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# Generalized thermoelastic functionally graded solid with a periodically varying heat source

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**Abstract**

This paper deals with the problem of thermoelastic interactions in a functionally graded isotropic unbounded medium due to the presence of periodically varying heat sources in the context of the linear theory of generalized thermoelasticity without energy dissipation (TEWOED). The governing equations of generalized thermoelasticity without energy dissipation (GN model type II) for a functionally graded materials (FGM) (i.e. material with spatially varying material properties)are established. The governing equations are expressed in Laplace–Fourier double transform domain and solved in that domain. Now, the inversion of the Fourier transform is carried out by using residual calculus, where poles of the integrand is obtained numerically in complex domain by using Laguerre's method and the inversion of Laplace transform is done numerically using a method based on Fourier series expansion technique. The numerical estimates of the displacement, temperature, stress and strain are obtained for a hypothetical material. The solution to the analogous problem for homogeneous isotropic material is obtained by taking nonhomogeneity parameter suitably. Finally the results obtained are presented graphically to show the effect of nonhomogeneity on displacement, temperature, stress and strain.


*MSC:* 74F

*Keywords:* Generalized thermoelasticity; Green–Naghdi model; Periodically varying heat source; Functionally graded material

## 1. Introduction

Thermoelasticity theories which admit a finite speed for thermal signals (second sound) have aroused much interest in the last three decades. These theories, known as generalized theories, involves hyperbolic type heat transport equation in contrast to the classical coupled thermoelasticity involving parabolic type (diffusion type) heat transport equation, which predicts infinite speed of propagation of thermal signals. Among the generalized theories the extended thermoelasticity (ETE) proposed by Lord and Shulman (1967) and the temperature rate dependent thermoelasticity (TRDTE) propsed by Green and Lindsay (1972) have been the subject

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of recent investigation. In view of experimental evidence in support of the finiteness of the speed of propagation of heat wave, generalized thermoelasticity theories are more acceptable than the conventional thermoelasticity theories in dealing with practical problems involving very short time intervals and high heat fluxes, like those occurring in laser units, energy channels and nuclear reactor, etc.

Later Green and Naghdi (1991) developed three models for generalized thermoelasticity of homogeneous and isotropic materials which are labeled as Models I, II and III. The nature of these theories are such that when the respective theories are linearized, Model I reduced to the classical heat conduction theory (based on Fourier's law). The linearized versions of Model II and III permit propagation of thermal waves at finite speed. Model II, in particular, exhibits a feature that is not present in the other established thermoelastic models as it does not sustain dissipation of thermal energy (Green and Naghdi, 1992, 1993). In this model the constitutive equations are derived by starting with the reduced energy equation and by including the thermal displacement gradient among other constitutive variables. The Green–Naghdi third model admits the dissipation of energy.

Functionally graded material (FGM) as a new kind of composites were initially designed as thermal barrier materials for aerospace structures, in which the volume fractions of different constituents of composites vary continuously from one side to another (Suresh and Mortensen, 1998). These novel nonhomogeneous materials have excellent thermo-mechanical properties to withstand high temperature and have extensive applications to important structures, such as aerospace, nuclear reactors, pressure vessels and pipes, chemicals plants, etc. The use of FGMs can eliminate or control thermal stresses in structural components (Aboudi et al., 1995; Wetherhold and Wang, 1996).

Sugano (1987) analyzed the one-dimensional transient thermal stress problem of nonhomogeneous plate where the thermal conductivity and Young's modulus vary exponentially, whereas Poisson's ratio and the coefficient of linear thermal expansion vary arbitrary in the thickness direction. Jeon et al. (1997) presented the analytical treatments for the steady thermoelastic problems of nonhomogeneous slabs, assuming that the shear modulus of elasticity, the thermal conductivity and the coefficient of linear thermal expansion vary with the power product form of axial coordinate variable. Sankar and Tzeng (2002) analyzed the two dimensional steady thermal stress problem of a functionally graded beam whose thermoealstic constants vary exponentially through the thickness. Vel and Batra (2002), and Qian and Batra (2004) analyzed the three dimensional steady or transient thermal stress problems of functionally graded rectangular plate whose material properties vary with the power product form through the thickness. On the other hand, since shell type structures are used in various industrial fields, the thermoelastic analysis of circular cylinders, spheres and cylindrical panels made of FGM becomes important. Obata and Noda (1994) analyzed the one-dimensional functionally graded hollow cylinder and hollow sphere using a perturbation method. Lutz and Zimmerman presented the exact solutions for one dimensional thermal stresses of functionally graded sphere (Lutz and Zimmerman, 1996) and cylinder (Zimmerman and Lutz, 1999) whose elastic modulus and co-efficient of linear thermal expansion vary linearly with radius. Ye et al. (2001) presented the exact solution for the axisymmetric thermoelastic problem of a uniformly heated functionally graded transversely isotropic cylindrical shell, assuming that the modulus of elasticity and the coefficient of linear thermal expansion vary with the power product form of radial co-ordinate variable. El-Naggar et al. (2002) analyzed the transient thermal stresses in a rotating nonhomogeneous orthotropic hollow cylinder using a finite difference method. Wang and Mai (2005) analyzed the transient one-dimensional thermal stresses in nonhomogeneous materials such as plates, cylinders and spheres using a finite element method. Ootao and Tanigawa (2006) studied exactly a one-dimensional transient thermoelstic problem of a functionally graded hollow cylinder whose thermal and thermoelastic constants are assumed to vary with the power product form of radial co-ordinate variable. Shao et al. (2007) solved a thermomechanical problem of a FGM hollow circular cylinder whose material properties are assumed to be temperature independent and vary continuously in the radial direction. To the authors' knowledge no transient thermoelastic problem of functionally graded materials have been solved employing generalized thermoelasticity without energy dissipation (GN Model type II).

The main object of the present work is to consider a one dimensional thermoelastic disturbance in an infinite isotropic functionally graded medium in the context of generalized thermoelasticity without energy dissipation (GN Model type II) in presence of distributed periodically varying heat sources. The material properties of the FGM is assumed to vary exponentially with space variable. The governing equations for this problem are taken into Laplace–Fourier transform domain. The solutions for displacement, temperature, stress and strain in Laplace transform domain is obtained by taking Fourier inversion which is carried out by using residual calculus, where the poles of the integrand is obtained numerically in complex domain by using Laguerre's method. Then the inversion of Laplace transform have been carried out numerically by applying a method of numerical inversion of Laplace transform based on Fourier series expansion technique (Honig and Hirdes, 1984). Numerical results for displacement, temperature, stress and strain in physical space–time domain have been obtained for a copper like material and have been presented graphically to show the effect of nonhomogeneity. It is observed that the results of associated homogeneous case may easily be recovered from our results by letting the nonhomogeneity parameter become zero.

## 2. Basic formulation

The constitutive equation is

$$\tau_{ij} = 2\mu e_{ij} + [\lambda\Delta - \gamma(\theta - \theta_0)]\delta_{ij}, \tag{2.1}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \Delta = e_{ii}. \tag{2.2}$$

Stress equation of motion in absence of body forces is

$$\rho\ddot{u}_i = \tau_{ij,j}. \tag{2.3}$$

Heat equation corresponding to generalized thermoelasticity without energy dissipation is

$$K^{\star}\theta_{,ii} + \rho\dot{Q} = \rho c_v\ddot{\theta} + \gamma\theta_0\ddot{\Delta}, \tag{2.4}$$

where $\gamma = (3\lambda + 2\mu)\alpha_t$.

With the effects of functionally graded solid, the parameters $\lambda$, $\mu$, $K^{\star}$, $\gamma$ and $\rho$ are no longer constant but become space-dependent. Thus we replace $\lambda$, $\mu$, $K^{\star}$, $\gamma$ and $\rho$ by $\lambda_0 f(\vec{x})$, $\mu_0 f(\vec{x})$, $K_0^{\star} f(\vec{x})$, $\gamma_0 f(\vec{x})$ and $\rho_0 f(\vec{x})$, respectively, where $\lambda_0$, $\mu_0$, $K_0^{\star}$, $\gamma_0$ and $\rho_0$ are assumed to be constants and $f(\vec{x})$ is a given nondimensional function of space variable $\vec{x} = (x, y, z)$. Then the corresponding Eqs. (2.1), (2.3) and (2.4) take the following form

$$\tau_{ij} = f(\vec{x})[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(\theta - \theta_0)\}\delta_{ij}] \tag{2.5}$$

$$f(\vec{x})\rho_0\ddot{u}_i = f(\vec{x})[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(\theta - \theta_0)\}\delta_{ij}]_{,j} + f(\vec{x})_{,j}[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(\theta - \theta_0)\}\delta_{ij}] \tag{2.6}$$

and

$$[K_0^{\star} f(\vec{x})\theta_{,i}]_{,i} + \rho_0 f(\vec{x})\dot{Q} = \rho_0 f(\vec{x})c_v\ddot{\theta} + \gamma_0 f(\vec{x})\theta_0\ddot{\Delta} \tag{2.7}$$

## 3. Formulation of the problem

We now consider a functionally graded infinite isotropic thermoelastic body at a uniform reference temperature $\theta_0$ in the presence of periodically varying heat sources distributed over a plane area. We shall consider one-dimensional disturbance of the medium, so that the displacement vector $\vec{u}$ and temperature field $\theta$ can be expressed in the following form

$$\begin{aligned} \vec{u} &= (u(x,t), 0, 0), \\ \theta &= \theta(x,t). \end{aligned} \tag{3.1}$$

It is assumed that material properties depend only on the $x$ co-ordinate. So, we can take $f(\vec{x})$ as $f(x)$.

In the context of linear theory of Generalized thermoelasticity based on Green–Naghdi Model II the equation of motion, heat equation and constitutive equation can be written as

$$f(x)\left[(\lambda_0+2\mu_0)\frac{\partial^2 u}{\partial x^2}-\gamma_0\frac{\partial\theta}{\partial x}\right]+\left[(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\gamma_0(\theta-\theta_0)\right]\frac{\partial f(x)}{\partial x}=\rho_0 f(x)\frac{\partial^2 u}{\partial t^2}, \tag{3.2}$$

$$\frac{\partial}{\partial x}\left[K_0^\star f(x)\frac{\partial\theta}{\partial x}\right]+\rho_0 f(x)\dot{Q}=\rho_0 f(x)c_v\ddot{\theta}+\gamma_0 f(x)\theta_0\ddot{\Delta}, \tag{3.3}$$

$$\tau_{xx}=f(x)[(\lambda_0+2\mu_0)e_{xx}-\gamma_0(\theta-\theta_0)], \tag{3.4}$$

where

$$e_{xx}=\frac{\partial u}{\partial x}. \tag{3.5}$$

Introducing the following nondimensional variables

$$x'=\frac{x}{l},\quad u'=\frac{\lambda_0+2\mu_0}{\gamma_0\theta_0 l}u,\quad t'=\frac{ct}{l},\quad \theta'=\frac{\theta-\theta_0}{\theta_0},$$
$$f'(x')=f(x),\quad \tau'_{x'x'}=\frac{\tau_{xx}}{\gamma_0\theta_0},\quad e'_{x'x'}=e_{xx}, \tag{3.6}$$

where $l$ is a standard length and $c$ is a standard speed, and omitting the primes Eqs. (3.2)–(3.5) can be re-written in non-dimensional form as

$$f(x)\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}\right)+\left(\frac{\partial u}{\partial x}-\theta\right)\frac{\partial f(x)}{\partial x}=f(x)\frac{1}{C_P^2}\frac{\partial^2 u}{\partial t^2}, \tag{3.7}$$

$$C_T^2\frac{\partial}{\partial x}\left[f(x)\frac{\partial\theta}{\partial x}\right]+f(x)Q_0=f(x)\frac{\partial^2\theta}{\partial t^2}+\varepsilon_T f(x)\frac{\partial^3 u}{\partial t^2\partial x}, \tag{3.8}$$

$$\tau_{xx}(x,t)=f(x)\left[\frac{\partial u}{\partial x}-\theta\right], \tag{3.9}$$

$$e_{xx}(x,t)=\frac{\gamma_0\theta_0}{\lambda_0+2\mu_0}\frac{\partial u}{\partial x}, \tag{3.10}$$

where

$$C_T^2=\frac{K_0^\star}{\rho_0 c_v c^2},\quad \varepsilon_T=\frac{\gamma_0^2\theta_0}{(\lambda_0+2\mu_0)\rho_0 c_v},\quad C_P^2=\frac{\lambda_0+2\mu_0}{\rho_0 c^2},\quad Q_0=\frac{l}{\theta_0 c_v c}\frac{\partial Q}{\partial t}.$$

We assume that the medium is initially at rest. The undisturbed state is maintained at reference temperature. Then we have

$$u(x,0)=\dot{u}(x,0)=\theta(x,0)=\dot{\theta}(x,0)=0. \tag{3.11}$$

### 3.1. Exponential variation of nonhomogeneity

We take $f(x)=e^{-kx}$, where $k$ is a dimensionless constant. Then corresponding equation reduce to

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}-k\left(\frac{\partial u}{\partial x}-\theta\right)=\frac{1}{C_P^2}\frac{\partial^2 u}{\partial t^2}, \tag{3.12}$$

$$C_T^2\left(\frac{\partial^2\theta}{\partial x^2}-k\frac{\partial\theta}{\partial x}\right)+Q_0=\frac{\partial^2\theta}{\partial t^2}+\varepsilon_T\frac{\partial^3 u}{\partial t^2\partial x}, \tag{3.13}$$

$$\tau_{xx}(x,t)=e^{-kx}\left[\frac{\partial u}{\partial x}-\theta\right], \tag{3.14}$$

$$e_{xx}(x,t)=\beta_1\frac{\partial u}{\partial x},\ \text{where}\quad \beta_1=\frac{\gamma_0\theta_0}{\lambda_0+2\mu_0}. \tag{3.15}$$

Let us define Laplace–Fourier double transform of a function $g(x,t)$ by

$$\bar{g}(x,p) = \int_0^\infty g(x,t)\mathrm{e}^{-pt}\,\mathrm{d}t, \ Re(p) > 0,$$
$$\hat{\bar{g}}(\alpha,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(x,p)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x. \tag{3.16}$$

Applying Laplace–Fourier double integral transform to Eqs. (3.12)–(3.15) and using the relation in (3.11) we get

$$\left(\alpha^2 + \frac{p^2}{C_P^2} - \mathrm{i}k\alpha\right)\hat{\bar{u}}(\alpha,p) = (\mathrm{i}\alpha + k)\hat{\bar{\theta}}(\alpha,p), \tag{3.17}$$

$$[C_T^2\alpha(\alpha - \mathrm{i}k) + p^2]\hat{\bar{\theta}}(\alpha,p) = \mathrm{i}\varepsilon_T\alpha p^2\hat{\bar{u}}(\alpha,p) + \hat{\bar{Q}}_0, \tag{3.18}$$

$$\hat{\bar{\tau}}_{xx}(\alpha,p) = -\mathrm{i}(\alpha + \mathrm{i}k)[\hat{\bar{u}}(\alpha + \mathrm{i}k,p) - \hat{\bar{\theta}}(\alpha + \mathrm{i}k,p)], \tag{3.19}$$

$$\hat{\bar{e}}_{xx}(\alpha,p) = -\mathrm{i}\beta_1\alpha\hat{\bar{u}}(\alpha,p). \tag{3.20}$$

Solving Eq. (3.17) and (3.18) for $\hat{\bar{u}}(\alpha,p)$ and $\hat{\bar{\theta}}(\alpha,p)$ we get

$$\hat{\bar{u}}(\alpha,p) = \frac{\hat{\bar{Q}}_0(\mathrm{i}\alpha + k)}{M(\alpha)}, \tag{3.21}$$

$$\hat{\bar{\theta}}(\alpha,p) = \frac{\hat{\bar{Q}}_0\left(\alpha^2 + \frac{1}{C_P^2} - \mathrm{i}\alpha k\right)}{M(\alpha)}, \tag{3.22}$$

where

$$M(\alpha) = C_T^2\alpha^4 - 2\mathrm{i}kC_T^2\alpha^3 + \left[p^2\left(1 + \varepsilon_T + \frac{C_T^2}{C_P^2}\right) - C_T^2k^2\right]\alpha^2 - \left[p^2\mathrm{i}k\left(1 + \varepsilon_T + \frac{C_T^2}{C_P^2}\right)\right]\alpha + \frac{p^4}{C_P^2},$$
$$= C_T^2(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4). \tag{3.23}$$

Now the expressions for stress and strain in Laplace–Fourier transform domain can be obtained from (3.19) and (3.20) using (3.21) and (3.22) as follows

$$\hat{\bar{\tau}}_{xx}(\alpha,p) = -\frac{p^2\hat{\bar{Q}}_0}{M(\alpha + \mathrm{i}k)}, \tag{3.24}$$

$$\hat{\bar{e}}_{xx}(\alpha,p) = \frac{\beta_1\hat{\bar{Q}}_0\alpha(\alpha - \mathrm{i}k)}{M(\alpha)}. \tag{3.25}$$

Thus the solution for displacement, temperature, stress and strain in Laplace transform domain can be obtained in terms of the following four integrals.

$$\bar{u}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0(\mathrm{i}\alpha + k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{M(\alpha)}, \tag{3.26}$$

$$\bar{\theta}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0\left(\alpha^2 + \frac{p^2}{C_P^2} - \mathrm{i}\alpha k\right)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{M(\alpha)}, \tag{3.27}$$

$$\bar{\tau}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{\bar{Q}}_0\alpha(\alpha + \mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{M(-\alpha)}, \tag{3.28}$$

$$\bar{e}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\beta_1\hat{\bar{Q}}_0\alpha(\alpha - \mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{M(\alpha)}, \tag{3.29}$$

where

$$\begin{aligned} M(\alpha+\mathrm{i}k) &= M(-\alpha) \\ &= C_T^2\alpha^4 + 2\mathrm{i}kC_T^2\alpha^3 + \left[p^2\left(1+\varepsilon_T+\frac{C_T^2}{C_P^2}\right) - C_T^2k^2\right]\alpha^2 + \left[p^2\mathrm{i}k\left(1+\epsilon_T+\frac{C_T^2}{C_P^2}\right)\right]\alpha + \frac{p^4}{C_P^2}, \\ &= C_T^2(\alpha-l_1)(\alpha-l_2)(\alpha-l_3)(\alpha-l_4). \end{aligned} \tag{3.30}$$

### 3.2. Periodically varying heat source

Now let us assume that heat source is distributed over the plane $x=0$ in the following form

$$\begin{aligned} Q_0 &= Q_0^{\star}\delta(x)\sin\left(\tfrac{\pi t}{\tau}\right) \quad \text{for} \quad 0 \leqslant t \leqslant \tau, \\ &= 0 \quad \text{for} \quad t > \tau, \end{aligned} \tag{3.31}$$

then

$$\hat{\bar{Q}}_0 = \frac{Q_0^{\star}\pi\tau(1+\mathrm{e}^{-p\tau})}{\sqrt{2\pi}(\pi^2+p^2\tau^2)}. \tag{3.32}$$

Thus the expressions for displacement, temperature, stress and strain in Laplace transform domain take the following form

$$\bar{u}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^{\star}\tau(1+\mathrm{e}^{-p\tau})(\mathrm{i}\alpha+k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{2(\pi^2+p^2\tau^2)M(\alpha)}, \tag{3.33}$$

$$\bar{\theta}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^{\star}\tau(1+\mathrm{e}^{-p\tau})\left(\alpha^2+\frac{p^2}{C_P^2}-\mathrm{i}\alpha k\right)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{2(\pi^2+p^2\tau^2)M(\alpha)}, \tag{3.34}$$

$$\bar{\tau}_{xx}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^{\star}\tau(1+\mathrm{e}^{-p\tau})\alpha(\alpha+\mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{2(\pi^2+p^2\tau^2)M(-\alpha)}, \tag{3.35}$$

$$\bar{e}_{xx}(x,p) = \int_{-\infty}^{\infty} \frac{\beta_1 Q_0^{\star}\tau(1+\mathrm{e}^{-p\tau})\alpha(\alpha-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha}{2(\pi^2+p^2\tau^2)M(\alpha)}. \tag{3.36}$$

Applying contour integration to the Eqs. (3.33)–(3.36) we obtain

$$\begin{aligned} \bar{u}(x,p) &= -\frac{\mathrm{i}Q_0^{\star}\pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j(\mathrm{i}\alpha_j+k)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x \geqslant 0 \\ &= \frac{\mathrm{i}Q_0^{\star}\pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j(\mathrm{i}\alpha_j+k)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x < 0, \end{aligned} \tag{3.37}$$

$$\begin{aligned} \bar{\theta}(x,p) &= -\frac{\mathrm{i}Q_0^{\star}\pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j\left(\alpha_j^2+\frac{p^2}{C_P^2}-\mathrm{i}\alpha_j k\right)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x \geqslant 0 \\ &= \frac{\mathrm{i}Q_0^{\star}\pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1 \\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j\left(\alpha_j^2+\frac{p^2}{C_P^2}-\mathrm{i}\alpha_j k\right)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x < 0, \end{aligned} \tag{3.38}$$

$$\bar{\tau}_{xx}(x,p) = -\frac{\mathrm{i}Q_0^\star \pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1\\ \mathrm{Im}(l_j)<0}}^{4} B_j l_j(l_j+ik)\mathrm{e}^{-\mathrm{i}l_j x} \quad \text{for } x \geqslant 0$$
$$= \frac{\mathrm{i}Q_0^\star \pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1\\ \mathrm{Im}(l_j)>0}}^{4} B_j l_j(l_j+\mathrm{i}k)\mathrm{e}^{-\mathrm{i}l_j x} \quad \text{for } x < 0, \tag{3.39}$$

$$\bar{e}_{xx}(x,p) = -\frac{\mathrm{i}\beta_1 Q_0^\star \pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)<0}}^{4} A_j\alpha_j(\alpha_j-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x \geqslant 0$$
$$= \frac{\mathrm{i}Q_0^\star \pi\tau(1+\mathrm{e}^{-p\tau})}{C_T^2(\pi^2+p^2\tau^2)} \sum_{\substack{j=1\\ \mathrm{Im}(\alpha_j)>0}}^{4} A_j\alpha_j(\alpha_j-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}\alpha_j x} \quad \text{for } x < 0, \tag{3.40}$$

where $A_j$s and $B_j$s are given by

$$A_j = \prod_{\substack{n=1\\ n\neq j}}^{4} \frac{1}{(\alpha_j-\alpha_n)},$$
$$B_j = \prod_{\substack{n=1\\ n\neq j}}^{4} \frac{1}{(l_j-l_n)}, \quad j=1,2,3,4. \tag{3.41}$$

## 4. Inversion of laplace transform

It is difficult to find the analytical inverse Laplace transform of the complicated solutions for the displacement, temperature, stress and strain in Laplace transform domain. So we have to resort to numerical computations. We now outline the numerical procedure to solve the problem. Let $\bar{f}(x,p)$ be the Laplace transform of a function $f(x,t)$.

Then the inversion formula for Laplace transform can be written as

$$f(x,t) = \frac{1}{2\pi\mathrm{i}} \int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \mathrm{e}^{pt}\bar{f}(x,p)\,\mathrm{d}p, \tag{4.1}$$

where d is an arbitrary real number greater than real part of all the singularities of $\bar{f}(x,p)$.

Taking $p = d+\mathrm{i}w$, the preceding integral takes the form

$$f(x,t) = \frac{\mathrm{e}^{\mathrm{d}t}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tw} f(x,d+\mathrm{i}w)\,\mathrm{d}w. \tag{4.2}$$

Expanding the function $h(x,t) = \mathrm{e}^{-\mathrm{d}t}f(x,t)$ in a Fourier series in the interval [0, $2T$] we obtain the approximate formula (Honig and Hirdes, 1984)

$$f(x,t) = f_\infty(x,t) + E_\mathrm{D}, \tag{4.3}$$

where

$$f_\infty(x,t) = \tfrac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \quad \text{for} \quad 0 \leqslant t \leqslant 2T \tag{4.4}$$

and

$$c_k = \frac{e^{dt}}{T}\left[e^{\frac{ik\pi t}{T}}\,\bar{f}\left(x, d + \frac{ik\pi t}{T}\right)\right]. \tag{4.5}$$

The discretization error $E_D$ can be made arbitrary small by choosing d large enough (Honig and Hirdes, 1984). Since the infinite series in Eq. (4.4) can be summed upto a finite number $N$ of terms, the approximate value of $f(x,t)$ becomes

$$f_N(x,t) = \tfrac{1}{2}c_0 + \sum_{k=1}^{N} c_k \quad \text{for} \quad 0 \leqslant t \leqslant 2T. \tag{4.6}$$

Using the preceding formula to evaluate $f(x,t)$ we introduce a truncation error $E_T$ that must be added to the discretization error to produce total approximation error.

Two methods are used to reduce the total error. First the 'Korrecktur' method to reduce the discretization error. Next the ε-algorithm is used to accelerate convergence (Honig and Hirdes, 1984).

The Korrecktur method uses the following formula to evaluate the function $f(x,t)$:

$$f(x,t) = f_\infty(x,t) - e^{-2dT} f_\infty(x, 2T+t) + E'_D, \tag{4.7}$$

where the discretization error $|E'_D| \ll |E_D|$. Thus, the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) - e^{-2dT} f_{N'}(x, 2T+t), \tag{4.8}$$

where $N'$ is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. (4.6). Let $N = 2q+1$, where q is a natural number and let $s_m = \sum_{k=1}^{m} c_k$ be the sequence of partial sum of series in (4.6).

We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + \frac{1}{\varepsilon_{p,m+1} - \varepsilon_{p,m}}, \quad p = 1, 2, 3, \ldots.$$

It can be shown that (Honig and Hirdes, 1984) the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$$

converges to $f(x,t) + E_D - \frac{c_0}{2}$ faster than the sequence of partial sums $s_m, m = 1,2,3,\ldots$.

The actual procedure used to invert the Laplace transform consists of using Eq. (4.8) together with the ε-algorithm. The values of $d$ and $T$ are choosen according to the criterion outlined in (Honig and Hirdes, 1984).

## 5. Numerical results and discussions

To get the solution for thermal displacement, temperature, stress and strain in space time domain we have to apply Laplace inversion formula to the Eqs. (3.37), (3.38), (3.39), and (3.40), respectively. This has been done numerically using a method based on Fourier series expansion technique mentioned above. To get the roots of the polynomial $M(\alpha)$ and $M(-\alpha)$ in complex domain we have used Laguerre's method. The numerical code has been prepared using Fortran 77 programming language. For the purpose of illustration we consider copper like material with material constants (Roychoudhuri and Dutta, 2005)

$$\varepsilon_T = 0.0168, \quad \lambda = 1.387 \times 10^{12}\ \text{dynes/cm}^2,$$
$$\mu = 0.448 \times 10^{12}\ \text{dynes/cm}^2, \quad \alpha_t = 1.67 \times 10^{-8}/^\circ\text{C}, \quad \theta_0 = 1^\circ\text{C}$$

Also we have taken $Q_0^\star = 1$, $\tau = 1$, $C_P = 1$, $C_T = 2$, so the faster wave is the thermal wave.

In order to study the effect of nonhomogeneity on thermal displacement, temperature, thermal stress and strain we now present our results in the form of graphs (Figs. 1–8). Fig. 1 depicts variation of thermal displacement $u$ versus distance $x$ for time $t = 0.4$ when the nonhomogeneity parameter $k = 0$, 0.5, 1.0, 1.5. It is observed that as the value of the nonhomogeneity parameter $k$ decreases the peak of thermal displacement also decreases. The effect of nonhomogeneity is seen in the interval $0 < x < 0.3$. Fig. 2 is plotted to show the variation of temperature $\theta$ with distance $x$. It is seen from figure that as the value of $k$ increases the magnitude of the temperature decreases for fixed $x$ and ultimately $\theta$ approaches to zero value. This is because heat source varies periodically with the time for a short duration. This can also be verified from the expression of $\bar{\theta}$

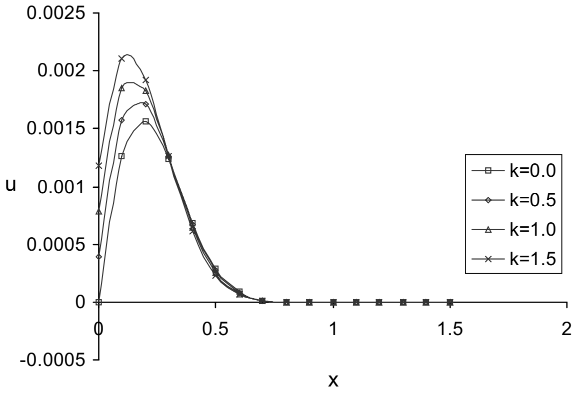


Fig. 1. Variation of displacement $u$ with distance $x$ for $t = 0.4$.

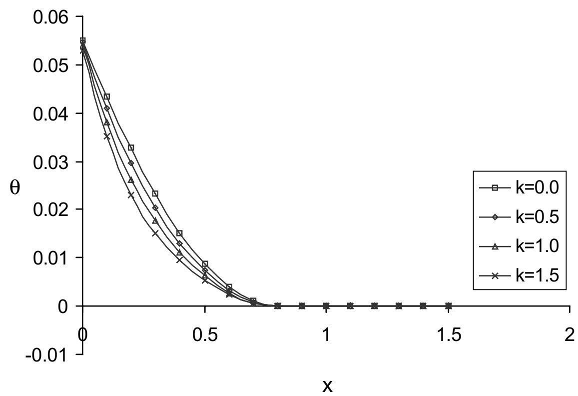


Fig. 2. Variation of temperature $\theta$ with distance $x$ for $t = 0.4$.

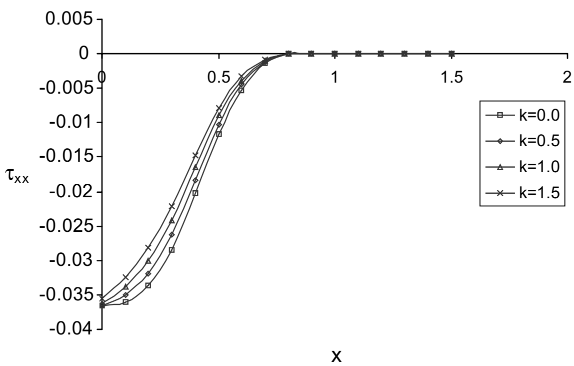


Fig. 3. Variation of stress $\tau_{xx}$ with distance $x$ for $t = 0.4$.

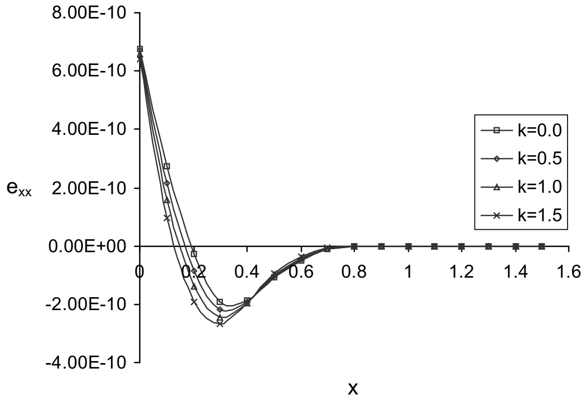


Fig. 4. Variation of strain $e_{xx}$ with distance $x$ for $t = 0.4$.

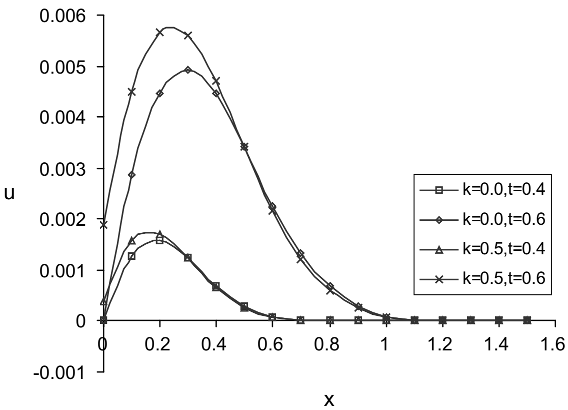


Fig. 5. Variation of displacement $u$ with distance $x$ for $t = 0.4$ and $t = 0.6$.

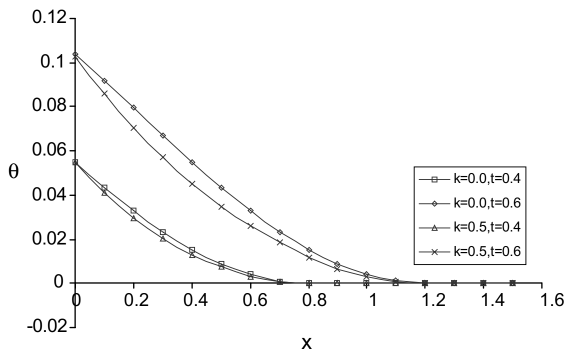


Fig. 6. Variation of temperature $\theta$ with distance $x$ for $t = 0.4$ and $t = 0.6$.

given in Eq. (3.38) involving $e^{-i\alpha_j x}$, $\text{Im}(\alpha_j) < 0$ for $x \geqslant 0$. Fig. 3 shows variation of thermal stress versus distance $x$. Here the stress takes negative values for $k = 0,0.5,1.0,1.5$ and the magnitude of stress increases as $k$ decreases for the particular value of $x$. The same was observed by Chandrasekharaiah and Srinath (1998) for the homogeneous case (k = 0). Fig. 4 gives the variation of thermal strain against distance $x$. The strain takes positive value in the range $0 < x < 0.2$ (k = 0), $0 < x < 0.18$ (k = 0.5), $0 < x < 0.16$ (k = 1.0),

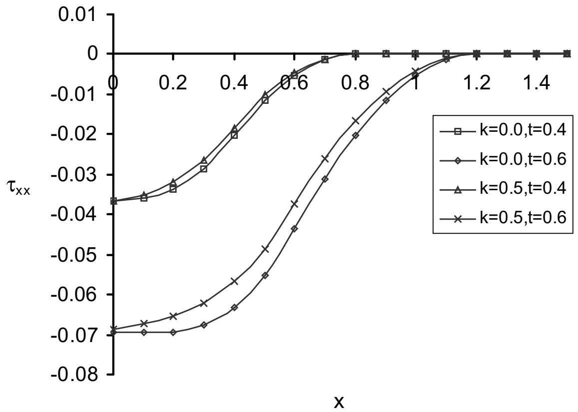


Fig. 7. Variation of stress $\tau_{xx}$ with distance $x$ for $t = 0.4$ and $t = 0.6$.

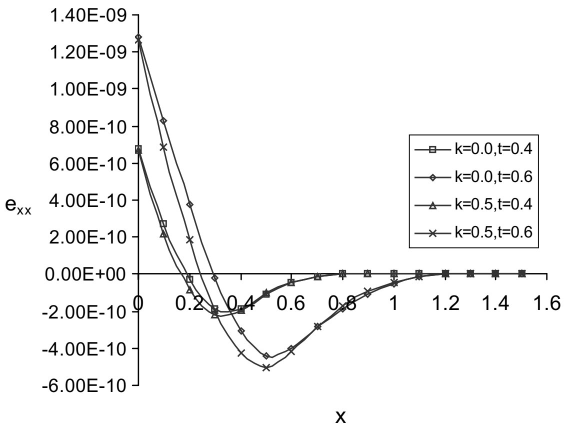


Fig. 8. Variation of strain $e_{xx}$ with distance $x$ for $t = 0.4$ and $t = 0.6$.

$0 < x < 0.12$ (k = 1.5) and then negative value and then finally diminishes to zero. This is also in conformity with the fact that strain should decrease with increasing distance $x$ from the plane $x = 0$ where the heat sources is active. The effect of nonhomogeneity is observed in the region $0 < x < 0.4$. Figs. 5–8 are plotted to show the variation of thermal displacement, temperature, thermal stress and strain respectively against $x$ for $k = 0{,}0.5$

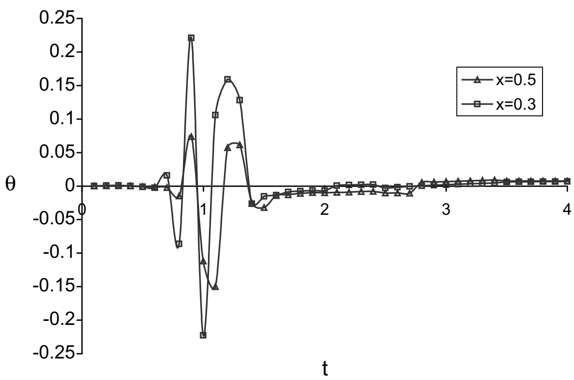


Fig. 9. Variation of temperature $\theta$ with time $t$ for $C_T = 7$, $k = 4$.

and $t = 0.4,0.6$. It is observed from these figures that as time $t$ increases the magnitude of displacement, temperature, stress and strain increase and the effect of nonhomogeneity also increases. Fig. 9 represents variation of temperature $\theta$ against time $t$ for $x = 0.3$, 0.5 and $C_T = 7$, $k = 4$. From the figure it is observed that the oscillation in temperature distribution occurs in the interval $0.5 < t < 1.5$ and after that the effect is almost zero.

In Figs. 1–8 it is observed that with the decrease of magnitude of the nonhomogeneity parameter the solution approaches to solution corresponding to the homogeneous problem and when the nonhomogeneity parameter $k = 0$, the results tally in magnitude with the corresponding results of Roychoudhuri and Dutta (2005), which was obtained analytically.

## 6. Conclusions

Method of Laplace–Fourier double transform has been applied to solve a generalized thermoelastic (TEWOED) problem of an isotropic functionally graded material with a periodically varying heat source. The material properties are assumed to vary exponentially with distance. The analysis of the results permits some concluding remarks.

1. The presence of nonhomogeneity parameter has significant effect on the solutions of displacement, temperature, stress and strain.
2. From the graphs it is clear that with the increase of nonhomogeneity parameter $k$ the peak of thermal displacement increases and the magnitude of temperature and stress decreases and for strain the effect of nonhomogeneity is observed in the interval $0 < x < 0.4$.
3. It is also observed that effect of nonhomogeneity on the solution increases with the increase of time.
4. With the decrease of the value of $k$ the solution approaches to the solution corresponding to the homogeneous problem.

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