the third regimes is indeed described by simple one-dimensional  $A + A \rightarrow 0$  dynamics is further corroborated by two numerical experiments :

- (i) If we turn off the  $r$ -th neighbour interaction at  $t = \tau_1$ , the slope of the persistence curve (in log-log scale) does not change much (Fig. 9), indicating that it is chiefly the nearest-neighbour interaction that drives the dynamics.
- (ii) Let us consider a periodic chain of  $L$  sites and sprinkle randomly some particles (excluding multiple occupancy at a site) with density  $\rho$ . Starting with a low ( $\sim 0.005$ ) value of  $\rho$ , we let the system evolve according to the usual  $A + A \rightarrow 0$  dynamics. The result is the (zero-dimensional) second regime (Fig. 10) for the first 100 iterations, followed by the (one-dimensional) third regime (Fig. 11). The second regime is found to follow Eq. 2 with  $\alpha = 1.55$  and the third regime shows the usual  $t^{-3/8}$  behaviour.

#### IV. DISCUSSION

(1) We shall first discuss a subtle issue regarding the dynamics in the third regime of the Ising model. When two “kinks” of Fig. 5 come closer than  $r$  to each other, a domain of length less than  $r$  is formed. The rules of update renders (i) every spin lying within this domain liable to flip (with probability 1/2) and (ii) the two spins at the ends of this domain bound to flip (with probability 1). The dynamics thus differs from the one for the  $A + A \rightarrow 0$  model, since once the domain is less than

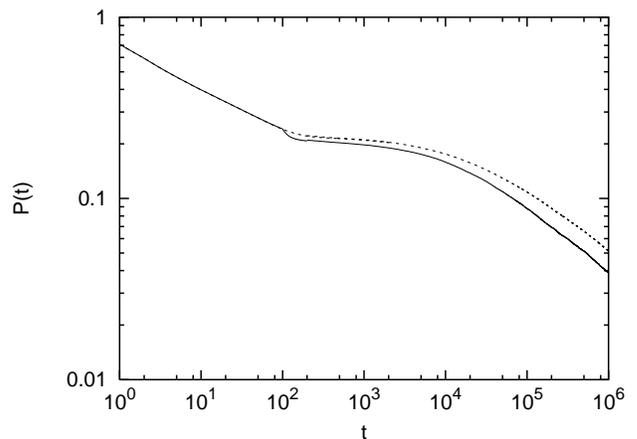


FIG. 9:

Plot of  $P(t)$  as a function of  $t$  for simultaneous nearest neighbor interaction and long-range interaction  $r = 20$ . After 100 iterations the long-range interaction is switched off (continuous line). The curve is for  $L = 10000$  and averaged over 1000 configurations. The dotted line corresponds to the usual case, when the  $r$ -th neighbour interaction continues for the entire range of time.

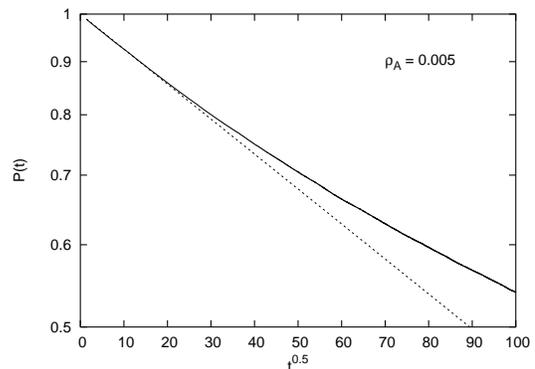


FIG. 10:

Plot of  $P(t)$  as a function of  $\sqrt{t}$  for  $A + A \rightarrow 0$  dynamics in log-linear scale. At  $t = 0$ , the  $A$  particles were randomly spaced with density  $\rho_A = 0.005$ . Here  $L = 4000$  and the results were averaged over 200 realisations. The initial portion ( $t < 400$ ) fits to Eq. 2 with  $\alpha = 1.55$ .

$r$  in length, it is swapped within the next  $r$  steps. But since we are primarily interested in the region  $t \gg r$ , this difference is not of much consequence. One can flip the spins *only* at the ends of the domain (keeping the ones within the domain unflipped) if we replace the energy expression of Eq. 3 by

$$E_k \equiv s_k[s_{k-1} + s_{k+1} + \kappa(s_{k-r} + s_{k+r})] \quad (5)$$

and choose  $\kappa < 1$ . We have checked that the persistence behaviour still remains almost the same.

(2) In the case of  $A + A \rightarrow 0$  model with initial density  $\rho_A = 0.005$ , we have shown a  $t^{-3/8}$  behaviour in

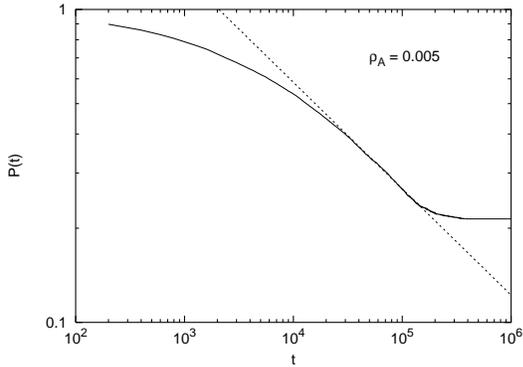


FIG. 11:

Plot of  $P(t)$  as a function of  $t$  for  $A + A \rightarrow 0$  dynamics in log-log scale. At  $t = 0$ , the  $A$  particles were randomly spaced with density  $\rho_A = 0.005$ . Here  $L = 4000$  and the results were averaged over 100 realisations. The algebraic portion fits to  $13.4t^{-0.34}$

Fig. 11. However, one comes across anomalous behaviour at somewhat larger system size (Fig. 12). Thus, after the

third regime continues for some time, one comes across another saturation region, with the saturation value *increasing* with increase of  $L$ . We did not observe such anomalous behaviour for the Ising system. Work is in progress on this issue.

(3) It is interesting to note that when the range  $r$  of the long-range interaction varies from site to site and takes any integer value chosen randomly between 2 and  $L$ , the persistence behaviour almost vanishes [9]. Since we have found the persistence behaviour to remain upto time  $\tau_1 \sim r^2$ , for a given value of  $r$  we conclude that the random nature of  $r$  tends to remove the persistence behaviour.

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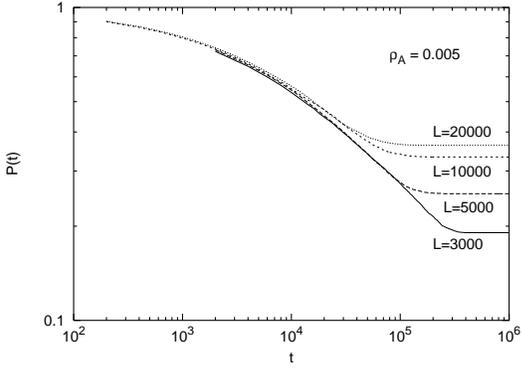


FIG. 12:

Plot of  $P(t)$  as a function of  $t$  for  $A + A \rightarrow 0$  dynamics in log-log scale. At  $t = 0$ , the  $A$  particles were randomly spaced with density  $\rho_A = 0.005$ . The size of the samples are indicated in the figure and the results were averaged over 50 to 200 realisations. The saturation region rises with increasing system size. Just before saturation, the curve does not show algebraic decay over any appreciably long region.