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Intersection graphs of ideals of rings

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ABSTRACT

In this paper, we consider the intersection graph $G(R)$ of nontrivial left ideals of a ring R. We characterize the rings R for which the graph $G(R)$ is connected and obtain several necessary and sufficient conditions on a ring R such that $G(R)$ is complete. For a commutative ring R with identity, we show that $G(R)$ is complete if and only if $G(R[x])$ is also so. In particular, we determine the values of n for which $G(\mathbb{Z}_n)$ is connected, complete, bipartite, planar or has a cycle. Next, we characterize finite graphs which arise as the intersection graphs of $\mathbb{Z}_n$ and determine the set of all non-isomorphic graphs of $\mathbb{Z}_n$ for a given number of vertices. We also determine the values of n for which the graph of $\mathbb{Z}_n$ is Eulerian and Hamiltonian.

1. Introduction

Let $F = \{S_i \mid i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ is the one-dimensional skeleton of the nerve of F, i.e., $G(F)$ is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j, $(i, j \in I)$ are adjacent (that is, S_i and S_j are joined by an edge) if and only if $S_i \neq S_j$ and $S_i \bigcap S_j \neq \emptyset$. For intersection graphs we have the following theorem due to Marczewski [9]:

Theorem 1.1. *Every simple graph is an intersection graph, that is, for any simple graph G there exists a family F of sets S_i, $i \in I$ such that G is isomorphic to the intersection graph $G(F)$.*

Naturally, it is more interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. For the last few decades several mathematicians studied such graphs on various algebraic structures. These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. The first step in this direction was taken by Bosak [2] in 1964. Then Csákány and Pollák [4] studied the graphs of subgroups of a finite group. Zelinka [11] continued the work on intersection graphs of nontrivial subgroups of finite abelian groups. Various constructions of graphs related to the ring structure are found in [1,3,6–8].

In our present paper, we consider the intersection graph of a family of nontrivial left ideals of a ring R. Let $\mathscr{L}(R)$ denote the set of all nontrivial (nonzero proper) left ideals of R. The intersection graph of $\mathscr{L}(R)$ is the undirected simple graph

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(without loops and multiple edges) whose vertices are in a one-to-one correspondence with all nontrivial left ideals of R and two distinct vertices are joined by an edge if and only if the corresponding left ideals of R have a nontrivial (nonzero) intersection. For convenience, we denote this intersection graph by $G(R)$ and if I be a nontrivial left ideal of R then the corresponding vertex v_I in $G(R)$ is also denoted by I. Clearly the set of vertices is empty for left simple rings. Thus in the following we shall consider only the graphs $G(R)$, where R is not left simple.

We characterize the rings R for which the graph $G(R)$ is connected and obtain several necessary and sufficient conditions on a ring R so that $G(R)$ is complete. One of these characterizations is obtained in terms of a certain class of rings R which generalize integral domains (cf. Theorem 2.14) and it is found that the same property carries over to the polynomial ring over R, as in the case of integral domains (cf. Theorem 2.18). In particular, we determine the values of n for which $G(\mathbb{Z}_n)$ is connected, complete, bipartite, planar, Eulerian, Hamiltonian or has a cycle. We also characterize finite graphs which are the intersection graphs of $\mathbb{Z}_n$ and determine the set of all non-isomorphic graphs of $\mathbb{Z}_n$ for a given number of vertices. Finally it should be noted that since the structure of ideals of the ring $\mathbb{Z}_n$ is same as that of the quotient ring of any principal ideal domain by a non-trivial ideal, most of the results of Sections 3–6 can be extended to these rings. Moreover, it is interesting enough to find that every simple undirected graph is an induced subgraph of $G(\mathbb{Z}_n)$ for some n.

We denote the set of all natural numbers, integers, rational numbers, non-negative integers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_0^+$ respectively. *Throughout the paper M denotes the set of all positive integers greater than one and which are not primes.*

2. Connected and complete graphs $G(R)$

Consider the ring $\mathbb{Z}_n$ of integers modulo n for $n \in M$. We know that $\mathbb{Z}_n$ is a principal ideal ring and each of these ideals is generated by $\overline{m} \in \mathbb{Z}_n$ where m is a factor of n. For convenience, we denote this ideal by (m). In the following we characterize the values of n for which $G(\mathbb{Z}_n)$ is connected.

Theorem 2.1. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is disconnected if and only if $n = pq$, where p and q are distinct primes.*

Fig. 1. $n = pq$.

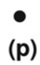


Fig. 2. $n = p^2$.

Proof. Let $n = pq$, where p and q are distinct primes. Then $\mathbb{Z}_n$ has only nontrivial ideals (p) and (q). Since $(p, q) = 1$, we have $(p) \cap (q) = (pq) = (0)$. So in the graph of $\mathbb{Z}_n$, the two vertices corresponding to the two ideals (p) and (q) are not adjacent. Since the graph of $\mathbb{Z}_n$ has only these two vertices it follows that the graph is disconnected (see Fig. 1).

Conversely, let the graph of $\mathbb{Z}_n$ be disconnected. To show that n is a product of two distinct primes, let $n = p_1p_2 \ldots p_k$, where p_i's are primes but may not be all distinct, for $i = 1, 2, \ldots, k$, $(k > 1)$. Let $k \geq 3$, $I = (p_1)$ and J be any nontrivial ideal of $\mathbb{Z}_n$. Then $J = (m)$, where $1 < m < n$, $m \mid n$.

Case 1: If $p_1 \mid m$, then $J \subseteq I$ which implies $I \cap J = J \neq (0)$. Then the vertices of the graph of $\mathbb{Z}_n$ corresponding to the two ideals I and J are adjacent.

Case 2: If p_1 does not divide m, then for any prime factor q of m, $(p_1, q) = 1$. Since $k \geq 3$, $n > p_1q$. Let $K = (q)$. Then $I \cap K \neq (0)$ as $\overline{p_1q} \in I \cap K$. But $q \mid m$ and so $\overline{m} \in (q) = K$ which implies $K \cap J \neq (0)$. Since I and K have nonzero intersection and J and K have nonzero intersection, the vertices of the graph of $\mathbb{Z}_n$ corresponding to the ideals I and J are connected by a path. Now let J and K be two vertices. Then from the above two cases we find that both the vertices are connected to I. Hence the vertices J and K are connected. Therefore the graph is connected. So for $k \geq 3$, the graph of $\mathbb{Z}_n$ is connected. Thus the only option left here is $k = 2$ and $p_1 \neq p_2$, as the graph of $\mathbb{Z}_{p^2}$ (where p is a prime) contains only a single point and hence connected (cf. Fig. 2). Therefore n is of the form pq, where p and q are distinct primes. □

We now find out conditions when the graph $G(R)$ of an arbitrary ring R is connected.

Example 2.2. Let K_1 and K_2 be two fields. Consider the direct product $K_1 \times K_2$ of these two fields. This direct product has only two nonzero proper ideals which are minimal. So the graph of this ring is disconnected.

Example 2.3. Consider the Klein's four group $K_4 = \{0, a, b, c\}$, where 0 is the identity, $a + a = b + b = c + c = 0$ and $a + b = b + a = c$, $a + c = c + a = b$, $b + c = c + b = a$. Define a multiplication on this abelian group by $x \cdot y = 0$. Then K_4 becomes a null ring. This ring contains three nontrivial ideals $\{a, 0\}$, $\{b, 0\}$, $\{c, 0\}$ which are minimal. Thus the graph of this ring is disconnected.

Theorem 2.4. *The intersection graph $G(R)$ of a ring R is disconnected if and only if R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal (as well as maximal).*

Proof. Let $G(R)$ be a disconnected graph. Suppose, $C_1, C_2, \ldots, C_k$ are components of $G(R)$. Because the graph is disconnected, $k \geq 2$. Let $I \in C_1$ and $J \in C_2$. Hence there is no path between I and J. This implies that $I \cap J = (0)$. Suppose that $I + J \neq R$. Then $I + J$ is a nontrivial left ideal of R such that $I, J \subseteq I + J$. Hence there is a path $(I, I + J, J)$ from I to J. This contradicts our assumption that I and J are in different components. Hence $I \cap J = (0)$ and $I + J = R$. We now show that I is a minimal left ideal of R. Let A be a left ideal of R such that $(0) \neq A \subseteq I$. Then the vertex A is adjacent to the vertex I and hence $A \in C_1$. This implies that A and J are not connected by a path. Hence arguing as above we have $A + J = R$. Let $i \in I$. Now $i = a + j$, where $a \in A, j \in J$. Then $i - a = j \in I \cap J = (0)$ and so $i = a \in A$ which implies $I \subseteq A$, i.e., $A = I$. Therefore I is a minimal left ideal of R. Moreover if there is any nontrivial left ideal $J \supseteq I$, then similarly one can show that $J = I$. Thus every member of C_1 is a minimal (as well as maximal) left ideal of R. Hence R contains at least two minimal left ideals and every nontrivial left ideal of R is minimal (as well as maximal). Converse part can be proved easily. □

It is interesting to note the following:

Corollary 2.5. *For any graph $G = G(R)$ of a ring R, whenever G is disconnected, it is a null graph (i.e., it has no edge).*

Now we will see an application of the previous theorem in the following example.

Example 2.6. Let R be a ring of 2×2 matrices over an infinite field K. Then R has an infinite number of proper left ideals. In this ring

$$I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \text{where } a, b \in K \right\} \quad \text{and} \quad J = \left\{ \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}, \text{where } c, d \in K \right\}$$

are minimal left ideals. Also these two left ideals are maximal. There do not exist any path between these two left ideals. Hence the graph $G(R)$ is disconnected. Then from the previous theorem it follows that every nontrivial left ideal of the ring R is minimal as well as maximal.

Theorem 2.7. *Let R be a commutative ring. Then the graph $G(R)$ is disconnected if and only if R is a direct product of two simple commutative rings, i.e., $R = R_1 \times R_2$ where each R_i $(i = 1, 2)$ is either a field or a null ring with prime number of elements.*

Proof. Let R be a commutative ring such that the graph $G(R)$ is disconnected. Then by Theorem 2.4, it follows that there are at least two maximal ideals I and J of R such that $I \cap J = (0)$ and $I + J = R$. Then R is isomorphic to the direct product of simple commutative rings R/I and R/J.

Conversely, let $R = R_1 \times R_2$ where each R_i $(i = 1, 2)$ is a simple commutative ring. If both R_1 and R_2 are fields, then R has only two nontrivial ideals, namely, $I = R_1 \times \left\{0_{R_2}\right\}$ and $J = \left\{0_{R_1}\right\} \times R_2$ where $I \cap J = (0_R)$. Hence R is disconnected. Suppose both R_1 and R_2 are null rings with prime number of elements. Let $(R_1, +) \cong (\mathbb{Z}_p, +)$ and $(R_2, +) \cong (\mathbb{Z}_q, +)$, where p, q are prime numbers. If p and q are distinct, then $(R, +) \cong (\mathbb{Z}_p \times \mathbb{Z}_q, +) \cong (\mathbb{Z}_{pq}, +)$ which has only two nontrivial subgroups, namely, $(\bar{p})$ and $(\bar{q})$. These two subsets are also only nontrivial ideals of the null ring R which have zero intersection. Hence R is disconnected.

Next suppose $p = q$. In this case also ideals of R are precisely the subgroups of $(R, +)$. Now we know that the intersection graph of nontrivial subgroups of the abelian group $(\mathbb{Z}_p \times \mathbb{Z}_p, +)$ is disconnected where all the vertices are isolated [11]. So $G(R)$ is disconnected.

Finally, consider the case where R_1 is a field and R_2 is a null ring where $(R_2, +) \cong (\mathbb{Z}_p, +)$ for some prime number p. It is a routine to verify that only nontrivial ideals of R are $(0_{R_1}) \times R_2$ and $R_1 \times (\bar{0})$. Hence in this case also $G(R)$ is disconnected. □

Corollary 2.8. *Let R be a commutative ring with identity. Then the graph $G(R)$ is disconnected if and only if R is a direct product of two fields.*

Next we characterize the values of n for which $G(\mathbb{Z}_n)$ is complete.

Theorem 2.9. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is complete if and only if $n = p^m$, where p is a prime number and $m \in N$, $(m > 1)$.*

Proof. Let $n = p^m$ for some prime p and $m \in \mathbb{N}$, $(m > 1)$. Then $\mathbb{Z}_n$ has (nontrivial) ideals (p^i), where $i = 1, 2, \ldots, m - 1$. All of them contain the element $\overline{p}^{m-1}$. Therefore in the graph of $\mathbb{Z}_n$ all vertices are adjacent to each other. Hence we get a complete graph.

Conversely, let $G(\mathbb{Z}_n)$ be complete. Suppose $n = {p_1}^{n_1}{p_2}^{n_2} \ldots {p_m}^{n_m}$, where p_i's are all distinct primes, $n_i \in \mathbb{N}$ and $m \geq 2$. So $({p_1}^{n_1}) \cap ({p_2}^{n_2} \ldots {p_m}^{n_m}) = (0)$. Thus the graph of $\mathbb{Z}_n$ cannot be complete. Hence $m = 1$ and so $n = {p_1}^{n_1}$, $(n_1 > 1)$, in which case the graph of $\mathbb{Z}_n$ is complete, as we have shown above. □

We note that the intersection graph of the ring $\mathbb{Z}$ of all integers is complete. In general, we have the following theorem:

Theorem 2.10. *The graph of an integral domain (but not a field) is complete.*

Proof. Let R be any integral domain which is not a field. Then in the graph of R, the vertex set is not empty. Let I and J be any two nontrivial ideals of R. Let $a \in I$, $b \in J$ where $a, b \neq 0$. Since R is an integral domain, we have $ab \neq 0$. So $I \cap J \neq (0)$ as $ab \in I \cap J$. Hence the graph of R is complete. $\square$

Note that the converse of this result is not true. For example, the ring $\mathbb{Z}_{p^4}$, where p is a prime number, is not an integral domain, but the corresponding intersection graph is complete. Note that it is an Artinian ring with only one minimal ideal. In fact, we obtain the following theorem:

Theorem 2.11. *The graph of a left Artinian ring R is complete if and only if R has a unique minimal left ideal.*

Proof. Let R be a left artinian ring which has a unique minimal left ideal, say I. Let J be a nonzero proper left ideal of R. Since R is left artinian, J contains a minimal left ideal I_0 of R. This implies that $I_0 = I$. Then $J \supseteq I$. Therefore for any nonzero proper left ideal J of R, we have $J \supseteq I$. Now, let J_1, J_2 be two nonzero proper left ideals of R. Therefore $J_1 \supseteq I, J_2 \supseteq I$, which gives $J_1 \cap J_2 \supseteq I \neq (0)$. Then the vertices of $G(R)$ corresponding to J_1 and J_2 are adjacent. Hence R is complete.

Conversely, let R be a left artinian ring and the graph $G(R)$ be complete. Because R is a left artinian ring, R has a minimal left ideal. We show that R has a unique minimal left ideal. If possible, let I and J be two distinct minimal left ideals of R, then $I \cap J = (0)$. So the vertices of $G(R)$ corresponding to I and J are not adjacent which contradicts the fact that $G(R)$ is complete. Therefore $I = J$. Hence R has a unique minimal left ideal. $\square$

It is interesting to note that if the ring R in the above theorem is commutative, then R also has a unique maximal ideal, i.e., R is a local ring and $\dim_K(M_1/M_1^2) = 1$, where M_1 is the unique maximal ideal of R and K is the residual field R/M_1.

Example 2.12. Let R be the polynomial ring $\mathbb{Q}[X]$ and I be the ideal generated by X^3. Then the quotient ring $\mathbb{Q}[X]/I$ is an infinite Artinian ring with only one minimal ideal J where J is generated by X^2. So $G(R)$ is complete. The only maximal ideal, M_1 of R is (X). So $K \cong \mathbb{Q}$ and $\dim_K(M_1/M_1^2) = \dim_{\mathbb{Q}}\big((X)/(X^2)\big) = 1$.

Example 2.13. Consider the ring R of 2×2 matrices over an infinite field. We know that this ring R is a left Artinian ring. Because R has more than one minimal left ideal, this graph is not complete.

In the following theorem we characterize commutative rings R with identity for which $G(R)$ is complete. Let I be an ideal of R. Then I is said to be *decomposable* if there exist ideals J, K of R such that $I = J + K$ and $J \cap K = (0)$. The ideal I is said to be *indecomposable* if it is not decomposable.

Theorem 2.14. *Let R be a commutative ring with identity. Then the following conditions are equivalent:*

(i) *$G(R)$ is complete;*
(ii) *for any two ideals I, J of R, $I \cap J = (0)$ implies $I = (0)$ or $J = (0)$;*
(iii) *for any $a, b \in R$, $(a) \cap (b) = (0)$ implies $a = 0$ or $b = 0$;*
(iv) *every nonzero ideal of R is indecomposable.*

Proof. The proof for (i) $\Longleftrightarrow$ (ii) follows from definition of a complete graph and (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) are obvious. $\square$

In this context, it is interesting to note that there are rings in which every nonzero principal ideals are indecomposable, but not all the nonzero ideals are so, i.e., the graphs of these rings are not complete.

Example 2.15. [1] Consider the ring $R = \mathbb{Q}[X, Y]/((X, Y)^2)$. In R, the ideal $(X, Y) = (X) + (Y)$ and $(X) \cap (Y) = (0)$. But the non-trivial principal ideals, namely, the ones of the form $(aX + bY)$ with $(a, b) \neq (0, 0)$ $a, b \in \mathbb{Q}$ are indecomposable.

Now the condition (iii) is a generalization of that of an integral domain. We know that if a commutative ring R with identity is an integral domain, then the polynomial ring $R[x]$ is also so. In the following we show that similar result holds for the above generalization also.

Lemma 2.16. *Let R be a commutative ring with identity such that $G(R)$ is complete. Then for any $a \in R \setminus \{0\}$ and $f \in R[x] \setminus \{0\}$, $(a) \cap (f) \neq (0)$.*

Proof. Let $a \in R \setminus \{0\}$ and $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x] \setminus \{0\}$, where $a_n \neq 0$. Since $G(R)$ is complete and $a, a_n \neq 0$, there exist $\alpha_1, \beta_1 \in R$ such that $c = a\alpha_1 = a_n\beta_1 \neq 0$. Then $f\beta_1 = a_0\beta_1 + a_1\beta_1x + a_2\beta_1x^2 + \cdots + cx^n \neq 0$. Let r be the

[1] Suggested by the learned referee.

highest nonnegative integer less than n such that $a_r\beta_1 \neq 0$. Then there are $\alpha_2, \beta_2 \in R$ such that $c_2 = c\alpha_2 = a_r\beta_1\beta_2 \neq 0$. So $f\beta_1\beta_2 = a_0\beta_1\beta_2 + a_1\beta_1\beta_2 x + a_2\beta_1\beta_2 x^2 + \cdots + c_2 x^r + c\beta_2 x^n \neq 0$ as $c_2 \neq 0$. Note that both c_2 and $c\beta_2$ are multiples of c. We repeat this process until we get $f\beta = \sum \lambda_i x^{r_i} \neq 0$ for some $\beta \in R$, where $c \mid \lambda_i$ for each i. Thus $f\beta \in (a) \cap (f)$ as $a \mid c$ and so we have $(a) \cap (f) \neq (0)$. □

Lemma 2.17. *Let R be a commutative ring with identity such that $G(R)$ is complete. Suppose $m \geqslant n$ be two positive integers such that $(f) \cap (g) \neq (0)$ for any two nonzero polynomials in $R[x]$ with $\deg f \leqslant m$ and $\deg g < n$ or $\deg f < m$ and $\deg g \leqslant n$. Then $(f) \cap (g) \neq (0)$ for any two nonzero polynomials in $R[x]$ with $\deg f = m$ and $\deg g = n$.*

Proof. Let $f = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$, $g = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n \in R[x]$, where $a_m, b_n \neq 0$. Now there exist $\alpha, \beta \in R$ such that $c = a_m\alpha = b_n\beta \neq 0$. Then $f\alpha, g\beta \in R[x] \setminus \{0\}$ such that both of them have the same leading coefficient, c. If $f\alpha = g\beta x^{m-n}$, then we are done. Otherwise $f\alpha - g\beta x^{m-n} \neq 0$ and $\deg(f\alpha - g\beta x^{m-n}) < m$. Thus by hypothesis, there are $h_1, h_2 \in R[x]$ such that

$$(f\alpha - g\beta x^{m-n})h_1 = g\beta h_2 \neq 0.$$

This implies $f\alpha h_1 = g\beta(x^{m-n}h_1 + h_2)$. Now if $f\alpha h_1 \neq 0$, then $0 \neq f\alpha h_1 = g\beta(x^{m-n}h_1 + h_2) \in (f) \cap (g)$. Otherwise let $f\alpha h_1 = 0$. Then $g\beta x^{m-n}h_1 \neq 0$, as $(f\alpha - g\beta x^{m-n})h_1 \neq 0$, which implies $g\beta h_1 \neq 0$. We remove all the terms t from h_1 for which $f\alpha t = g\beta t = 0$. Let the reduced polynomial be h_3 and still we have $f\alpha h_3 = 0$ and $g\beta h_3 \neq 0$. Let λ be the leading coefficient of h_3. Then $c\lambda = 0$ as $f\alpha h_3 = 0$. Now if $f\alpha\lambda = 0$, then since h_3 is reduced, we certainly have $g\beta\lambda \neq 0$. But $\deg(g\beta\lambda) < n$, as $c\lambda = 0$. Then there exist $h_4, h_5 \in R[x]$ such that $f\alpha h_4 = g\beta\lambda h_5 \neq 0$. Otherwise $f\alpha\lambda \neq 0$. Then $\deg(f\alpha\lambda) < m$ (as $c\lambda = 0$). So there exist $h_6, h_7 \in R[x]$ such that $f\alpha\lambda h_6 = g\beta h_7 \neq 0$. Thus in either case, we have $(f) \cap (g) \neq (0)$, as required. □

Theorem 2.18. *Let R be a commutative ring with identity. Then $G(R)$ is complete if and only if $G(R[x])$ is complete.*

Proof. Let R be a commutative ring with identity such that $G(R)$ is complete. Let $f, g \in R[x] \setminus \{0\}$ with $\deg f = m$ and $\deg g = n$. We show that

$$(f) \cap (g) \neq (0). \quad (1)$$

Consider the set $\mathbb{N} \times \mathbb{N}$. It is easy to verify that $\mathbb{N} \times \mathbb{N}$ is well-ordered with the lexicographic ordering which is defined by

$$(a, b) \leqslant (c, d) \quad \text{if and only if either } a \leqslant c \text{ or } a = c, b \leqslant d.$$

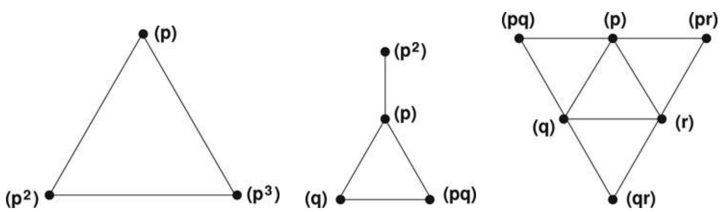
We show that (1) is true for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ by transfinite induction. By Lemma 2.16, (1) is true for $(m, n) = (0, 0)$. Now let (1) be true for any $(m, n) < (r, s)$. We show that (1) also holds for $(m, n) = (r, s)$. Now by induction hypothesis we have (1) is true for all $m < r, n \leqslant s$ and $m \leqslant r, n < s$. Then by Lemma 2.17, (1) is true for $m = r, n = s$. This completes the induction and proves that (1) is true for arbitrary $(m, n) \in \mathbb{N} \times \mathbb{N}$.

Conversely, let R be a commutative ring with identity such that $G(R[x])$ is complete. Let $a, b \in R \setminus \{0\}$. Since $G(R[x])$ is complete, we have $h_1, h_2 \in R[x]$ such that $ah_1 = bh_2 \neq 0$. We remove all the terms t from h_1 so that $at = 0$ as well as remove all the terms τ from h_2 so that $b\tau = 0$. Let h_3 and h_4 be obtained from h_1 and h_2 respectively after such reduction. Then $ah_3 = bh_4 \neq 0$. Let α and β be the leading coefficients of h_3 and h_4 respectively. Since h_3 and h_4 are reduced, we have $a\alpha = b\beta \neq 0$. Thus $(a) \cap (b) \neq (0)$ in R. Therefore $G(R)$ is complete. □

3. Bipartition and planarity condition of $G(\mathbb{Z}_n)$

In this section we consider the problem of characterizing rings $\mathbb{Z}_n$ for which intersection graphs of $\mathbb{Z}_n$ are bipartite or planar.

Lemma 3.1. *Let $n \in M$. The graph of $\mathbb{Z}_n$ contains a cycle (of length 3) if $n = p^4, p^2q$ or pqr, where p, q, r are distinct primes.*



(a) $n = p^4$. (b) $n = p^2q$. (c) $n = pqr$.

Fig. 3.

Proof. Follows directly from Fig. 3. □

Theorem 3.2. *Let $n \in M$. The graph of $\mathbb{Z}_n$ has a cycle if and only if $n = mt$, where $m = p^4, p^2q$ or pqr, p, q, r are distinct primes and $t \in \mathbb{N}$.*

Fig. 4. $n = p^3$.

Proof. Let $n = mt$, where $m = p^4, p^2q$ or pqr, p, q, r are distinct primes and t is any positive integer. Then the graph of $\mathbb{Z}_n$ will contain a subgraph isomorphic to $\mathbb{Z}_m$. Then by the above lemma the graph of $\mathbb{Z}_n$ contains a cycle.

Conversely, suppose that the graph of $\mathbb{Z}_n$ contains a cycle. Let $n = p_1p_2 \dots p_k$, $(k > 1)$, where p_i's are prime numbers but may not be all distinct. For $k < 3$, the number of vertices of $G(Z_n)$ is less than 3. Hence it cannot contain a cycle. Let $k = 3$. Then n is of the form p^3, p^2q or pqr, where p, q, r are distinct primes. If $n = p^3$, then the graph of $\mathbb{Z}_n$ has only two vertices and so it does not contain any cycle (cf. Fig. 4). In other two cases graphs of $\mathbb{Z}_n$ contain cycles and satisfy the required condition. If $k \geq 4$, then n is always a multiple of m, where $m = p^4, p^2q$ or pqr for some distinct primes p, q, r. Hence the result follows. □

Corollary 3.3. *Let $n \in M$. The graph of $\mathbb{Z}_n$ does not contain a cycle if and only if $n = p^2, pq$ or p^3, where p and q are distinct primes. In all other cases, it contains a cycle of length 3.*

The above corollary has an important consequence. Since we know [10] that a simple graph is bipartite if and only if it has no odd cycle, we immediately have the following result:

Theorem 3.4. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is bipartite if and only if $n = pq$ or p^3, where p and q are distinct primes.*

In the following we determine the values of n for which $G(\mathbb{Z}_n)$ is planar.

Lemma 3.5. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is planar if $n = p^5, p^2q$ or pqr, where p, q and r are distinct primes.*

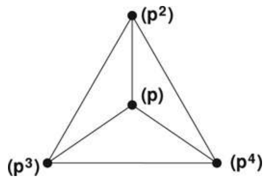


Fig. 5. $n = p^5$.

Proof. Follows directly from Figs. 3(b), (c) and 5. □

Corollary 3.6. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is planar if $n = p_1p_2 \dots p_k$, $1 < k \leq 3$, where each p_i is a prime number (may not be all distinct).*

Proof. Since any graph of $\mathbb{Z}_n$ for $n = p_1p_2 \dots p_k$ (where $1 < k \leq 3$ and each p_i is prime) is isomorphic to a subgraph of any one of the three graphs of Lemma 3.5, the result follows. □

Lemma 3.7. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is non-planar if $n = p^3q, p^2q^2, p^2qr$ or $pqrs$, where p, q, r, s are distinct primes.*

Proof. The subgraph shown in Fig. 6(a) is contained in graphs of $\mathbb{Z}_n$ for $n = p^3q, p^2q^2, p^2qr$, whereas Fig. 6(b) is a subgraph of $\mathbb{Z}_{pqrs}$ (p, q, r, s are distinct primes). Both subgraphs are isomorphic to K_5 and hence they are non-planar. □

Corollary 3.8. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is non-planar if $n = p_1p_2 \dots p_k$, where $k \geq 5$ and each p_i is prime (may not be all distinct), unless $k = 5$ and $p_1 = p_2 = \dots = p_5$.*

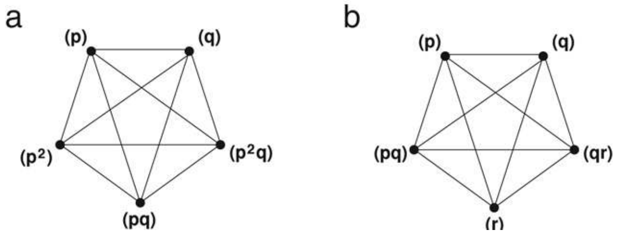


Fig. 6.

Proof. Let $k = 5$, but n is not of the form p^5 (where p is prime). Then n is one of the following forms: p^4q, p^3q^2, p^3qr, p^2q^2r, p^2qrs or $pqrst$, where p, q, r, s, t are distinct primes. But all of them are multiples of some of the forms of n described in Lemma 3.7. Thus graphs of $\mathbb{Z}_n$ for these values of n contain K_5 as a subgraph and hence they are non-planar. If $k > 5$, then n is either a multiple of some of the forms of n described in Lemma 3.7 or n is a multiple of p^6, in which case also the graph of $\mathbb{Z}_n$ contains K_5, as the graph of $\mathbb{Z}_{p^6}$ is K_5. □

Theorem 3.9. *Let $n \in M$. The graph of $\mathbb{Z}_n$ is planar if and only if n is one of the forms: p^i $(2 \leqslant i \leqslant 5)$, pq, p^2q, pqr, where p, q, r are distinct primes.*

Proof. Follows immediately from Lemma 3.5, Corollary 3.6, Lemma 3.7 and Corollary 3.8. □

4. Counting principles

In the following we wish to count degrees of vertices of the graph $G = G(\mathbb{Z}_n)$ of the ring $\mathbb{Z}_n$ for any natural number $n \in M$. Let T_n be the total number of vertices of G. Then T_n is the number of nonzero proper ideals of $\mathbb{Z}_n$. We know that $\mathbb{Z}_n$ is a principal ideal ring and each of these ideals is generated by $\overline{m} \in \mathbb{Z}_n$ where m is a factor of n, except 1 and n itself. For convenience, as before, we denote the ideal $(\overline{m})$ by (m) and T_n by T. Let $n = p_1^{n_1}p_2^{n_2}\cdots p_m^{n_m} \in M$, where $m, n_j \in \mathbb{N}$, $(m, n_1) \neq (1, 1)$, p_j are distinct primes $(j = 1, 2, \ldots, m)$. Then

$$T = (n_1 + 1)(n_2 + 1)\cdots(n_m + 1) - 2. \tag{2}$$

Let $I = (p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}p_{k+1}^{i_{k+1}}\cdots p_m^{i_m})$ be a nonzero proper ideal of $\mathbb{Z}_n$, where $0 \leqslant k < m, 0 \leqslant i_j < n_j, i_j \in \mathbb{Z}, j = k+1, k+2, \ldots, m$ (provided in the case $k = 0$, not all i_j are zero).

Let us count the degree of the vertex v_I of G corresponding to the ideal I of $\mathbb{Z}_n$. Now any vertex corresponding to the ideal $J \neq I$, which is not adjacent to v_I must contain the factor $p_{k+1}^{n_{k+1}}\cdots p_m^{n_m}$. So the total number of such vertices is one less than the total number of factors of $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$, i.e.,

$$(n_1 + 1)(n_2 + 1)\cdots(n_k + 1) - 1.$$

Note that the term -1 is coming due to the fact that the ideal

$$(p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}p_{k+1}^{n_{k+1}}\cdots p_m^{n_m}) = (n) = (0).$$

Also the vertex v_I is not adjacent to itself. So if N is the number of vertices of G which are not adjacent to v_I, then

$$N_I = \begin{cases} (n_1 + 1)(n_2 + 1)\cdots(n_k + 1), & \text{if } k \neq 0; \\ 1, & \text{if } k = 0. \end{cases} \tag{3}$$

Therefore the degree d_I of the vertex v_I is given by

$$d_I = T - N_I. \tag{4}$$

Now when $N_I = 1, k = 0$. Then I is of the form $(p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m})$, where $0 \leqslant i_j < n_j, i_j \in \mathbb{Z}, j = 1, 2, \ldots, m$, (but not all $i_j = 0$, in which case $I = (1) = \mathbb{Z}_n$). Therefore the number of vertices of degree $T - 1$ is

$$n_1n_2\ldots n_m - 1. \tag{5}$$

In general, the number of vertices of degree $T - (n_{r_1} + 1)(n_{r_2} + 1)\cdots(n_{r_s} + 1)$, $(r_i \in \{1, 2, \ldots, m\}, i = 1, 2, \ldots, s, 0 < s < m)$ is

$$\prod_{\substack{r=1 \\ r\notin\{r_1,r_2,\ldots,r_s\}}}^{m} n_r. \tag{6}$$

Let us explain how to calculate the total number of vertices of G of a given degree, say d. Let ν_d be the total number of vertices of G of degree d. Clearly d must be less than T.

Step I: If $d = T - 1$, then $\nu_d = n_1 n_2 \dots n_m - 1$.

Step II: For $d < T - 1$, factorize $T - d$ by means of factors in the collection $\{n_i + 1 \mid i = 1, 2, \dots, m\}$. For each such factorization $T - d = (n_{r_1} + 1)(n_{r_2} + 1) \cdots (n_{r_s} + 1)$, $(r_i \in \{1, 2, \dots, m\}, i = 1, 2, \dots, s, 0 < s < m)$, compute the product given by (6). If no such factorization is found, then $\nu_d = 0$.

Step III: Add all the products obtained in step II to get the value of ν_d.

Example 4.1. Let us consider the graph G of the ring $\mathbb{Z}_{pqr}$, where p, q, r are distinct prime numbers (cf. Fig. 3(c)). Now $n_1 = n_2 = n_3 = 1$ and so $T = 6$. Let $I = (p)$. Then $N_I = 2$ and so $d_I = T - N_I = 4$. Similarly, $d_{(q)} = d_{(r)} = 4$. Again for $J = (pq)$, $N_J = 4$. Hence $d_J = 2$. Likewise $d_{(qr)} = d_{(pr)} = 2$. On the other hand, the possible values of N are $1, 1 + 1, (1 + 1)(1 + 1)$, i.e., 1, 2 or 4. Thus possible degrees of vertices of G are $6 - 1, 6 - 2, 6 - 4$, i.e., 5, 4 or 2. Now the number of vertices of degree 5, 4 and 2 are respectively, $n_1 n_2 n_3 - 1 = 0$, $n_2 n_3 + n_3 n_1 + n_1 n_2 = 3$, $n_3 + n_2 + n_1 = 3$. Thus G is a graph of 6 vertices of which 3 are of degree 4 and the other 3 are of degree 2.

Now let us calculate the number of edges of the graph $G = G(\mathbb{Z}_n)$. Let e be the total number of edges of G. We know that $2e$ is the sum of degrees of all vertices of G. Thus it follows from (6) that

$$\begin{aligned}
2e &= \{(T-1)(n_1 n_2 \cdots n_m - 1)\} + \{(T - (n_1+1)) n_2 n_3 \cdots n_m + (T - (n_2+1))\, n_1 n_3 \cdots n_m \\
&\quad + \cdots + (T - (n_m + 1))\, n_1 n_2 \cdots n_{m-1}\} \\
&\quad + \{(T - (n_1+1)(n_2+1)) n_3 n_4 \cdots n_m + (T - (n_1+1)(n_3+1)) n_2 n_4 \cdots n_m \\
&\quad + \cdots + (T - (n_{m-1}+1)(n_m+1))\, n_1 n_2 \cdots n_{m-2}\} \\
&\quad + \cdots + \{(T - (n_1+1)(n_2+1) \cdots (n_{m-1}+1))\, n_m + (T - (n_1+1)(n_2+1) \cdots (n_{m-2}+1)(n_m+1))\, n_{m-1} \\
&\quad + \cdots + (T - (n_2+1) \cdots (n_m+1))\, n_1\} \\
&= 1 - T + T\,\{n_1 n_2 \cdots n_m + (n_2 n_3 \cdots n_m + n_1 n_3 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-1}) \\
&\quad + (n_3 n_4 \cdots n_m + n_2 n_4 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-2}) + \cdots + (n_m + n_{m-1} + \cdots + n_1)\} \\
&\quad - \{n_1 n_2 \cdots n_m + (n_1+1) n_2 \cdots n_m + n_1 (n_2+1) n_3 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-1}(n_m+1) \\
&\quad + (n_1+1)(n_2+1) n_3 \cdots n_m + (n_1+1) n_2 (n_3+1) n_4 \cdots n_m + n_1 \cdots n_{m-2}(n_{m-1}+1)(n_m+1) \\
&\quad + \cdots + (n_1+1)(n_2+1) \cdots (n_{m-1}+1) n_m + \cdots + n_1 (n_2+1)(n_3+1) \cdots (n_m+1)\} \\
&= 1 - T + T(T+1) - \{(2^m - 1)\, n_1 n_2 \cdots n_m + (2^{m-1} - 1)\,(n_2 n_3 \cdots n_m + n_1 n_3 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-1}) \\
&\quad + (2^{m-2} - 1)\,(n_3 n_4 \cdots n_m + \cdots + n_2 n_4 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-2}) \\
&\quad + \cdots + (2-1)\,(n_m + n_{m-1} + \cdots + n_1)\} \\
&= T^2 + 1 + \{n_1 n_2 \cdots n_m + (n_2 n_3 \cdots n_m + n_1 n_3 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-1}) \\
&\quad + (n_3 n_4 \cdots n_m + n_2 n_4 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-2}) + \cdots + (n_m + n_{m-1} + \cdots + n_1)\} \\
&\quad - \{2^m\, n_1 n_2 \cdots n_m + 2^{m-1}\,(n_2 n_3 \cdots n_m + n_1 n_3 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-1}) \\
&\quad + 2^{m-2}\,(n_3 n_4 \cdots n_m + \cdots + n_2 n_4 \cdots n_m + \cdots + n_1 n_2 \cdots n_{m-2}) \\
&\quad + \cdots + 2\,(n_m + n_{m-1} + \cdots + n_1)\} \\
&= T^2 + 1 + (T+1) + \{(2n_1+1)(2n_2+1) \cdots (2n_m+1) - 1\} \\
&= T^2 + T + 2 - (t-1) \\
&= 1 + T + T^2 - t,
\end{aligned}$$

where t is the total number of vertices of the graph $G(\mathbb{Z}_{n^2})$.

Let us denote the number of vertices and edges of $G = G(\mathbb{Z}_n)$ by T_n and e_n respectively. Then

$$2e_n = 1 + T_n + T_n^2 - T_{n^2}. \tag{7}$$

So we get that

$$e_n = \binom{T_n}{2} - \frac{1}{2}\left(T_{n^2} - 2T_n - 1\right). \tag{8}$$

It is interesting to note that the maximum possible edges of G is $\binom{T_n}{2}$ (when it is complete) and so $T_{n^2} - 2T_n - 1 \geqslant 0$. Thus we obtain the following inequality:

$$T_{n^2} > 2T_n. \tag{9}$$

Example 4.2. Consider the graph $G = G(\mathbb{Z}_n)$ for $n = p^2qr$, where p, q, r are distinct prime numbers. In this case $T_n = (2+1)(1+1)(1+1) - 2 = 10$ and $T_{n^2} = (4+1)(2+1)(2+1) - 2 = 43$. So the total number of edges in G is $\frac{1}{2}(1 + 10 + 10^2 - 43) = 34$.

5. Characterization of Eulerian and Hamiltonian graphs

Now we apply the above results to characterize the values of n for which graphs of $\mathbb{Z}_n$ are Eulerian. We know [10] that a simple connected graph G is Eulerian if and only if every vertex of G is of even degree.

Theorem 5.1. *Let $n \in M$. Then the graph of $\mathbb{Z}_n$ is Eulerian if and only if $n = p_1p_2\dots p_m$ or $n = {p_1}^{n_1}{p_2}^{n_2}\dots{p_m}^{n_m}$, where each n_i is even ($n_i \in \mathbb{N}, i = 1, 2, \dots, m$) and p_i's are distinct primes.*

Proof. *Case* 1: Let $n = p_1p_2\dots p_m$, where p_i's are distinct primes. Then T, the total number of vertices of $G = G(\mathbb{Z}_n)$ is $(1+1)(1+1)\dots(1+1) - 2 = 2^m - 2$. Let $I = (p_{i_1}p_{i_2}\dots p_{i_k})$, ($i_j \in \{1, 2, \dots, m\}, j = 1, 2, \dots, k, 0 < k < m$), be any nonzero proper ideal of $\mathbb{Z}_n$. Then $N_I = (1+1)(1+1)\dots(1+1) = 2^k$. Hence $d_I = T - N_I = 2^m - 2 - 2^k$ which is even. Thus G is an Eulerian graph.

Case 2: Let $n = {p_1}^{n_1}{p_2}^{n_2}\dots{p_m}^{n_m}$, where n_i's are all even and p_i's are distinct primes. Then $T = (n_1+1)(n_2+1)\dots(n_m+1)-2$, which is odd. Let $I = ({p_1}^{n_1}{p_2}^{n_2}\dots{p_k}^{n_k}{p_{k+1}}^{i_{k+1}}\dots{p_m}^{i_m})$, where $0 \leqslant k < m, 0 \leqslant i_j < n_j, i_j \in \mathbb{Z}, j = k+1, k+2, \dots, m$ (provided in the case $k = 0$, not all i_j are zero). Then $N_I = 1$ or $(n_1+1)(n_2+1)\cdots(n_k+1)$ according as $k = 0$ or not. In either case N_I is odd, as each n_i is even. Thus $d_I = T - N_I$ is even and hence, in this case also, G is an Eulerian graph.

Conversely, let the graph G of $\mathbb{Z}_n$ be Eulerian for $n = {p_1}^{n_1}\dots{p_m}^{n_m} \in M$, where $m, n_j \in \mathbb{N}$, $(m, n_1) \neq (1, 1)$, p_i's are distinct primes ($i = 1, 2, \dots, m$). Now G is Eulerian for $n_i = 1$ for all $i = 1, 2, \dots, m$ by the case I above. Also the same is true for all n_i's are even. So let us now consider the case when not all n_i's are even and at least one n_i is greater than 1. Then $T = (n_1+1)(n_2+1)\dots(n_m+1) - 2$ is even. Let $I = ({p_1}^{n_1-1}{p_2}^{n_2-1}\dots{p_m}^{n_m-1})$. Then $N_I = 1$ (as $k = 0$). Thus $d_I = T - 1$ which is odd. This contradicts the fact that G is Eulerian. Hence the result follows. □

As no general characterization of Hamiltonian graphs exist, it is interesting to consider the problem of determining conditions on n under which a graph $G(\mathbb{Z}_n)$ is Hamiltonian. In the sequel, we prove the following theorem:

Theorem 5.2. *All the graphs of $\mathbb{Z}_n (n \in M)$ are Hamiltonian except when n is one of the following forms: p^2, pq, p^2q, p^3, where p, q are distinct primes.*

First we have a sufficient condition.

Lemma 5.3.[2] *Let $n = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k} \in M$, where $k, n_j \in \mathbb{N}$, $(k, n_1) \neq (1, 1)$, p_j are distinct primes ($j = 1, 2, \dots, k$). Then the graph $G(\mathbb{Z}_n)$ is Hamiltonian if*

$$(n_1+1)(n_2+1)\cdots(n_k+1) \geqslant 5 \tag{10}$$

$$n_1n_2\dots n_k - 1 \geqslant k. \tag{11}$$

Proof. Let $G = G(\mathbb{Z}_n)$ such that (10) and (11) are satisfied. By (2), the condition (10) merely says that G has at least three vertices. Now for each $j = 1, 2, \dots, k$, let G_j be the induced subgraph of G by the set of vertices corresponding to the ideals of $\mathbb{Z}_n$ containing $p_j^{n_j-1}\prod_{\substack{i=1\\i\neq j}}^{k} p_i^{n_i} = \lambda_j$ (say). Since each of these ideals contains $\lambda_j \neq 0$, G_j is a complete subgraph and clearly G is the union of all these complete subgraphs. Also the intersection of all such G_j is the complete subgraph H induced by the set of vertices corresponding to the ideals of $\mathbb{Z}_n$ containing $p_1^{n_1-1}p_2^{n_2-1}\cdots p_k^{n_k-1} = \lambda$ (say). Since the vertices of H are those corresponding to the ideals generated by the factors of λ (other than 1), we have H has $n_1n_2\dots n_k - 1$ vertices. At this point one should note that if $H = G$,[3] then G is complete and since by (10) G contains at least 3 vertices, it is Hamiltonian. So let H be a proper subgraph of G. In this case we define inductively the subgraph H_1 of G induced by the vertices of G_1 which are not in H, then the subgraph H_2 of G induced by the vertices of G_2 which are not in H and H_1, and so on. If any H_j becomes empty in this process we delete it from the list and relabel the list accordingly. Suppose $H, H_1, H_2, \dots, H_t$ exhaust all the vertices of G for some $1 \leqslant t \leqslant k$. Now since each of these subgraphs are complete, each of them contains a spanning path. Let $V(H) = \{v_1, v_2, \dots, v_m\}$ (where $m = n_1n_2\dots n_k - 1$) be the set of vertices of H. If $m = 1$, then by (11), $k = 1$. So $n_1 = m + 1 = 2$ and $n = p_1^2$ which contradict the condition (10). Thus $m > 1$.

We construct a spanning cycle of G as follows:

Starting from v_1 we go to an end of the spanning path of H_1, then after traversing the path we leave it to move to v_2. Again from v_2 we go to an end of the spanning path of H_2, traverse the path and move to v_3 from the other end.[4] Since $t \leqslant k \leqslant m$, we continue this process to exhaust all the vertices of $H_1, H_2, \dots, H_t$ and finally reach to some vertex v of H. If $t = m$, then $v = v_1$ and the cycle is complete. Otherwise we complete the remaining spanning path $(v_{t+1}v_{t+2}\dots v_m)$ of H and make an end by joining v_m with v_1.

Thus the above process builds up a Hamiltonian cycle of G and so G is Hamiltonian as required. □

2 Contributed by the learned referee.

3 By (10) and (11), it follows that this case arises when and only when $k = 1$. But then by (10), $n = p_1^{n_1}$, where $n_1 \geqslant 4$.

4 If the path contains a single vertex, then from that point we return back to v_3.

We note that the condition (10) is necessary as otherwise the number of vertices of G is less than 3 and no (Hamiltonian) cycle is possible. So we get that $G(\mathbb{Z}_n)$ is not Hamiltonian if n is of the form p^2, pq or p^3 for distinct primes p and q. On the other hand the condition (11) is not necessary as $G(\mathbb{Z}_{pqr})$ is Hamiltonian (cf. Fig. 3(c)) for distinct primes p, q, r; but $n = pqr$ does not satisfy (11). However (11) is always satisfied if each $n_i \geqslant 2$ for then $n_1 n_2 \dots n_k \geqslant 2^k \geqslant k + 1$ for all $k \geqslant 1$. Thus it follows from the above lemma that the graph $G(\mathbb{Z}_n)$ is Hamiltonian when $k = 1, n_1 \geqslant 4$ and $k \geqslant 2, n_i \geqslant 2$ for all $i = 1, 2, \dots, k$.

Lemma 5.4. *Let $n = p\alpha$ where p is a prime number such that $p \nmid \alpha$ and $\alpha \in M$. If $G(\mathbb{Z}_\alpha)$ is Hamiltonian, then $G(\mathbb{Z}_n)$ is Hamiltonian.*

Proof. Let $G(\mathbb{Z}_\alpha)$ be Hamiltonian for some $\alpha \in M$ and p be a prime number such that $p \nmid \alpha$. Let $n = p\alpha$. Then vertices of $G(\mathbb{Z}_n)$ correspond to the principal ideals generated by the non-trivial ($\neq 1$ or n) factors of n. Now the set of such factors of n is $V_1 \cup V_2 \cup \{p\}$, where $V_1 = \{\alpha_i \in \mathbb{N} \setminus \{1\} \mid \alpha_i \mid \alpha, \}$ and $V_2 = \{p\alpha_i \in \mathbb{N} \setminus \{1\} \mid \alpha_i \mid \alpha, \alpha_i \neq \alpha\}$. Now the subgraph H_1 (say) of $G(\mathbb{Z}_n)$ induced by the set of vertices corresponding to the ideals generated by elements of V_1 is complete as each of these ideals contains α. Again since $(\alpha_i) \cap (\alpha_j) \neq \{0\}$ in $\mathbb{Z}_\alpha$ if and only if $(p\alpha_i) \cap (p\alpha_j) \neq \{0\}$ in $\mathbb{Z}_n$, the subgraph H_2 (say) of $G(\mathbb{Z}_n)$ induced by the set of vertices corresponding to the ideals generated by elements of V_2 is isomorphic to the graph $G(\mathbb{Z}_\alpha)$. Hence this subgraph has a spanning path,[5] say, $(p\alpha_1, p\alpha_2, \dots, p\alpha_t)$ for some $t \in \mathbb{N}$. Certainly $t \geqslant 3$ as $G(\mathbb{Z}_\alpha)$ is Hamiltonian. Moreover the vertex corresponding to the ideal (p) is adjacent to all the vertices of H_2 as $p\alpha_i \in (p)$ for all i. Then the cycle

$$(p\alpha_1, p, p\alpha_2, p\alpha_3, \dots, p\alpha_t, \alpha_t, \alpha, \alpha_{t-1}, \alpha_{t-2}, \dots, \alpha_1, p\alpha_1)$$

is a Hamiltonian cycle of $G(\mathbb{Z}_n)$. □

Now from the above discussion we know that the graph $G(\mathbb{Z}_{p_1^{n_1} p_2^{n_2}})$ is Hamiltonian for $n_1, n_2 \geqslant 2$. So from the above lemma it follows that $G(\mathbb{Z}_n)$ is Hamiltonian for $n = p_1^{n_1} p_2^{n_2} p_3 p_4 \dots p_k$, where $k \geqslant 2$ and $n_1, n_2 \geqslant 2$. Similarly the graph $G(\mathbb{Z}_{p_1^{n_1} p_2})$ is Hamiltonian for $n_1 \geqslant 3$ by Lemma 5.3 as $n_1 \cdot 1 \geqslant 3 > 2$ and hence $G(\mathbb{Z}_n)$ is Hamiltonian for $n = p_1^{n_1} p_2 p_3 p_4 \dots p_k$, where $k \geqslant 2$ and $n_1 \geqslant 3$. Again we note that the graph $G(\mathbb{Z}_{pqr})$ is Hamiltonian (cf. Fig. 3(c)) and below we present a spanning cycle of the graph $G(\mathbb{Z}_{p^2 qr})$:

$$(p, p^2 q, p^2, p^2 r, pr, pqr, r, q, qr, pq, p).$$

Then by Lemma 5.4, we have $G(\mathbb{Z}_n)$ is Hamiltonian for $n = p_1^{n_1} p_2 p_3 p_4 \dots p_k$ for $k \geqslant 3$ and for $n_1 = 1$ or 2. Finally that the graph $G(\mathbb{Z}_{p^2 q})$ is not Hamiltonian for distinct primes p, q is clear from Fig. 3(b). Thus we have proved that $G(\mathbb{Z}_n)$ is Hamiltonian for all values of $n \in M$ except the forms described in Theorem 5.2.

6. Another counting principle

Now let us put the problem in the opposite direction. Suppose it is given that G is a graph of the ring $\mathbb{Z}_n$ for some $n \in M$. Let T be the number of vertices of G. We wish to describe all non-isomorphic graphs G with T vertices which are graphs of $\mathbb{Z}_n$ for some $n \in M$. By (2), it is clear that such a graph G is completely determined by the factorization of the positive integer $T + 2$, i.e., if $T + 2 = m_1 m_2 \dots m_\nu$, for some $m_j \in \mathbb{N}, 1 < m_j \leqslant T + 2, j = 1, 2, \dots, \nu$, then G is the graph of $\mathbb{Z}_n$ where $n = p_1^{m_1 - 1} p_2^{m_2 - 1} \cdots p_\nu^{m_\nu - 1}$ for any set of distinct primes $p_1, p_2, \dots, p_\nu$.[6] Indeed, suppose $T + 2$ has two *different* factorizations, namely

$$T + 2 = a_1 a_2 \dots a_t = b_1 b_2 \dots b_k$$

for some $1 < a_i, b_j \leqslant T + 2, a_i, b_j \in \mathbb{N}, i = 1, 2, \dots, t, j = 1, 2, \dots, k, t, k \in \mathbb{N}$. Then the graphs corresponding to them are $G(\mathbb{Z}_n)$ and $G(\mathbb{Z}_m)$ respectively, where $n = p_1^{a_1 - 1} p_2^{a_2 - 1} \cdots p_t^{a_t - 1}$ and $m = q_1^{b_1 - 1} q_2^{b_2 - 1} \cdots q_k^{b_k - 1}$ for some sets of distinct primes $p_1, p_2, \dots, p_t$ and $q_1, q_2, \dots, q_k$. Now the factorizations are different means that either some $a_i \notin \{b_1, b_2, \dots, b_k\}$ or some $b_j \notin \{a_1, a_2, \dots, a_t\}$. By (6), we know that in the former case, $G(\mathbb{Z}_n)$ has some vertices of degree $T - a_i$ but $G(\mathbb{Z}_m)$ does not have any such vertex. Thus $G(\mathbb{Z}_n)$ can not be isomorphic to the graph $G(\mathbb{Z}_m)$. Similar thing happens in the other case. We summarize the following fact in the next theorem and then illustrate the same by an example.

An *unordered factorization* of a natural number $n > 1$ is a representation of n as a product of positive integers greater than 1, the order of the factors in the product being irrelevant. The total number of unordered factorization of n is denoted by $p^*(n)$.

Theorem 6.1. *The number of non-isomorphic graphs of $\mathbb{Z}_n$ (for various $n \in M$) for a given number T of vertices is equal to the number of (unordered) factorizations of $T + 2$.*

[5] For convenience we are writing only the generators of the ideals corresponding to the vertices of the cycle.

[6] Since lattices of ideals of rings $\mathbb{Z}_n$ and $\mathbb{Z}_m$ are isomorphic for $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and $m = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}$, where p_j and q_j ($j = 1, 2, \dots, k$) are two different sets of distinct primes, the graph $G(\mathbb{Z}_n)$ is isomorphic to the graph $G(\mathbb{Z}_m)$.

Example 6.2. Suppose $T = 4$. Then $T + 2 = 6$ which has two different factorizations, namely, $6 = 3.2 = 6$. So either $G = G(\mathbb{Z}_{p^2q})$ or $G = G(\mathbb{Z}_{p^5})$, where p, q are distinct primes (cf. Fig. 3(b) and 5). These are the only non-isomorphic graphs of 4 vertices, among the graphs of $\mathbb{Z}_n$ for some $n \in M$. Note that in the graph $G(\mathbb{Z}_{p^2q})$, the number of vertices of degree $T - 1\,(=3)$ is $2.1 - 1 = 1$, the number of vertices of degree $T - (2 + 1)\,(=1)$ is 1 and the number of vertices of degree $T - (1 + 1)\,(=2)$ is 2, whereas in the graph $G(\mathbb{Z}_{p^5})$, the number of vertices of degree $T - 1\,(=3)$ is $5 - 1 = 4$ and there are no vertices of degree 1 or 2.

We have the following recursion formula for finding $p^*(n)$:

Theorem 6.3 ([5]). *Let $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ be the prime factorization of a natural number n, where p_i are distinct primes, $n_i \in \mathbb{N}$ for all $i = 1, 2, \dots, k$, $(k \in \mathbb{N})$, with $n_1 = \min\{n_i \mid i = 1, 2, \dots, k\}$. Then*

$$p^*(n)n_1 = \sum_{r=1}^{n_1} r \cdot \left(\sum_{d \mid \frac{n}{p_1^{n_1}}} \lambda(p_1^r d) p^*\left(\frac{n}{p_1^r d}\right) \right), \tag{12}$$

where $\lambda(m) = \sum_{i \mid c(m)} \frac{1}{i}$, $c(m) = \gcd\{r, r_2, \dots, r_k\}$ for $m = p_1^r p_2^{r_2} \dots p_k^{r_k}$, $0 < r \leqslant n_1$ and $0 \leqslant r_j \leqslant n_j$, $r, r_j \in \mathbb{N}$, $j = 2, 3, \dots, k$.

Corollary 6.4 ([5]). *Let n be a natural number and p be a prime number such that $p \nmid n$. Then*

$$p^*(np) = \sum_{d \mid n} p^*(d). \tag{13}$$

In particular, for any two distinct prime numbers p, q and for any natural number n,

$$p^*(pq^n) = \sum_{i=0}^{n} p(i). \tag{14}$$

where $p(i)$ denotes the number of (additive) partitions of i.

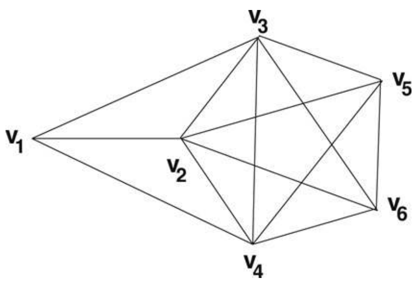
Example 6.5. Let $T = 70$. Then $T + 2 = 3^2.2^3$. Now the divisors of 2^3 are 1, 2, 4, 8. So we have

$$\begin{aligned} p^*(72) \cdot 2 &= \sum_{i=1}^{2} i \cdot \left(\sum_{d \mid 8} \lambda(3^i d) p^*\left(\frac{72}{3^i d}\right) \right), \\ &= \{p^*(24) + p^*(12) + p^*(6) + p^*(3)\} + 2 \cdot \{\lambda(3^2)p^*(8) + \lambda(3^2 \cdot 2)p^*(4) \\ &\quad + \lambda(3^2 \cdot 2^2)p^*(2) + \lambda(3^2 \cdot 2^3)p^*(1)\}. \end{aligned}$$

Now $p^*(2) = p^*(3) = 1$, $p^*(4) = p^*(2^2) = p(2) = 2$, $p^*(8) = p^*(2^3) = p(3) = 3$. Again $p^*(6) = p^*(3 \cdot 2) = 1 + p(1) = 2$, $p^*(12) = p^*(3 \cdot 2^2) = 1 + p(1) + p(2) = 4$, $p^*(24) = p^*(3 \cdot 2^3) = 1 + p(1) + p(2) + p(3) = 7$ by (14). Thus we have $p^*(72) \cdot 2 = \{7 + 4 + 2 + 1\} + 2 \cdot \left\{(1 + \frac{1}{2}) \cdot 3 + 1 \cdot 2 + (1 + \frac{1}{2}) \cdot 1 + 1 \cdot 1\right\} = 14 + 18 = 32$. Hence $p^*(72) = 16$. Thus there are 16 non-isomorphic graphs of 70 vertices, which are graphs of $\mathbb{Z}_n$ for some $n \in M$.

We wish to make a final comment that the structure of the graph $G(\mathbb{Z}_n)$ as described in the proof of Lemma 5.3 (which is, in fact, a contribution of the learned referee) enabled us to characterize these graphs in a more precise way. A very interesting consequence of this characterization is that every simple (undirected) graph becomes an induced subgraph of $G(\mathbb{Z}_n)$ for some $n \in M$ (details of which is long and beyond the scope of this paper). We briefly illustrate this in the following example:

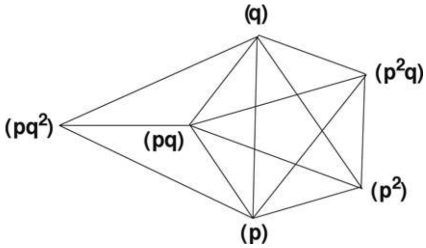
Example 6.6. Let G be the following graph with the vertex set $V = \{v_1, v_2, \dots, v_6\}$:



The maximal cliques of G are: $C_1 = \{v_1, v_2, v_3, v_4\}$ and $C_2 = \{v_2, v_3, v_4, v_5, v_6\}$. Assign distinct primes p_1 and p_2 to the cliques C_1 and C_2 respectively.[7] Then for each $v \in V$, label the vertex v with the number $\prod_{v \in C_i} p_i$ (step I). Now relabel the vertices by using different powers in order to make the labeling distinct (step II). Now consider a number n which is a common multiple of all the labels such that it is greater than each of them. We choose here $n = p^2q^2$. Then we assign the final label to each vertex which is the quotient of n with the corresponding last label (step III).

Vertices	v_1	v_2	v_3	v_4	v_5	v_6
Step I	p	pq	pq	pq	q	q
Step II	p	pq	p^2q	pq^2	q	q^2
Step III	pq^2	pq	q	p	p^2q	p^2

It can be easily verified that the graph G is isomorphic to the following subgraph induced by the labeled vertices of the graph $G(\mathbb{Z}_{p^2q^2})$:



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[7] For convenience we write $p = p_1$ and $q = p_2$.