

Maximal qubit violation of n -local inequalities in quantum network

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Source independent quantum networks are considered as a natural generalization to the Bell scenario where we investigate the nonlocal properties of quantum states distributed and measured in a network. Considering the simplest network of entanglement swapping, recently Gisin et. al. and Andreoli et. al. independently provided a systematic characterization of the set of quantum states leading to violation of the so-called 'bilocal' inequality. In this work, we consider the complexities in the quantum networks with an arbitrary number of parties distributed in chain-shaped and star-shaped networks. We derive the maximal violation of the ' n -local' inequality that can be achieved by arbitrary two-qubit states for such chain and star-shaped networks. This would further provide us deeper understanding of quantum correlations in complex structures.

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I. INTRODUCTION

Since its foundation in the early decades of the last century, quantum theory has been elevated to the status of the most precisely tested and successful theory in the history of science. Yet, many of its consequences have puzzled most of the scientists. Many counter-intuitive features of quantum theory like quantum entanglement[2], plays a central role in the foundation of the theory as well as in the development of the quantum information and computation theory[17]. Another important feature is non-locality, shown initially through Bell's theorem that expresses the nature of quantum correlations exist between distant particles/parties of a composite quantum system, thus precluding its explanation by any local hidden variable(LHV) model.

Not surprisingly many generalizations of Bell's theorem[3] have been pursued over the years. Immediate natural generalizations of this simple scenario include more measurements per party[5], sequential measurements[6], several measurement outcomes[7], increasing number of parties[8, 9] and stronger notions of quantum nonlocality[10, 11]. A common feature of all those generalizations is the correlations between the parties assumed to be mediated by the same source of states. However, in a quantum network scenario, the correlations between the distance nodes are not restricted by a single source, rather it is assumed that there are many independent sources distribute entanglement in a non-trivial way across the whole network and thus generates a strong correlation among several nodes. The simplest network scenario is provided by entanglement swapping[12], where two distant parties, say, Alice and Charlie, share entangled states with a central node Bob. Upon measuring in an entangled basis and conditioning on his outcomes, Bob can generate entanglement and non-local correlations among the two other distant parties even though they had no direct interactions prior to the measurement. To compare classical and quantum correlations in this scenario, it is natural to consider classical models consisting of two independent hidden variables, and we come to a notion of bilocal assumption[15, 16]. The bilocal scenario and generalizations to networks with an increasing number(n) of independent sources of states, called n -local scenario[13–17], allows us the emergence of a new kind of non-local correlations.

The main goal of this work is to provide the maximal violation of the n -local inequalities for the n -local scenario using arbitrary qubit states shared between two parties from different independent sources like the bilocal inequalities given by Andreoli et. al. and Gisin et. al. independently[1, 13]. Here, we consider a chain-shaped network and a star-shaped network[4, 14].

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II. CHARACTERIZING BILOCAL CORRELATION

In this section, we will briefly describe the basic structure of the bilocal network (see Figure 1). Here, Alice and Bob are sharing a quantum state produced by the source S_1 , and Bob and Charlie are sharing a quantum state produced by another source S_2 . One can generalize this notion to networks with an increasing number(n) of independent sources of states, the so-called n -local scenario, allow one to study a new kind of correlations. For instances, correlations

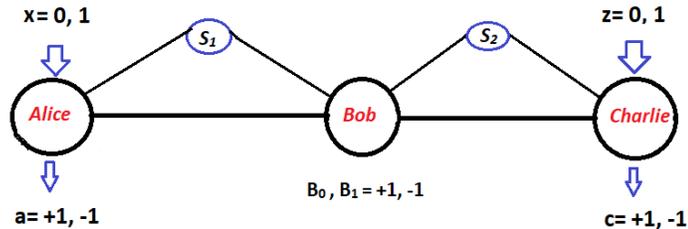


FIG. 1: Bilocal Network scenario

that appear classical, according to the usual local hidden variable (LHV) models can also display non-classicality if the independence of the sources is taken into account. More specifically, a bilocal correlation consists of three distant observers Alice, Bob and Charlie correlated by two independent sources of states. In the quantum case, Bob shares two pairs of entangled particles, one with Alice and another with Charlie. Now Alice, Bob and Charlie perform measurements X , Y , and Z with outcomes respectively a , b , and c . The special feature of this scenario is that Bob is carrying two different particles produced by two independent sources with different hidden variables, say, (λ_1, λ_2) . Any probability distribution compatible with the bilocality assumption, i.e., independence of the sources can be decomposed as,

$$p(a, b, c|x, y, z) = \int d\lambda_1 d\lambda_2 p(\lambda_1) p(\lambda_2) p(a|x, \lambda_1) p(b|y, \lambda_1, \lambda_2) p(c|z, \lambda_2) \quad (1)$$

with binary inputs of Alice and Charlie $x = 0, 1$ and $z = 0, 1$ and with outputs ± 1 in each case. The middle party, Bob always performs the same measurement with input y , with four possible outcomes, e.g., the BSM(Bell State Measurement), denoting Bob's outcome by two bits $B_0 = \pm 1$ and $B_1 = \pm 1$. It follows that any bilocal hidden variable model described by the previous equation must full-fill the following bilocal inequality,

$$S_{biloc} = \sqrt{|I|} + \sqrt{|J|} \leq 2 \quad (2)$$

where

$$I = \langle (A_0 + A_1) B_0 (C_0 + C_1) \rangle$$

$$J = \langle (A_0 - A_1) B_1 (C_0 - C_1) \rangle$$

and $\langle \rangle$ denotes the expectation value of multiple experimental runs.

III. MOTIVATION

From the previous section, we learn that any correlation distributed between Alice, Bob and Charlie will be said non- bilocal if it violates the inequality (2). In[1], Gisin et. al. showed that all the pure entangled states violate this bilocal inequality. They also showed that for mixed states distributed over the three parties from two different sources

there is a relation between bilocal inequality violation and CHSH inequality violation for each independent source. They proved the following relation,

$$S_{biloc}^{max} \leq \sqrt{S_{AB(CHSH)}^{max} S_{BC(CHSH)}^{max}} \quad (3)$$

So, the violation of bilocal inequality implies that either ρ_{AB} , or ρ_{BC} , or both must violate CHSH Inequality[3], where ρ_{AB} and ρ_{BC} be two mixed states shared between Alice, Bob and Bob, Charlie respectively with the generic form,

$$\rho_{AB} = \frac{1}{4}(\mathbb{1} + \vec{m}_A \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\sigma} \cdot \vec{m}_B + \sum_{mn} t_{mn} \sigma_m \otimes \sigma_n)$$

and

$$\rho_{BC} = \frac{1}{4}(\mathbb{1} + \vec{m}_B \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\sigma} \cdot \vec{m}_C + \sum_{mn} t_{mn} \sigma_m \otimes \sigma_n)$$

Here, the vector \vec{m}_A (\vec{m}_B) represents the Bloch vector of Alice's (Bob's) reduced state, while t_{mn} (with $m, n \in x, y, z$) is the terms in the correlation matrix, and similarly for ρ_{BC} . Maximizing S_{biloc} , they proved that the minimum criteria for bilocal violation are either ρ_{AB} or ρ_{BC} be non-local. In support of this statement, F. Andreoli et. al., took the state $|\phi^+\rangle\langle\phi^+|$, $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (one of the four Bell states) between Alice and Bob and a generic two qubit state between Bob and Charlie and showed that the generic state while satisfying CHSH inequality, $|\phi^+\rangle\langle\phi^+| \otimes \rho_{BC}$, violates bilocal inequality(Figure 2). Observing these results, one should ask what would be the corresponding results

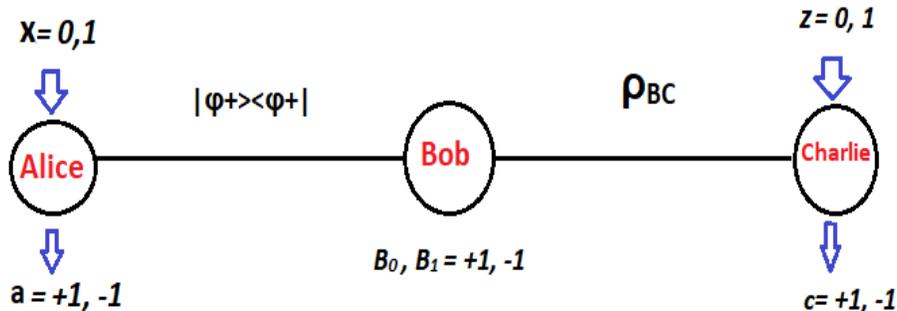


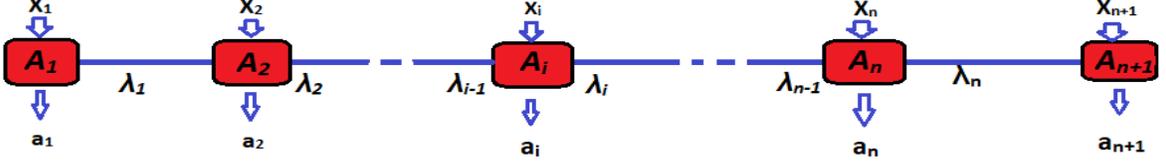
FIG. 2: Alice and Bob are sharing $|\phi^+\rangle\langle\phi^+|$ and a generic two-qubit state is shared between Bob and Charlie.

if one chooses more complicated quantum network scenario with such generic states. We have tried to answer this question in two specific complicated network scenarios.

IV. N- LOCAL SCENARIO

We first consider the n -local chain-shaped scenario, depicted in figure 3. There are n sources S_i ($i=1,\dots,n$) and $n+1$ parties A_i ($i=1,\dots,n+1$) arranged in a linear chain such that any two neighbouring parties share a common source. For each i ($i=1,\dots,n+1$), party A_i can perform two dichotomic measurements $x_i = x_i^k$ ($k=1,2$) where $x_i^k \in \{0,1\}$ on the respective subsystems they have received and obtain outcomes $a_i = a_i^j$ ($j=1,2$) where $a_i^j \in \{0,1\}$. Except the extreme two parties A_1 and A_{n+1} , each of the remaining $n-1$ parties who receive two subsystems, the measurement will be joint measurement operating on both the subsystems simultaneously[14]. In this scenario, Bell's locality assumption takes the form:

$$P(a_1, \dots, a_{n+1}|x_1, \dots, x_{n+1}) = \int d\lambda \rho(\lambda) \prod_{i=1}^{n+1} P(a_i|x_i, \lambda) \quad (4)$$

FIG. 3: n -local chain shaped Network scenario

Here λ is the joint hidden state. For each party A_i , the input x_i and the corresponding output a_i is determined by the local probability distributions $P(a_i|x_i, \lambda)$. The hidden state follows the distribution $\rho(\lambda)$, satisfying the normalization condition $\int \rho(\lambda) d\lambda = 1$, where we assume that the measurement choices of each party are independent of λ . Now, suppose each of the sources S_i be characterized by the hidden state λ_i , Moreover, we assume that for each $i \in \{2, \dots, n\}$, the output of the party A_i depend on the hidden states λ_{i-1}, λ_i and its input x_i , whereas for A_1 and A_{n+1} the outputs depend on their corresponding inputs and λ_1 and λ_n , respectively. So, we can write,

$$P(a_1, a_{n+1}|x_1, \dots, x_{n+1}) = \int d\lambda_1 \dots \int d\lambda_n \rho(\lambda_1, \dots, \lambda_n) P(a_1|x_1) \\ \times \prod_{i=2}^n P(a_i|x_i, \lambda_{i-1}, \lambda_i) P(a_{n+1}|x_{n+1}, \lambda_n).$$

Without any further assumption this equation can jump to usual Bell local equation, when all the different hidden states will be the same. So, for n -local assumption, the distribution of the hidden states with independent sources, we can factorize the probability distribution as;

$$\rho(\lambda_1, \dots, \lambda_n) = \prod_i \rho_i(\lambda_i)$$

. The independence of the sources and probability distributions as factorized above, together define the assumption of n -locality. To study whether a given correlation is n -local or not there is a non-linear inequality exactly same like bilocal inequality which is referred to as n -local inequalities. Here, we only consider the case, where the two extreme parties have binary inputs and outputs and with one input and four outputs for the remaining $n-1$ parties (i.e., P^{14} scenario). Another picture is each party has two inputs and two outputs (P^{22} Scenario). We observe that for $n=2$, the scenario exactly like bilocal [16] for which Gisin et. al. [1] proved the inequality in equation (3). Here, we try to extend the result for this n -local network with P^{14} scenario.

To check the n -local correlation we have the following inequality;

$$\sqrt{|I|} + \sqrt{|J|} \leq 2 \quad (5)$$

where,

$$I = \sum_{x_1, x_{n+1}=0,1} \langle A_{1,x_1} A_2^0 \dots A_n^0 A_{n+1,x_{n+1}} \rangle \quad (6)$$

$$J = \sum_{x_1, x_{n+1}=0,1} (-1)^{x_1+x_{n+1}} \langle A_{1,x_1} A_2^1 \dots A_n^1 A_{n+1,x_{n+1}} \rangle \quad (7)$$

and A^0, A^1 be the joint measurements of the middle parties with outputs $A^0 = \pm 1$ and $A^1 = \pm 1$. From the above inequality, one would be able to observe that if a correlation violates the inequality then the correlation is non- n -local and also it can be seen [14] that for quantum correlations the inequality is violated. So, for this n -local scenario we find the minimum number of sources which is CHSH non-local and violate the n -local inequality. And later we extend this result to star-shaped n -local scenario also, where there is a central node connected with n number of parties with n independent sources.

V. VIOLATION CRITERIA FOR LINEAR CHAIN NETWORK

In [1], for bilocal network, sharing two generic mixed states ρ_{AB} and ρ_{BC} in Sec:III, provided by two independent sources, the maximum value of the left-hand side of the bilocality inequality (5) is given by,

$$S_{biloc}^{max} = 2\sqrt{\sqrt{\Lambda_1^A \gamma_1^C} + \sqrt{\Lambda_2^A \gamma_2^C}} \quad (8)$$

where, $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq 0$ are the three eigenvalues of the matrix $R^{AB} = t^{AB\dagger} t^{BC}$ formed by the correlation matrix $t^{AB} = U^{AB} R^{AB}$, denoting U^{AB} is a unitary matrix. Similarly, $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq 0$ for R^{BC} .

Now, maximal CHSH value for ρ_{AB} is $S_{AB}^{max} = 2\sqrt{\Lambda_1 + \Lambda_2}$ and similarly for ρ_{BC} . So, we have, $S_{BC}^{max} = 2\sqrt{\gamma_1 + \gamma_2}$. It directly follows $S_{biloc}^{max} \leq \sqrt{S_{AB}^{max} S_{BC}^{max}}$, which proves that the violation of the bilocality inequality implies that either ρ_{AB} or ρ_{BC} or both must violate CHSH inequality. In[5], they showed, if Alice and Bob share a maximally entangled state(Bell State) while Bob and Charlie share a generic quantum state then bilocal inequality violation can be easily obtained without CHSH violation of ρ_{BC} .

$$S_{biloc}^{max}(|\phi^+\rangle\langle\phi^+| \otimes \rho_{BC}) = \sqrt{\sqrt{\gamma_1} + \sqrt{\gamma_2}} \geq \sqrt{\gamma_1 + \gamma_2} = S_{BC}^{max(CHSH)}$$

Considering a little bit more complex network than above, we introduce first one more party (instead of three, now four), called Alice, Bob1, Bob2, Charlie as in the figure 4. Here, the number of independent sources are three and we can call this situation as 3-local (trilocal) scenario, a special case of n-local scenario. The middle two parties, i.e., Bob1 and Bob2 again will do Bell State measurements with one input and four outputs denoted by $B_1^0 = \pm 1, B_1^1 = \pm 1, B_2^0 = \pm 1, B_2^1 = \pm 1$. The inequality for this 3-local scenario is,

$$\sqrt{|I|} + \sqrt{|J|} \leq 1$$

where,

$$I = \frac{1}{4} \langle (A_0 + A_1) B_1^0 B_2^0 (C_0 + C_1) \rangle$$

and

$$J = \frac{1}{4} \langle (A_0 - A_1) B_1^1 B_2^1 (C_0 - C_1) \rangle$$

Now, we take $\rho_{AB_1} = |\phi^+\rangle\langle\phi^+|$, $\rho_{B_2C} = |\phi^+\rangle\langle\phi^+|$ and $\rho_{B_1B_2}$ as a generic quantum state as in figure 4. If we choose measurement choices of Alice and Charlie be any Bloch vector $(\sin\alpha, 0, \cos\alpha)$, $(\sin\alpha', 0, \cos\alpha')$ and $(\sin\theta, 0, \cos\theta)$, $(\sin\theta', 0, \cos\theta')$ and $B_1^0 = B_2^0 = \sigma_z \otimes \sigma_z$ and $B_1^1 = B_2^1 = \sigma_x \otimes \sigma_x$, we can determine the bilocal value for $\rho_{AB_1} \otimes \rho_{B_1B_2} \otimes \rho_{B_2C}$ with this measurements settings. After maximizing the bilocal value we get $S_{3-local}^{max} = \sqrt{\sqrt{\Lambda_1} + \sqrt{\Lambda_2}}$, where Λ_1 and Λ_2 are the two maximum eigenvalues of the matrix $R^{B_1B_2} = (t^{B_1B_2})^\dagger t^{B_1B_2}$. So, it can be easily seen that to violate 3-local inequality $\rho_{B_1B_2}$ need not be non-local.

Similarly if we take $\rho_{AB_1} = |\phi^+\rangle\langle\phi^+|$ with $\rho_{B_1B_2}$ and ρ_{B_2C} be generic states, we get the bilocal value, $\sqrt{\sqrt{\Lambda_1 \gamma_1} + \sqrt{\Lambda_2 \gamma_2}}$, where Λ_1 and Λ_2 are two maximum eigenvalues of the matrix $R^{B_1B_2} = (t^{B_1B_2})^\dagger t^{B_1B_2}$ and similarly γ_1 and γ_2 for R^{B_2C} .

Now, we take all the three states in generic form and calculate the bilocal value for arbitrary two measurement choices of Alice and Charlie. The quantity I can be expressed as follows:

$$\begin{aligned} I &= \frac{1}{4} Tr[(a + a') \cdot \vec{\sigma} \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes (c + c') \cdot \vec{\sigma} \rho_{AB_1} \otimes \rho_{B_1B_2} \otimes \rho_{B_2C}] \quad (9) \\ &= \frac{1}{4} Tr[(a + a') \cdot \vec{\sigma} \otimes \sigma_z \rho_{AB_1}] Tr[\sigma_z \otimes \sigma_z \rho_{B_1B_2}] Tr[\sigma_z \otimes (c + c') \cdot \vec{\sigma} \rho_{B_2C}] \\ &= \frac{1}{4} \sum_{i=x,y,z} (a_i + a'_i) t_{iz}^{AB_1} \cdot t_{zz}^{B_1B_2} \cdot \sum_{j=x,y,z} t_{zj}^{B_2C} (c_j + c'_j) \end{aligned}$$

And similarly,

$$J = \frac{1}{4} Tr[(a - a') \cdot \vec{\sigma} \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes (c - c') \cdot \vec{\sigma} \rho_{AB_1} \otimes \rho_{B_1B_2} \otimes \rho_{B_2C}] \quad (10)$$

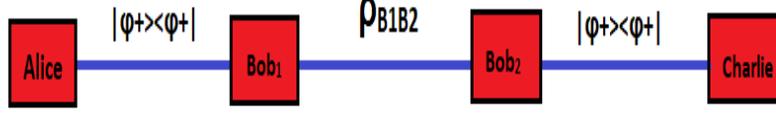


FIG. 4: 3-local chain shaped Network scenario

$$\begin{aligned}
&= \frac{1}{4} \text{Tr}[(a - a') \cdot \vec{\sigma} \otimes \sigma_x \rho_{AB_1}] \text{Tr}[\sigma_x \otimes \sigma_x \rho_{B_1 B_2}] \text{Tr}[\sigma_x \otimes (c - c') \cdot \vec{\sigma} \rho_{B_2 C}] \\
&= \frac{1}{4} \sum_{i=x,y,z} (a_i - a'_i) t_{ix}^{AB_1} t_{xx}^{B_1 B_2} \cdot \sum_{j=x,y,z} t_{xj}^{B_2 C} (c_j - c'_j)
\end{aligned}$$

As said earlier, the correlation matrix can be written in the polar decomposition form [1]. Here, we can denote the eigenvalues of the matrix $R^{B_1 B_2}$ as, $\beta_1 \geq \beta_2 \geq \beta_3$. Next, to prove our aim, we maximize $S_{3\text{-local}}$ with respect to the Bloch vectors \vec{a} , \vec{a}' , \vec{c} and \vec{c}' . Considering the form of the vector $\vec{a} = (\sin\alpha, 0, \cos\alpha)$, $\vec{a}' = (\sin\alpha', 0, \cos\alpha')$ and $\vec{c} = (\sin\theta, 0, \cos\theta)$, $\vec{c}' = (\sin\theta', 0, \cos\theta')$, we can easily find the maximum value by imposing the conditions $\partial_\alpha S = 0$, $\partial_{\alpha'} S = 0$, $\partial_\theta S = 0$ and $\partial_{\theta'} S = 0$ and obtain [1],

$$S_{3\text{-local}}^{\max} = \sqrt{\sqrt{\Lambda_1 \gamma_1 \beta_1} + \sqrt{\Lambda_2 \gamma_2 \beta_2}} \quad (11)$$

Consequently, the states ρ_{AB_1} , $\rho_{B_1 B_2}$ and $\rho_{B_2 C}$ can violate the 3-local inequality if and only if $\sqrt{\Lambda_1 \gamma_1 \beta_1} + \sqrt{\Lambda_2 \gamma_2 \beta_2} > 1$.

For this inequality a criterion analogous to the Horodecki criteria[2] could be found easily by applying Cauchy-Schwarz inequality:

$$S_{3\text{-local}}^2 < S_{AB_1}^{\max(\text{CHSH})} S_{B_1 B_2}^{\max(\text{CHSH})} S_{B_2 C}^{\max(\text{CHSH})} \quad (12)$$

We can easily extend this result for n-local scenario. If we take $\rho_{AB_1} \otimes \rho_{B_1 B_2} \otimes \dots \otimes \rho_{B_{n-2} B_{n-1}} \otimes \rho_{B_{n-1} C}$ for the n -local chain network captured in the fig[3]. We find for this scenario,

$$I = \frac{1}{4} \sum_{i=x,y,z} (a_i + a'_i) t_{iz}^{AB_1} \times t_{zz}^{B_1 B_2} t_{zz}^{B_2 B_3} \dots t_{zz}^{B_{n-1} B_n} \times \sum_{j=x,y,z} t_{zj}^{B_n C} (c_j + c'_j)$$

and

$$J = \frac{1}{4} \sum_{i=x,y,z} (a_i - a'_i) t_{ix}^{AB_1} \times t_{xx}^{B_1 B_2} t_{xx}^{B_2 B_3} \dots t_{xx}^{B_{n-1} B_n} \times \sum_{j=x,y,z} t_{xj}^{B_n C} (c_j - c'_j)$$

Calculating the value of $S_{n\text{-local}}$ and maximizing, we finally obtain,

$$S_{n\text{-local}} = \sqrt{\sqrt{\Lambda_1 \gamma_1 \beta_1 \dots} + \sqrt{\Lambda_2 \gamma_2 \beta_2 \dots}} \quad (13)$$

All the eigenvalues come from the matrix R as previously discussed.

VI. VIOLATION CRITERIA FOR STAR-SHAPED NETWORK

In figure 5, we consider the star-shaped network composed of $n+1$ parties, where a central node (referred to as Bob) shares a quantum state with each of the n nodes (referred to as several Alice), i.e., states are provided by n independent sources. Here, we restrict our attention to the special case where each of n Alice's perform dichotomic measurements with two outputs. The inputs are denoted by $x_i \in \{0, 1\}$ for each i^{th} Alice and the outcomes are denoted by $a_i \in \{0, 1\}$. The central node measures one input y with 2^n outputs labeled by string $b = b^1 b^2 b^3 \dots b^n$ where $b^j \in \{0, 1\}$ for $j = 1, 2, 3, \dots, n$.

Similar to the bilocal case, we can say, a star-shaped network scenario is n -local if the joint probability distribution on the star-network can be written as,

$$P(a_1 \dots a_n | x_1 \dots x_n y) = \int \left(\prod_{i=1}^n d\lambda_i \rho(\lambda_i) P(a_i | x_i, \lambda_i) \right) P(b | y, \lambda_1 \dots \lambda_n) \quad (14)$$

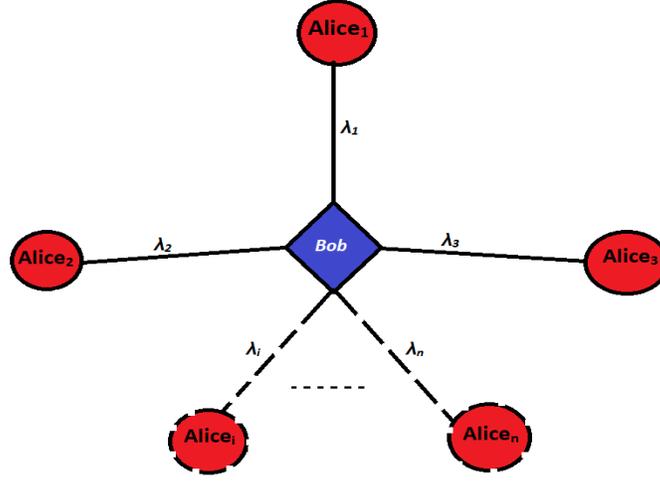


FIG. 5: n -local star shaped Network scenario

Following this, we have an inequality [4] that satisfied by the above kind of probability distributions and said as a n -local correlations. These inequalities are analogous to the Bell inequality in the star network scenario and they are like,

$$S_{star} = \sum_{j=1}^{2^{n-1}} I_j^{1/n} \leq 2^{n-2} \quad (15)$$

where,

$$I_j = \frac{1}{2^n} \sum_{x_1 \dots x_n} (-1)^{g_j(x_1, \dots, x_n)} \langle A_{x_1}^1 \dots A_{x_n}^n B^j \rangle, \quad (16)$$

each depending on a function $g_j(x_1, \dots, x_n)$. These 2^{n-1} functions contain an even number of x_i . For example, when $n = 2$,

$$g_1(x_1, x_2) = 0,$$

$$g_2(x_1, x_2) = x_1 + x_2$$

and for $n = 3$,

$$g_1(x_1, x_2, x_3) = 0, g_2(x_1, x_2, x_3) = x_1 + x_2,$$

$$g_3(x_1, x_2, x_3) = x_1 + x_3, g_4(x_1, x_2, x_3) = x_2 + x_3.$$

Again, for $n = 4$,

$$g_1(x_1, x_2, x_3) = 0, g_2(x_1, x_2, x_3) = x_1 + x_2$$

$$g_2(x_1, x_2, x_3) = x_1 + x_3, g_2(x_1, x_2, x_3) = x_1 + x_4, g_2(x_1, x_2, x_3) = x_2 + x_3$$

$$g_2(x_1, x_2, x_3) = x_2 + x_4, g_2(x_1, x_2, x_3) = x_3 + x_4, g_2(x_1, x_2, x_3) = x_1 + x_2 + x_3 + x_4$$

Now, we set the measurement choices of each Alice in a rotated bases, $(A_0^i + A_1^i) = 2 \cos \alpha \vec{n}$ and $(A_0^i - A_1^i) = 2 \sin \alpha \vec{n}'$, and for the central node, Bob, we use generalized Bell State Measurements[3]. The generalized n-party Bell basis $|\psi_{r_1 \dots r_n}\rangle_{r_1 \dots r_n}$, for $r_1 \dots r_n \in 0, 1$ came from,

$$|\psi_{r_1 \dots r_n}\rangle = Z^{r_1} \otimes X^{r_2} \otimes \dots \otimes X^{r_n} |GHZ_n\rangle \quad (17)$$

where

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

We can directly jump to the generalized Bell state measurements B^j , i.e., $B^j = M_0^j - M_1^j$, where,

$$M_b^j = \sum_{r_1 \dots r_n} \delta_{b, b^j(r_1 \dots r_n)} |\psi_{r_1 \dots r_n}\rangle \langle \psi_{r_1 \dots r_n}|$$

So, collecting all the elements, we can simply calculate the value of I_j over a quantum state ρ as,

$$I_j = \frac{1}{2^n} \sum (-1)^{g_j(x_1 \dots x_n)} \text{tr}([A_{x_1}^1 \otimes \dots \otimes A_{x_n}^n] \otimes (M_0^j - M_1^j) \rho) \quad (18)$$

As $B^j = M_0^j - M_1^j$. Here, for $n = 3$,

$$b^1(r, s, t) = r, b^2(r, s, t) = r \oplus s \oplus 1,$$

$$b^3(r, s, t) = r \oplus t \oplus 1, b^4(r, s, t) = r \oplus s \oplus t \oplus 1,$$

And for $n = 4$ [3],

$$b^1(r_1, r_2, r_3, r_4) = r_1, b^1(r_1, r_2, r_3, r_4) = r_2 \oplus r_3 \oplus 1, b^1(r_1, r_2, r_3, r_4) = r_2 \oplus r_4 \oplus 1,$$

$$b^1(r_1, r_2, r_3, r_4) = r_3 \oplus r_4 \oplus 1, b^1(r_1, r_2, r_3, r_4) = r_2 \oplus 1, b^1(r_1, r_2, r_3, r_4) = r_3 \oplus 1, b^1(r_1, r_2, r_3, r_4) = r_4 \oplus 1$$

$$b^1(r_1, r_2, r_3, r_4) = r_2 \oplus r_3 \oplus r_4.$$

Thus, for the generic quantum state $\rho = \rho_{A_1 B} \otimes \dots \otimes \rho_{A_n B}$, we calculate the value of S from eqn(16) and maximizing the the quantity S with respect to α and using Lagrange Multipliers, we have

$$S = 2^{n-2} \sqrt{\left(\prod_1^n t_1^{A_i}\right)^{\frac{1}{n}} + \left(\prod_1^n t_2^{A_i}\right)^{\frac{1}{n}}}$$

where $t_1^{A_i}$ and $t_2^{A_i}$ are the two greater eigenvalues of the correlation matrix $T_{\rho_{A_i B}}^T T_{\rho_{A_i B}}$. Here we used the argument that $\vec{n} \perp \vec{n}'$ and $T_{\rho_{A_i B}}^T T_{\rho_{A_i B}}$ is a symmetric matrix and diagonalizable.

VII. CONCLUSION

Generalizations of Bell's theorem to complex networks offer a new theoretical and experimental ground for further understanding of quantum correlations and its practical applications in information processing tasks. Similar to usual Bell-type scenarios, understanding the set of quantum correlations we can achieve and in particular, what are the optimal quantum violation of Bell inequalities, is of primal importance here. In this work, we simply generalize the result of the maximal violation of bilocal inequality to n -local inequality. We set up a scenario with n independent sources distributed in two different ways. First, the chain-shaped network where all parties have two hands except the two extreme parties and we obtain an analogous inequality given by Gisin[1] and Andreoli[13] et. al. Secondly, the n independent sources are distributed in a manner that a central node connected with many other parties called star-shaped network with one input and 2^n outputs of the central party. Each result always provides the bilocal result when we take $n = 2$, i.e., two independent sources. So, we got a more general violation by the two-qubit states for the more general network scenario with n -local inequality. It should be a possible future interest to find this bound with more efficient measurement choices and also with more complex networks.

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