Applied Mathematical Modelling 33 (2009) 3287–3298

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Generalized thermo-visco-elastic problem of a spherical shell with three-phase-lag effect

Avijit Kar *, M. Kanoria

Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata 700 009, India

ARTICLE INFO

Article history:
Received 23 October 2007
Received in revised form 18 October 2008
Accepted 22 October 2008
Available online 7 November 2008

Keywords:
Generalized thermo-visco-elasticity
Energy dissipation
Laplace transform
Step input temperature
Vector–matrix differential equation
Three-phase-lag model

ABSTRACT

This problem deals with the thermo-visco-elastic interaction due to step input of temperature on the stress free boundaries of a homogeneous visco-elastic isotropic spherical shell in the context of generalized theories of thermo-elasticity. Using the Laplace transformation the fundamental equations have been expressed in the form of vector–matrix differential equation which is then solved by eigen value approach. The inverse of the transformed solution is carried out by applying a method of Bellman et al. [R. Bellman, R.E. Kolaba, J.A. Lockette, Numerical Inversion of the Laplace Transform, American Elsevier Publishing Company, New York, 1966]. The stresses are computed numerically and presented graphically in a number of figures for copper material. A comparison of the results for different theories (TEWED (GN-III), three-phase-lag method) is presented. When the body is elastic and the outer radius of the shell tends to infinity, the corresponding results agree with the result of existing literature.

1. Introduction

The classical theory of thermo-elasticity involving infinite speed of propagation of thermal signals, contradicts the physical facts. During the last three decades, non-classical theories involving finite speed of heat transportation in elastic solids have been developed to remove this paradox. In contrast to the conventional coupled thermo-elasticity theory [1], which involves a parabolic-type heat transport equation, these generalized theories involving a hyperbolic-type heat transport equation are supported by experiments exhibiting the actual occurrence of wave-type heat transport in solids, called second sound effect. The extended thermo-elasticity theory (ETE) proposed by Lord and Shulman [1], incorporates a flux-rate term into Fourier's law of heat conduction, and formulates a generalized form that involves a hyperbolic-type heat transport equation admitting finite speed of thermal signals. Green and Lindsay [2] developed temperature-rate-dependent thermo-elasticity (TRDTE) theory by introducing relaxation time factors that does not violate the classical Fourier law of heat conduction and this theory also predicts a finite speed for heat propagation. The closed-form solutions for thermo-elastic problems in generalized theory of thermo-elasticity have been obtained by [3]. Hetnarski and Ignaczak [4] studied the response of semi-space to a short laser pulse in the context of generalized thermo-elasticity.

Most engineering materials such as metals possess a relatively high rate of thermal damping and thus are not suitable for use in experiments concerning second sound propagation. But, given the state of recent advances in material science, it may be possible in the foreseeable future to identify (or even manufacture for laboratory purposes) an idealized material for the purpose of studying the propagation of thermal waves at finite speed. The relevant theoretical developments on the subject are due to Green and Naghdi [5–7], and provide sufficient basic modifications in the constitutive equations that permit

* Corresponding author. Tel.: +91 9433775538, 9830713650.
E-mail address: akmathematics@yahoo.com (A. Kar).

0307-904X/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.apm.2008.10.036

treatment of a much wider class of heat flow problems, labelled as types I–III. The natures of these three types of constitutive equations are such that when the respective theories are linearized, type-I is the same as the classical heat equation (based on Fourier's law), whereas the linearized versions of types-II and -III theories permit propagation of thermal waves at finite speed. The entropy flux vector in types-II and -III (i.e. thermo-elasticity without energy dissipation (TEWOED) and thermo-elasticity with energy dissipation (TEWED)) models are determined in terms of the potential that also determines stresses. When Fourier conductivity is dominant the temperature equation reduces to classical Fourier law of heat conduction and when the effect of conductivity is negligible the equation has undamped thermal wave solutions without energy dissipation. Kar and Kanoria [8] investigated thermo-elastic stress wave propagation in an unbounded body with a spherical hole following the theories developed in [2,7]. Several investigations relating to generalized thermo-elasticity theories (TEWOED(GN-II) and TEWED(GN-III)) have been presented by [8–16].

Tzou [17] have developed a new model called dual-phase-lag model for heat transport mechanism by considering micro-structural effects into the delayed response in time in the macroscopic formulation by taking into account that increase of the lattice temperature is delayed due to phonon–electron interactions on the macroscopic level. Tzou [17] introduced two-phase-lags to both the heat flux vector and the temperature gradient. According to this model, classical Fourier's law $\overrightarrow{q} = -K\overrightarrow{\nabla}T$ has been replaced by $\overrightarrow{q}(P, t + \tau_q) = -K\overrightarrow{\nabla}T(P, t + \tau_T)$, where the temperature gradient $\overrightarrow{\nabla}T$ at a point P of the material at time $t + \tau_T$ corresponds to the heat flux vector $\overrightarrow{q}$ at the same point at time $t + \tau_q$. Here, K is the thermal conductivity of the material. The delay time τ_T is interpreted as that caused by the microstructural interactions and is called the phase-lag of the temperature gradient. The other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the phase-lag of the heat flux. For $\tau_q = \tau_T = 0$, this is identical with classical Fourier's law. If $\tau_q = \tau$ and $\tau_T = 0$, Tzou [18] refers to the model as single-phase-lag model.

The most recent and relevant development in themo-elasticity theory is three-phase-lag model [19]. Roychoudhuri established this model by introducing three-phase-lags in the heat flux vector, the temperature gradient and the displacement gradient. According to this model $\overrightarrow{q}(P, t + \tau_q) = -[K\overrightarrow{\nabla}T(P, t + \tau_T) + K^*\overrightarrow{\nabla}v(P, t + \tau_v)]$, where $\overrightarrow{\nabla}v$ $(\dot{v} = T)$ is the thermal displacement gradient and K^* is the additional material constant. To study some practical relevant problems and have found that in heat transfer problems involving very short time intervals and in the problems of very high heat fluxes, the hyperbolic equation gives significantly different results than the parabolic equation. According to this phenomenon the lagging behavior in the heat conduction in solid should not be ignored particularly when the elapsed times during a transient process are very small, say about 10^{-7} second or the heat flux is very much high. Three-phase-lag model is very useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon–electron interactions, phonon-scattering etc., where the delay time τ_q captures the thermal wave behavior (a small scale response in time), the phase-lag τ_T captures the effect of phonon–electron interactions (a microscopic response in space), the other delay time τ_v is effective since, in the three-phase-lag model, the thermal displacement gradient is considered as a constitutive variable whereas in the conventional thermo-elasticity theory temperature gradient is considered as a constitutive variable.

The study of viscoelastic behavior is of interest in several contexts. First, materials used in engineering applications may exhibit viscoelastic behavior as an unintentional side effect. Second, the mathematics underlying visco-elasticity theory is of interest within the applied mathematics community. Third, visco-elasticity is of interest in some branches of material science, metallurgy and solid-state-physics. Fourth, the causal links between visco-elasticity and microstructure is exploited in the use of viscoelastic tests as an inspection tools. In reality all materials deviate from Hooke's law in various ways, for example, by exhibiting viscous-like as well as elastic characteristics. Viscoelastic materials are those for which the relationship between stress and strain depends on time. All materials exhibit some viscoelastic response. In common metals such as steel, aluminum, copper etc. at room temperature and small strain, the behavior does not deviate much from linear elasticity. Synthetic polymer, wood as well as metals at high temperature display significant viscoelastic effects.

The Kelvin–Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behavior of material. This model represents the delayed elastic response subjected to stress where the deformation is time-dependent. Mishra [20] studied magneto–thermo–mechanical interaction in an aeolotropic viscoelastic cylinder subject to periodic loading considering Kelvin–Voigt model of linear viscoelasticity. Several investigations relating to thermo-visco-elasticity theory have been presented by [21–24,28].

The main object of the present paper is to study the thermo-visco-elastic stresses in an isotropic visco-elastic homogeneous spherical shell due to step input of temperatures on the stress free boundaries of the shell in the context of TEWED (GN-III) [6] and three-phase-lag model [19] of generalized thermo-visco-elasticity. Using the Laplace transformation the fundamental equations have been expressed in the form of vector–matrix differential equation which is then solved by eigen value approach. The inversion of Laplace transform is done following [25]. The results obtained theoretically have been computed numerically and are presented graphically for Copper material. A complete and comprehensive analysis and comparison of results of the above theories are presented.

2. Basic equations and constitutive relations

We consider a homogeneous isotropic thermo-visco-elastic spherical shell of inner radius a and outer radius b in an undisturbed state and initially at uniform temperature T_0. We introduce spherical polar co-ordinates (r, θ, ϕ) with the centre of the cavity as the origin.

In the present problem (due to spherical symmetry) the displacement and temperature are assumed to be functions of r and time t only. The stress–strain–temperature relation in the present problem are (Kelvin–Voigt type [26])

$$\tau_{ij} = \lambda\left(1 + t_0\frac{\partial}{\partial t}\right)\Delta\delta_{ij} + 2\mu\left(1 + t_0\frac{\partial}{\partial t}\right)e_{ij} - \gamma T\delta_{ij}, \tag{1}$$

and generalized heat conduction equation in three-phase-lag model is

$$K^*\nabla^2 T + \tau_v^*\nabla^2\dot{T} + K\tau_T\nabla^2\ddot{T} = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)(\rho C_e\ddot{T} + \gamma T_0\ddot{\Delta}), \tag{2}$$

where $\tau_{ij}(i,j = r,\theta)$ is the stress tensor, Δ is the dilatation, T is the temperature increase over the reference temperature T_0, $\gamma = (3\lambda + 2\mu)\alpha_t$, λ and μ are the Lame's constants, α_t is the coefficient of linear thermal expansion of the material, K is the coefficient of thermal conductivity, K^* is the additional material constant, ρ is the mass density, C_e is the specific heat of the solid at constant strain, t_0, τ_T and τ_q are the mechanical relaxation time, the phase-lag of temperature gradient and the phase-lag of heat flux, respectively. Also $\tau_v^* = K + \tau_v K^*$, where τ_v is the phase-lag of thermal displacement gradient and δ_{ij} is the Kronecker delta.

Eqs. (1) and (2), when $\tau_T = \tau_q = \tau_v = 0$, reduce to the equations of thermo-elasticity with energy dissipation (TEWED (GN-III)) for the viscous case.

If $\underset{\sim}{\mathbf{u}} = [u(r,t),0,0]$ be the displacement vector, then

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = e_{\phi\phi} = \frac{u}{r}. \tag{3}$$

The stress equation of motion in spherical polar co-ordinates is given by

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho\frac{\partial^2 u}{\partial t^2}. \tag{4}$$

Introducing the following dimensionless quantities:

$$U = \frac{(\lambda+2\mu)u}{a\gamma T_0}; \quad (R,S) = \left(\frac{r}{a},\frac{b}{a}\right); \quad (\sigma_R,\sigma_\theta) = \frac{1}{\gamma T_0}(\tau_{rr},\tau_{\theta\theta}); \quad \Theta = \frac{T}{T_0}; \quad \eta = \frac{Gt}{a}; \quad G^2 = \frac{\lambda+2\mu}{\rho};$$

Eqs. (1), (2) and (4) become

$$\sigma_R = \left(1 + \frac{t_0 G}{a}\frac{\partial}{\partial\eta}\right)\frac{\partial U}{\partial R} + 2\left(\lambda_1 + \lambda_2\frac{\partial}{\partial\eta}\right)\frac{U}{R} - \Theta, \tag{5}$$

$$\sigma_\theta = \left(1 + \frac{t_0 G}{a}\frac{\partial}{\partial\eta}\right)\frac{U}{R} + \left(\lambda_1 + \lambda_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial U}{\partial R} + \frac{U}{R}\right) - \Theta, \tag{6}$$

$$\left[a_0 + a_1\frac{\partial}{\partial\eta} + a_2\frac{\partial^2}{\partial\eta^2}\right]\left(\frac{\partial^2\Theta}{\partial R^2} + \frac{2}{R}\frac{\partial\Theta}{\partial R}\right) = \left\{1 + b_1\frac{\partial}{\partial\eta} + \frac{1}{2}b_1^2\frac{\partial^2}{\partial\eta^2}\right\}\left[\frac{\partial^2\Theta}{\partial\eta^2} + \epsilon\frac{\partial^2}{\partial\eta^2}\left(\frac{\partial U}{\partial R} + \frac{2U}{R}\right)\right], \tag{7}$$

and

$$\left(1 + \frac{t_0 G}{a}\frac{\partial}{\partial\eta}\right)\left[\frac{\partial^2 U}{\partial R^2} + \frac{2}{R}\frac{\partial U}{\partial R} - \frac{2U}{R^2}\right] = \frac{\partial\Theta}{\partial R} + \frac{\partial^2 U}{\partial\eta^2}, \tag{8}$$

where $\lambda_1 = \frac{\lambda}{\lambda+2\mu}$, $\lambda_2 = \frac{\lambda_1 t_0 G}{a}$, $a_0 = \frac{K^*}{\rho C_e G^2}$, $a_1 = \frac{\tau_v^*}{a\rho C_e G}$, $a_2 = \frac{\tau_T K}{a^2\rho C_e}$, $b_1 = \frac{\tau_q G}{a}$ and $\epsilon = \frac{\gamma^2 T_0}{\rho C_e(\lambda+2\mu)}$ are dimensionless constants. ϵ being the thermo-elastic coupling constant.

The boundary conditions are given by

$$\sigma_R = 0 \quad \text{on} \quad R = 1, \quad S\ \eta \geqslant 0, \tag{9}$$

$$\Theta = \chi_1 H(\eta), \quad \text{on} \quad R = 1, \quad \eta > 0, \tag{10}$$

$$\quad = \chi_2 H(\eta) \quad \text{on} \quad R = S, \quad \eta > 0, \tag{11}$$

where χ_1 and χ_2 are dimensionless constants and $H(\eta)$ is the Heaviside unit step function. The above conditions indicate that for time $\eta \leqslant 0$ there is no temperature ($\Theta = 0$). Thermal shocks are given on the boundaries of the shell ($R = 1, S$) immediately after time $\eta = 0$. Thermal stresses in the elastic medium due to the application of these thermal shocks are calculated. We assume that the medium is at rest and undisturbed initially. The initial and regularity conditions can be written as

$$U = \frac{\partial U}{\partial\eta} = \frac{\partial^2 U}{\partial\eta^2} = 0 \quad \text{and} \quad \Theta = \frac{\partial\Theta}{\partial\eta} = \frac{\partial^2\Theta}{\partial\eta^2} = 0 \quad \text{at} \quad \eta = 0, \quad R \geqslant 1, \tag{12}$$

$$U = \Theta = \frac{\partial U}{\partial\eta} = \frac{\partial\Theta}{\partial\eta} = 0 \quad \text{when} \quad R \to \infty. \tag{13}$$

3. Method of solution

Let

$$\{\overline{U}(R,p), \overline{\Theta}(R,p)\} = \int_0^\infty \{U(R,\eta), \Theta(R,\eta)\} e^{-p\eta} d\eta,$$

with $Re(p) > 0$ denote the Laplace transform of U and Θ, respectively.

On taking Laplace transform, Eqs. (7) and (8) reduce to

$$\frac{d^2\overline{\Theta}}{dR^2} + \frac{2}{R}\frac{d\overline{\Theta}}{dR} = a_3\left[\overline{\Theta} + \epsilon\left(\frac{d\overline{U}}{dR} + \frac{2\overline{U}}{R}\right)\right], \tag{14}$$

and

$$\frac{d^2\overline{U}}{dR^2} + \frac{2}{R}\frac{d\overline{U}}{dR} - \frac{2\overline{U}}{R^2} = a_4\left(\frac{d\overline{\Theta}}{dR} + p^2\overline{U}\right), \tag{15}$$

where $a_3 = \frac{p^2(1+b_1p+\frac{1}{2}b_1^2p^2)}{a_0+a_1p+a_2p^2}$ and $a_4 = \frac{a}{a+t_0pG}$.

Differentiating Eq. (14) with respect to R and using Eq. (15) we get

$$\frac{d^2}{dR^2}\left(\frac{d\overline{\Theta}}{dR}\right) + \frac{2}{R}\frac{d}{dR}\left(\frac{d\overline{\Theta}}{dR}\right) - \frac{2}{R^2}\left(\frac{d\overline{\Theta}}{dR}\right) = a_3\left[\epsilon p^2 a_4\overline{U} + (1+\epsilon a_4)\frac{d\overline{\Theta}}{dR}\right]. \tag{16}$$

Eqs. (15) and (16) can be written in the form

$$L(\overline{U}) = a_4p^2\overline{U} + a_4\frac{d\overline{\Theta}}{dR}, \tag{17}$$

and

$$L\left(\frac{d\overline{\Theta}}{dR}\right) = \epsilon a_3a_4p^2\overline{U} + a_3(1+\epsilon a_4)\frac{d\overline{\Theta}}{dR}, \tag{18}$$

where ϵ is the thermo-elastic coupling constant and

$$L \equiv \frac{d^2}{dR^2} + \frac{2}{R}\frac{d}{dR} - \frac{2}{R^2}. \tag{19}$$

From Eqs. (17) and (18) we have the vector–matrix differential equation as follows:

$$L\widetilde{V} = \widetilde{A}\widetilde{V}, \tag{20}$$

where

$$\widetilde{V} = \left[\overline{U}, \frac{d\overline{\Theta}}{dR}\right]^T, \quad \widetilde{A} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{21}$$

and $C_{11} = a_4p^2, C_{12} = a_4, C_{21} = \epsilon a_3a_4p^2, C_{22} = a_3(1+\epsilon a_4)$.

4. Solution of the vector–matrix differential equation

Let

$$\widetilde{V} = \widetilde{X}(m)\omega(R,m), \tag{22}$$

where m is a scalar, $\widetilde{X}$ is a vector independent of R and $\omega(R,m)$ is a non-trivial solution of the scalar differential equation

$$L\omega = m^2\omega. \tag{23}$$

Let $\omega = R^{-1/2}\omega_1$. Therefore, from Eq. (23) we have

$$\frac{d^2\omega_1}{dR^2} + \frac{1}{R}\frac{d\omega_1}{dR} - \left(\frac{9}{4R^2} + m^2\right)\omega_1 = 0. \tag{24}$$

The solution of the Eq. (23) is

$$\omega = [A_1I_{3/2}(mR) + A_2K_{3/2}(mR)]/\sqrt{R}. \tag{25}$$

Using Eqs. (22) and (23) into Eq. (20) we get

$$\widetilde{A}\widetilde{X} = m^2\widetilde{X}, \tag{26}$$

where $\widetilde{X}(m)$ is the eigenvector corresponding to the eigenvalue m^2. The characteristic equation corresponding to $\widetilde{A}$ can be written as

$$m^4 - (C_{11} + C_{22})m^2 + (C_{11}C_{22} - C_{12}c_{21}) = 0. \tag{27}$$

The roots of the characteristic Eq. (27) are of the form $m^2 = m_1^2$ and $m^2 = m_2^2$, where

$$m_1^2 + m_2^2 = C_{11} + C_{22}, \quad m_1^2 m_2^2 = C_{11}C_{22} - C_{12}C_{21}. \tag{28}$$

The eigenvectors $X(m_j), j = 1,2$ corresponding to eigenvalues $m_j^2, j = 1,2$ can be calculated as

$$\widetilde{X}(m_j) = \begin{bmatrix} X_1(m_j) \\ X_2(m_j) \end{bmatrix} = \begin{bmatrix} C_{12} \\ -(C_{11} - m_j^2) \end{bmatrix}, \quad j = 1,2. \tag{29}$$

Therefore, the Eq. (22) can be written as

$$\widetilde{V} = \widetilde{X}(m_j)[A_1 I_{3/2}(m_1 R) + B_1 K_{3/2}(m_1 R)]/\sqrt{R} + \widetilde{X}(m_j)[A_1 I_{3/2}(m_2 R) + B_1 K_{3/2}(m_2 R)]/\sqrt{R}, \quad j = 1,2. \tag{30}$$

From equations in (21) we can write

$$\overline{U} = \sum_{i=1,2} C_{12}[A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)]/\sqrt{R}, \tag{31}$$

and

$$\frac{d\overline{\Theta}}{dR} = -\sum_{i=1,2}(C_{11} - m_i^2)[A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)]/\sqrt{R}, \tag{32}$$

where $I_{3/2}(m_i R)$ and $K_{3/2}(m_i R)$ are the modified Bessel functions of order 3/2 of first and second kind, respectively. A_i's and B_i's $i = 1,2$ are independent of R but depend on p and are to be determined from the boundary conditions.

Using the recurrence relations of modified Bessel functions [27] we obtain, from the Eq. (32)

$$\overline{\Theta} = \sum_{i=1,2} \frac{(C_{11} - m_i^2)}{m_i}[A_i I_{1/2}(m_i R) + B_i K_{1/2}(m_i R)]/\sqrt{R}, \tag{33}$$

since

$$\frac{1}{R^{1/2}} P_{3/2}(mR) = -\frac{d}{dR}\left[\frac{P_{1/2}(mR)}{mR^{1/2}}\right], \tag{34}$$

where $P = I, K$. Taking Laplace transform of the Eqs. (5) and (6) we get

$$\bar{\sigma}_R = \sum_{i=1,2} A_i\left[a_5 I_{3/2}(m_i R) - \frac{a_4 p^2}{m_i} R I_{1/2}(m_i R)\right]\Big/ R^{3/2} + \sum_{i=1,2} B_i\left[a_5 K_{3/2}(m_i R) - \frac{a_4 p^2}{m_i} R K_{1/2}(m_i R)\right]\Big/ R^{3/2}, \tag{35}$$

and

$$\begin{aligned}\bar{\sigma}_\Theta = &\sum_{i=1,2} A_i\left[a_6 I_{3/2}(m_i R) - \left\{a_7 m_i + \frac{a_4 p^2 - m_i^2}{m_i}\right\} R I_{1/2}(m_i R)\right]\Big/ R^{3/2} \\ &+ \sum_{i=1,2} B_i\left[a_6 K_{3/2}(m_i R) - \left\{a_7 m_i + \frac{a_4 p^2 - m_i^2}{m_i}\right\} R K_{1/2}(m_i R)\right]\Big/ R^{3/2}\end{aligned} \tag{36}$$

where $a_5 = 2a_4(\lambda_1 + \lambda_2 p - \frac{t_0 p G}{a} - 1)$, $a_6 = \left\{(\lambda_1 + \lambda_2 p)(1 - 2a_4) + \frac{1}{a_4}\right\}$, $a_7 = (\lambda_1 + \lambda_2 p)a_4$. Using the boundary conditions $\bar{\sigma}_R = 0$ on $R = 1$, $R = S$ and $\overline{\Theta} = \frac{\chi_1}{p}$ on $R = 1$, $\overline{\Theta} = \frac{\chi_2}{p}$ on $R = S$. and using the recurrence relations [27] from Eqs. (33) and (34) we obtain

$$\begin{aligned} A_1 W_{11} + A_2 W_{12} + B_1 W_{13} + B_2 W_{14} &= 0, \\ A_1 W_{21} + A_2 W_{22} + B_1 W_{23} + B_2 W_{24} &= 0, \\ A_1 W_{31} + A_2 W_{32} + B_1 W_{33} + B_2 W_{34} &= \frac{\chi_1}{p}, \\ A_1 W_{41} + A_2 W_{42} + B_1 W_{43} + B_2 W_{44} &= \frac{\chi_2}{p}, \end{aligned} \tag{37}$$

where

$$W_{1i} = a_5 P_{3/2}(m_j) - \frac{a_4 p^2}{m_j} P_{1/2}(m_j),$$
$$W_{2i} = a_5 P_{3/2}(m_j S) - \frac{a_4 p^2}{m_j} P_{1/2}(m_j S),$$
$$W_{3i} = \frac{a_4 p^2 - m_j^2}{m_j} P_{1/2}(m_j),$$
$$W_{4i} = \frac{a_4 p^2 - m_j^2}{m_j S^{1/2}} P_{1/2}(m_j S), \tag{38}$$

where $P = I$ for $i = j = 1, 2$; $P = K$ for $i = 3, j = 1$ and $i = 4, j = 2$.

From (37) the values of A_1, A_2, B_1 and B_2 are given as

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{\chi_1}{p} \\ \frac{\chi_2}{p} \end{pmatrix}. \tag{39}$$

Eq. (27) can be written as

$$m^4 - (a_3 + a_4 p^2 + \epsilon a_3 a_4) m^2 + a_3 a_4 p^2 = 0. \tag{40}$$

Therefore, the positive roots of Eq. (40) are

$$m_1, m_2 = \frac{1}{2}\left(\sqrt{\alpha} \pm \sqrt{\beta}\right), \tag{41}$$

where

$$\alpha, \beta = \left(\sqrt{a_3} \pm \sqrt{a_4} p\right)^2 + \epsilon a_3 a_4. \tag{42}$$

Therefore, m_1 and m_2 are real and positive quantities.

5. Special case (when the body is non-viscous and infinite)

For non-viscous material $t_0 = 0$, i.e., $\lambda_2 = 0$ and $a_4 = 0$. Therefore, $a_5 = -\frac{4\mu}{\lambda+2\mu}$, $a_6 = \frac{2\mu}{\lambda+2\mu}$, and $a_7 = \frac{\lambda}{\lambda+2\mu}$. Hence, Eqs. (35) and (36) reduce to

$$\bar{\sigma}_R = \sum_{i=1,2} A_i \left[-\frac{4\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \frac{p^2}{m_i} R I_{1/2}(m_i R) \right] \Big/ R^{3/2} + \sum_{i=1,2} B_i \left[-\frac{4\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{p^2}{m_i} R K_{1/2}(m_i R) \right] \Big/ R^{3/2}, \tag{43}$$

and

$$\bar{\sigma}_\Theta = \sum_{i=1,2} A_i \left[\frac{2\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu) m_i} R I_{1/2}(m_i R) \right] \Big/ R^{3/2}$$
$$+ \sum_{i=1,2} B_i \left[\frac{2\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu) m_i} R K_{1/2}(m_i R) \right] \Big/ R^{3/2}, \tag{44}$$

Moreover, for large value of b i.e. for large value of S, $K_0(m_i S)$ and $K_1(m_i S)$ tend to zero. Thus for large value of b the asymptotic expression of $\bar{\sigma}_R(I)$ and $\bar{\sigma}_\theta(I)$ are given as

$$\bar{\sigma}_R(I) = \frac{\chi_2}{p} \frac{\sqrt{S}}{R^2} \times \frac{e^{-m_2(S-R)} \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_1 S}\right) + \frac{p^2}{m_1} S \right] \times \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_2 R}\right) + \frac{p^2}{m_2} R \right] - e^{-m_1(S-R)} \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_2 S}\right) + \frac{p^2}{m_2} S \right] \times \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_1 R}\right) + \frac{p^2}{m_1} R \right]}{\frac{p^2 - m_1^2}{m_1} \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_2 S}\right) + \frac{p^2}{m_2} S \right] - \frac{p^2 - m_2^2}{m_2} \left[\frac{4\mu}{\lambda+2\mu}\left(1 - \frac{1}{m_1 S}\right) + \frac{p^2}{m_1} S \right]}, \tag{45}$$

$\to 0$ as $S \to \infty$

and

$$\bar{\sigma}_\theta(I) = \frac{\chi_2}{p}\frac{\sqrt{S}}{R^2}\frac{e^{-m_1(S-R)}\left[\frac{2\mu}{\lambda+2\mu}\left(1-\frac{1}{m_1R}\right)-\frac{\lambda m_1^2+(\lambda+2\mu)(p^2-m_1^2)}{(\lambda+2\mu)m_1}R\right]\times\left[\frac{4\mu}{\lambda+2\mu}\left(1-\frac{1}{m_2S}\right)+\frac{p^2}{m_2}S\right]-e^{-m_2(S-R)}\left[\frac{2\mu}{\lambda+2\mu}\left(1-\frac{1}{m_2R}\right)-\frac{\lambda m_2^2+(\lambda+2\mu)(p^2-m_2^2)}{(\lambda+2\mu)m_2}R\right]\times\left[\frac{4\mu}{\lambda+2\mu}\left(1-\frac{1}{m_1S}\right)+\frac{p^2}{m_1}S\right]}{\frac{p^2-m_1^2}{m_1}\left[\frac{4\mu}{\lambda+2\mu}\left(1-\frac{1}{m_2S}\right)+\frac{p^2}{m_2}S\right]-\frac{p^2-m_2^2}{m_2}\left[\frac{4\mu}{\lambda+2\mu}\left(1-\frac{1}{m_1S}\right)+\frac{p^2}{m_1}S\right]}, \tag{46}$$

$\to 0$ as $S \to \infty$.

Therefore, as $b \to \infty$

$$\bar{\sigma}_R = \sum_{i=1,2} B_i\left[-\frac{4\mu}{\lambda+2\mu}K_{3/2}(m_iR) - \frac{p^2}{m_i}RK_{1/2}(m_iR)\right]\Big/R^{3/2}, \tag{47}$$

$$\bar{\sigma}_\theta = \sum_{i=1,2} B_i\left[\frac{2\mu}{\lambda+2\mu}K_{3/2}(m_iR) - \frac{\lambda m_i^2+(\lambda+2\mu)(p^2-m_i^2)}{(\lambda+2\mu)m_i}RK_{1/2}(m_iR)\right]\Big/R^{3/2}, \tag{48}$$

where B_i's $(i = 1, 2)$ are given as

$$B_1 = -\frac{\chi_1}{p} \times \frac{m_1\left[4\mu m_2K_{3/2}(m_2) + (\lambda+2\mu)p^2K_{1/2}(m_2)\right]}{4\mu\left[(p^2-m_2^2)m_1K_{3/2}(m_1)K_{1/2}(m_2) - (p^2-m_1^2)m_2K_{1/2}(m_1)K_{3/2}(m_2)\right] + (\lambda+2\mu)p^2(m_1^2-m_2^2)K_{1/2}(m_1)K_{1/2}(m_2)}, \tag{49}$$

and

$$B_2 = \frac{\chi_1}{p} \times \frac{m_2\left[4\mu m_1K_{3/2}(m_1) + (\lambda+2\mu)p^2K_{1/2}(m_1)\right]}{4\mu\left[(p^2-m_2^2)m_1K_{3/2}(m_1)K_{1/2}(m_2) - (p^2-m_1^2)m_2K_{1/2}(m_1)K_{3/2}(m_2)\right] + (\lambda+2\mu)p^2(m_1^2-m_2^2)K_{1/2}(m_1)K_{1/2}(m_2)}. \tag{50}$$

The values of m_1 and m_2 are the same for this problem and those in [8] though the dimensionless forms are different. Therefore, the above results are equivalent to those in [8].

6. Numerical results and discussions

To get the solutions for displacement, temperature distribution and stresses in the space-time domain we have to apply the Laplace inversion formula to the Eqs. (31), (33), (35) and (36), respectively, which have been done numerically using the method of [25] for fixed value of the space variable and for $\eta = \eta_i$, $i = 1(1)7$, where η_i's are computed from roots of the shifted Legendre polynomial of 7° (see Appendix) with $S = 4$. The computations for the state variables are carried out for different values of $R(R \geqslant 1)$ and values of η_i = 0.0257750, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612, 3.67119. The materials chosen for numerical evaluation are copper material. The physical data for copper are taken as [29].

$$\rho = 8.96\ \text{g/cm}^3, \quad \epsilon = 0.0186, \quad T_0 = 20\ ^\circ\text{C},$$
$$\lambda = 1.387\times10^{12}\ \text{dy/cm}^2, \quad \mu = 0.448\times10^{12}\ \text{dy/cm}^2,$$
$$C_e = 0.23\ \text{cal/g}\ ^\circ\text{C}, \quad K = 0.92\ \text{cal/cm}\ ^\circ\text{C s},$$

and the hypothetical values of relaxation time parameters are taken as

$$t_0 = 1.0\times10^{-7}\ \text{s},\ \tau_q = 2.0\times10^{-7}\ \text{s},\ \tau_T = 1.5\times10^{-7}\ \text{s},\ \tau_v = 1.0\times10^{-7}\ \text{s}.$$

In the case of G–N theory K^* is an additional material constant depending on the material. For copper material K^* is taken as $K^* = \frac{C_e(\lambda+2\mu)}{4}$ [10].

The results of the numerical evaluation of the thermo-elastic stress variations, temperature distribution and displacement are illustrated in Figs. 1–6. The variation of the stresses, temperature and displacement are observed when the step input of temperatures $\chi_1 = 4$ and $\chi_2 = 3$ are applied on the inner boundary $R = 1$ and outer boundary $S = 4$ of the shell, respectively in three-phase-lag model and TEWED (GN-III) model. Almost oscillatory natures are observed for the profiles of the stress components (σ_R and σ_θ), temperature distribution and displacement. It is also observed that the qualitative behavior are almost same for both the models (three-phase-lag model and TEWED (GN-III) model). Fig. 1a and 1b show the variation of the thermo-elastic radial stress σ_R against radial distance R for time $\eta = 1.21$ and 0.026, respectively. In Fig. 1a, the amplitude of oscillation for thermo-elastic radial stress σ_R is more pronounced in the case of three-phase-lag model in comparison with TEWED (GN-III) model. It is also observed that due to the presence of viscosity term in the three-phase-lag model the amplitude of the thermo-elastic radial stress σ_R has appreciably decreased for viscous case in comparison with non-viscous case. Similar behavior is also observed for thermo-elastic radial stress σ_R in the case of TEWED (GN-III) model.

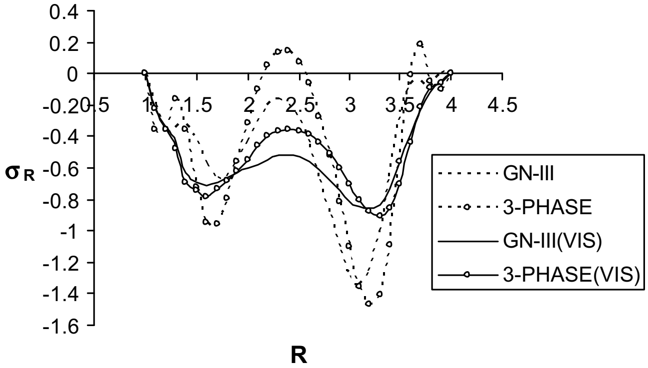

Fig. 1a. Radial stress vs. R for time $\eta = 1.21$.

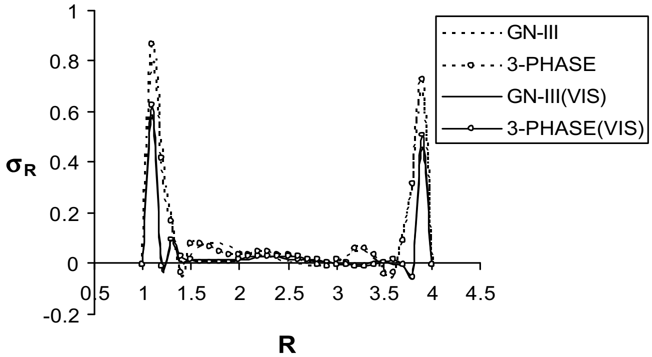

Fig. 1b. Radial stress vs. R for time $\eta = .026$.

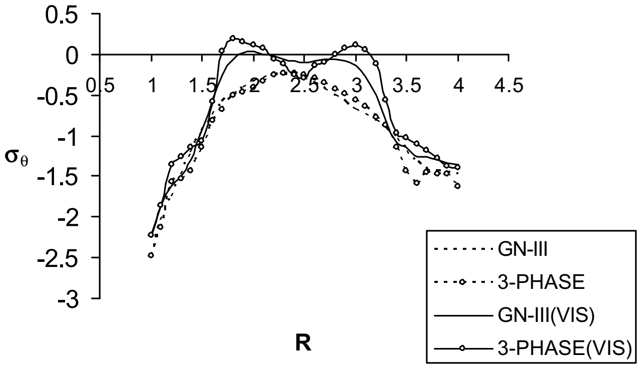

Fig. 2a. Hoop stress vs. R for time $\eta = 1.21$.

In Fig. 1b, when time is small ($\eta = .026$) i.e. at early stage of wave propagation both the models give close results, whereas for comparatively large time (Fig. 1a) the waves propagate with different speeds. It is clear from figures that the numerical results for the radial stress are found to satisfy the theoretical boundary conditions.

Figs. 2a and 2b are plotted for thermo-elastic hoop stress σ_θ against radial distance R for time $\eta = 1.21$ and .026, respectively for three-phase-lag model and TEWED (GN-III) model. It is clear from Fig. 2a that the maximum stress occurs at the inner boundary. It is also clear that oscillatory nature is more prominent in the case of three-phase-lag model in comparison with TEWED (GN-III) model when viscosity term is encountered, whereas for non-viscous case both the models give close results for time $\eta = 1.21$. It is observed from Fig. 2b that at the early stage ($\eta = .026$) the maximum stress occurs near the boundaries and it almost disappears in the interior of the shell for both the models (both viscous and non-viscous case), whereas for comparatively large time (Fig. 2a) the wave propagates with different speeds (viscous case).

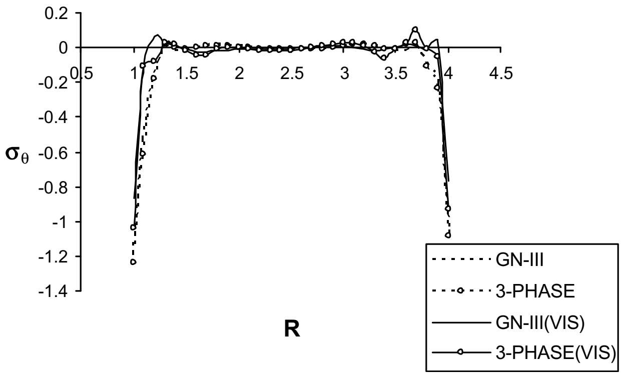

Fig. 2b. Hoop stress vs. R for time $\eta = .026$.

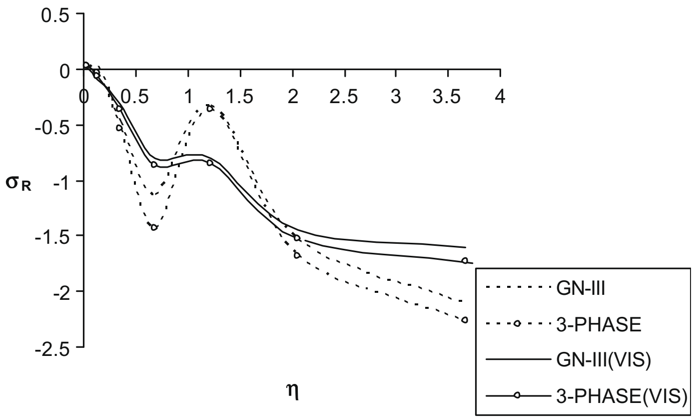

Fig. 3. Radial stress vs. η for $R = 1.4$.

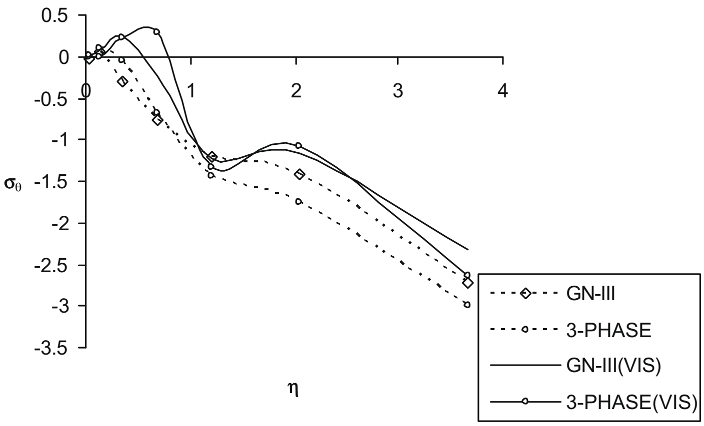

Fig. 4. Hoop stress vs. η for $R = 1.4$.

Figs. 3 and 4 depict the variations of σ_R and σ_θ versus time η for $R = 1.4$, respectively. It is seen that the stress waves propagate with time and magnitudes of both the stresses in three-phase-lag model are large in comparison with TEWED (GN-III) model for both viscous and non-viscous case.

Fig. 5 shows the graphs of temperature distribution (Θ) against the radial distance R for fixed time $\eta = 1.21$. Here, it is observed that the magnitude of temperature distribution in three-phase-lag model is slightly greater than that corresponding to the TEWED (GN-III) model for both viscous and non-viscous case. Also the magnitude of temperature distribution (Θ)

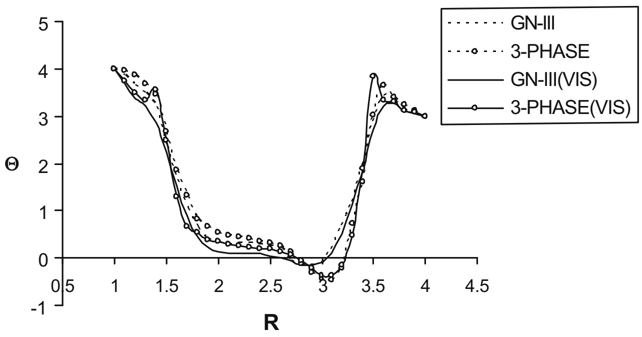

Fig. 5. Radial variation of temperature for time $\eta = 1.21$.

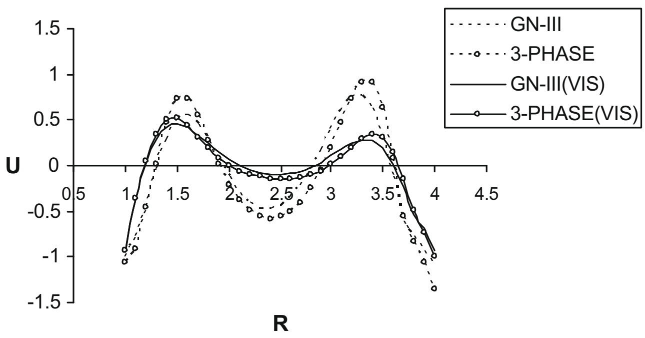

Fig. 6. Radial variation of displacement for time $\eta = 1.21$.

is large for non-viscous case in comparison with viscous case. It is also seen that the numerical results satisfy the boundary conditions ($\chi_1 = 4$ on $R = 1$ and $\chi_2 = 3$ on $S = 4$), which are in agreement with our theoretical results.

Fig. 6 is plotted for radial variation of the displacement for fixed time $\eta = 1.21$. Here, we observe that the amplitudes of oscillation is greater for the three-phase-lag model in comparison with the TEWED (GN-III) model for both the viscous and non-viscous case. It is also observed that in the viscous case the amplitude of thermo-elastic displacement is appreciably decreased for both the models in comparison with non-viscous case. For all above numerical calculations FORTRAN-77 programming Language has been used.

Acknowledgements

We are grateful to Prof. S.C. Bose of the Department of Applied Mathematics, University of Calcutta for his kind help and guidance in the preparation of the paper. We also express our sincere thanks to the reviewers for their valuable suggestions for the improvement of the paper. Avijit Kar is grateful to the Council of Scientific and Industrial Research (C.S.I.R), New Delhi for the award of a fellowship.

Appendix

Let the Laplace transform of $\sigma_i(R, \eta)$ be given by

$$\bar{\sigma}_j(R, p) = \int_0^\infty e^{-p\eta} \sigma_j(R, \eta) d\eta. \tag{A.1}$$

We assume that $\sigma_j(R, \eta)$ is sufficiently smooth to permit the use of the approximate method we apply.

Putting $x = e^{-\eta}$ in equation (A.1) we obtain

$$\bar{\sigma}_j(R, p) = \int_0^1 x^{p-1} g_j(R, x) dx, \tag{A.2}$$

where

$$g_j(R, x) = \sigma_j(R, -\log x). \tag{A.3}$$

Applying the Gaussian quadrature rule to the equation (A.2) we obtain the approximate relation

$$\sum_{i=1}^{n} W_i x_i^{p-1} g_j(R, x_i) = \bar{\sigma}_j(R, p), \tag{A.4}$$

where x_i's$(i = 1, 2, \ldots, n)$ are the roots of the shifted Legendre polynomial and W_i's$(i = 1, 2, \ldots, n)$ are the corresponding weights [Bellman] and $p = 1(1)n$.

For $p = 1(1)n$), the equations (A.4) can be written as

$$\begin{aligned}
&W_1 g_j(R, x_1) + W_2 g_j(R, x_2) + \cdots + W_n g_j(R, x_n) = \bar{\sigma}_j(R, 1)\\
&W_1 x_1 g_j(R, x_1) + W_2 x_2 g_j(R, x_2) + \cdots + W_n x_n g_j(R, x_n) = \bar{\sigma}_j(R, 2)\\
&\vdots\\
&W_1 x_1^{n-1} g_j(R, x_1) + W_2 x_2^{n-1} g_j(R, x_2) + \cdots + W_n x_n^{n-1} g_j(R, x_n) = \bar{\sigma}_j(R, n)
\end{aligned}$$

Therefore,

$$\begin{pmatrix} g_j(R, x_1) \\ g_j(R, x_2) \\ .. \\ .. \\ .. \\ g_j(R, x_n) \end{pmatrix} = \begin{pmatrix} W_1 & W_2 & \cdot & \cdot & \cdot & W_n \\ W_1 x_1 & W_2 x_2 & \cdot & \cdot & \cdot & W_n x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ W_1 x_1^{n-1} & W_2 x_2^{n-1} & \cdot & \cdot & \cdot & W_n x_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \bar{\sigma}_j(R, 1) \\ \bar{\sigma}_j(R, 2) \\ \cdot \\ \cdot \\ \cdot \\ \bar{\sigma}_j(R, n) \end{pmatrix}. \tag{A.5}$$

(As the matrix is the product of diag$\{W_i\}$ multiplied by Vander Monde matrix, it can be shown that the matrix is non-singular.)

Hence $g_j(R, x_1)$, $g_j(R, x_2)$, …, $g_j(R, x_n)$ are known.

For $n = 7$ we have

Roots of the shifted	Corresponding Weights
Legendre Polynomial	
2.5446043828620886E−2	6.4742483084434816E−2
1.2923440720030282E−1	1.3985269574463828E−1
2.9707742431130145E−1	1.9091502525255938E−1
5.0000000000000000E−1	2.0897959183673466E−1
7.0292257568869853E−1	1.9091502525255938E−1
8.7076559279969706E−1	1.3985269574463828E−1
9.7455395617137909E−1	6.4742483084434816E−2

From equations in (A.5) we can calculate the discrete values of $g_j(R, x_i)$ i.e., $\sigma_j(R, \eta_i)$; $(i = 1, 2, \ldots, 7)$ and finally using interpolation we obtain the stress components $\sigma_i(R, \eta)$; $(i = R, \theta)$.

References

[1] H.W. Lord, Y. Shulman, A generalized dynamic theory of thermoelasticity, J. Mech. Phys. Solid 15 (1967) 299–309.
[2] A.E. Green, K.A. Lindsay, Thermoelasticity, J. Elast. 2 (1972) 1–7.
[3] R.B. Hetnarski, J. Ignaczak, Generalized thermoelasticity: closed form solutions, J. Therm. Stress 16 (1993) 473–498.
[4] R.B. Hetnarski, J. Ignaczak, Generalized thermoelasticity: response of semi-space to a short laser pulse, J. Therm. Stress 17 (1994) 377–396.
[5] A.E. Green, P.M. Naghdi, A re-examination of the basic postulate of thermo-mechanics, Proc. Roy. Soc. Lond. 432 (1991) 171–194.
[6] A.E. Green, P.M. Naghdi, An undamped heat wave in an elastic solid, J. Therm. Stress 15 (1992) 253–264.
[7] A.E. Green, P.M. Naghdi, Thermoelasticity without energy dissipation, J. Elast. 31 (1993) 189–208.
[8] A. Kar, M. Kanoria, Thermoelastic interaction with energy dissipation in an unbounded body with a spherical hole, Int. J. Solid Struct. 44 (2007) 2961–2971.
[9] S.K. Roychoudhuri, P.S. Dutta, Thermoelastic interaction without energy dissipation in an infinite solid with distributed periodically varying heat sources, Int. J. Solid Struct. 42 (2005) 4192–4203.
[10] J.N. Sharma, R.S. Chouhan, On the problem of body forces and heat sources in thermoelasticity without energy dissipation, Ind. J. Pure Appl. Math. 30 (1999) 595–610.
[11] S.K. Roychoudhuri, N. Bandyopadhyay, Thermoelastic wave propagation in a rotating elastic medium without energy dissipation, Int. J. Math. Math. Sci. 1 (2004) 99–107.
[12] A. Kar, M. Kanoria, Thermoelastic interaction with energy dissipation in an infinitely extended thin plate containing a circular hole, Far East J. Appl. Math. 24 (2006) 201–217.
[13] A. Kar, M. Kanoria, Thermoelastic interaction with energy dissipation in a transversely isotropic thin circular disc, Euro. J. Mech. A: Solid 26 (2007) 969–981.
[14] D.S. Chandrasekharaiah, Thermoelastic plane waves without energy dissipation, Mech. Res. Commun. 23 (1996) 549–555.

[15] D.S. Chandrasekharaiah, A note on the uniqueness of solution in the linear theory of thermoelasticity without energy dissipation, J. Elast. 43 (1996) 279–283.
[16] S. Mukhopadhyay, Thermoelastic interactions without energy dissipation in an unbounded medium with a spherical cavity due to a thermal shock at the boundary, J. Therm. Stress 25 (2002) 877–887.
[17] D.Y. Tzou, A unified field approach for heat conduction from macro to micro scales, ASME J. Heat Transfer 117 (1995) 8–16.
[18] S.K. Roychoudhuri, One-dimensional thermoelastic waves in elastic half-space with dual-phase-lag effects, J. Mech. Mater. Struct. 2 (2007) 489–503.
[19] S.K. Roychoudhuri, On a thermoelastic three-phase-lag model, J. Therm. Stress 30 (2007) 231–238.
[20] J.C. Mishra, S.C. Samanta, A.K. Chakrabarty, Magneto–thermo–mechanical interaction in an aeolotropic viscoelastic cylinder permeated by magnetic field subjected to a periodic loading, Int. J. Eng. Sci. 29 (1991) 1209–1216.
[21] S. Mukhopadhyay, Effects of thermal relaxations on thermoviscoelastic interactions in an unbounded body with a spherical cavity subjected to a periodic loading on the boundary, J. Therm. Stress 23 (2000) 675–684.
[22] M.A. Ezzat, A.S. El-Karamany, A.A. Samaan, State-space formulation to generalized thermoviscoelasticity with thermal relaxation, J. Therm. Stress 24 (2001) 823–846.
[23] A. Baksi, R.K. Bera, L. Debnath, Eigen value approach to study the effect of rotation and relaxation time in two-dimensional problems of generalized thermoelasticity, Int. J. Eng. Sci. 42 (2004) 1573–1585.
[24] A. Baksi, R.K. Bera, Relaxation effects on plane wave propagation in a rotating magneto–thermo–viscoelastic medium, J. Therm. Stress 29 (2006) 753–769.
[25] R. Bellman, R.E. Kolaba, J.A. Lockette, Numerical Inversion of the Laplace Transform, American Elsevier Publishing Company, New York, 1966.
[26] R.S. Lakes, Viscoelastic Solids, CRC Press, Washington, USA, 1998.
[27] J.N. Watson, Theory of Bessel Function, second ed., Cambridge University Press, 1980.
[28] J.C. Mishra, A.K. Chakrabarty, S.C. Samanta, S.C. Mishra, Thermally induced vibration in an infinite anisotropic viscoelastic solid with a cylindrical cavity in the presence of a uniform axial magnetic field, Int. J. Eng. Sci. 31 (1992) 1047–1052.
[29] <www.matweb.com>.