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Fuzzy topology via fuzzy geometric logic with graded consequence

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ABSTRACT

In this paper fuzzy geometric logic and fuzzy geometric logic with graded consequence have been introduced. Graded fuzzy topological system and fuzzy topological space with graded inclusion are obtained via fuzzy geometric logic with graded consequence. As an offshoot, the notion of graded frame has been developed. This work is a two-fold many-valued generalization of Vickers' scheme 'Topology via Logic' and naturally emerges from observational semantics.

1. Introduction

The ultimate aim of this article is to introduce fuzzy geometric logic with graded consequence. Three closely related notions namely, graded fuzzy topological system, fuzzy topology with graded inclusion and graded frame shall also be introduced. As a matter of fact all these notions are interwoven. Mathematically speaking, these may be called generalizations of corresponding classical notions to the many-valued context. But many-valuedness has taken place in two layers as will be apparent in the sequel.

Geometric logic has been discussed in various works such as [9,15,19,26,27,29,30]. However for our purpose the reference point shall be Vickers' books and papers namely [26,27,29,30]. The formulae of geometric logic are based on two propositional connectives viz. $\wedge$, the binary conjunction and $\bigvee$, the arbitrary disjunction over arbitrary set of formulae including null set. As a special case the binary disjunction $\vee$ is obtained. Besides, the logic has an existential quantifier $\exists$. It is noteworthy that geometric logic does not have negation, implication or universal quantifier. Also in this logic sequents of the form $\alpha \vdash \beta$ are derived from a set (may be the null set) of sequents. These special sequents have exactly one formula on either side of the symbol $\vdash$ (turnstile), the intention of the symbol being, as usual, that β follows from α. The related notion of topological system is a triple $(X, \models, A)$ where X is a non-empty set, A is a frame and $\models$ is a binary relation from X to A called 'satisfiability'. The frame A is a partially ordered set closed with respect to finite meet ($\wedge$) and arbitrary join ($\bigvee$) and where $\wedge$ distributes over $\bigvee$. As a consequence it becomes a bounded lattice. Any topology is an instance of a frame. In a sense topological system is a generalization of topological space. The pioneering works by Vickers mentioned before deal with the relationship among geometric logic, topological system and topology. X may be interpreted as the set of objects

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or situations while A may be interpreted as properties or assertions. By $x \models a$ we may mean that the object (situation) x satisfies the property (assertion) a. In this paper we shall ascribe grades to this relational complex at various levels.

Fuzzy topological space was introduced in [5] and has since been studied extensively [2,3,11,17,18,21]. Fuzzy topological systems have also been defined and studied earlier [12,25] with some variation in the definition. This notion with other names has been studied in categorical framework also [7,24]. Categorical relationship among the categories of fuzzy topological spaces, fuzzy topological systems and frames has been the topic of great interest to which area the present authors have contributed too [12–14]. In the present work another level of generalization (that is, introduction of many-valuedness) shall take place giving rise to graded fuzzy topological systems (vide Section 6) and fuzzy topological space with graded inclusion (vide Section 6). It will also be required to generalize the notion of frame to graded frame (vide Section 6).

Although fuzzy topological systems have been discussed in earlier literature, to the knowledge of present authors, no notion of fuzzy geometric logic or something akin to it has so far been recorded. We consider the notion of fuzzy geometric logic as the main contribution of the present paper. There have been two steps in this case of generalization also, the first step, fuzzy geometric logic (vide Section 2) and the second, fuzzy geometric logic with graded inclusion (vide Section 5).

At this point some motivation behind generalizing the classical notions and geometric logic to multi-valuedness is called for. In [25] we get an example where the satisfiability relation $\models$ of a topological system needs a generalization. Taking a physical interpretation of topological systems by Vickers [30] in which an assertion like "starts 01010" is satisfied by a computer program generating bits of 0's and 1's, the authors present examples where all the generated symbols 'are not identical to either "1" or "0" but rather similar to these' [25]. In such a situation the program would satisfy the above assertion to some degree.

Secondly, geometric logic is endowed with an informal observational semantics [27]: whether what has been observed does satisfy (match) an assertion or not. In fact, from the stand point of observation, negative and implicational propositions and universal quantification face ontological difficulties. On the other hand arbitrary disjunction needs to be included (cf. [27] for an elegant discussion on this issue). Now, observations of facts and assertions about them may corroborate with each other partially. It is a fact of reality and in such cases it is natural to invoke the concept of 'satisfiability to some extent or to some degree'. As a result the question whether some assertion follows from some other assertion might not have a crisp answer 'yes'/'no'. It is likely that in general the 'relation of following' or more technically speaking, the consequence relation turnstile ($\vdash$) may be itself many-valued or graded (vide Definition 5.1). For an introduction to the general theory of graded consequence relation we refer to [1,4]. This theory falls within the broad category of fuzzy logic but is not exactly the same as that developed in [6,10,22,23]. Thus, we have adopted graded satisfiability as well as graded consequence (cf. Sections 6 and 5 respectively) in the present work.

Thirdly, exploration of a relationship between many-valued logics (Łukasiewicz n-valued logic $Ł_n^c$) and fuzzy topology has already been in the agenda [12,20]. We, in [12], have investigated the relationship between $Ł_n^c$ and fuzzy topological systems but following [20], by $Ł_n^c$ we have understood $Ł_n^c$-algebra. It has been imperative to link with fuzzy topological systems (and fuzzy topological spaces as a result), a many-valued logic similar to classical topological systems and geometric logic. This goal has been achieved here with the introduction of a general fuzzy geometric logic. In our case, of course, a further generalization has been made by taking the consequence relation as many-valued also and this in turn gives rise to a generalization of the algebraic structure frame to graded frame.

While preparing the final draft of the paper we have come across with a very recent publication [8] which has some overlap with our notions of fuzzy topological space with graded inclusion, graded frame and graded fuzzy topological system. The similarity and differences of these notions with those in [8] shall be pointed out at appropriate places (cf. Note 1, Note 2, Note 3). However our main focus is on fuzzy geometric logics which is of no concern of the paper mentioned.

The present paper is organized as below.

In Section 2 fuzzy geometric logic is introduced and its soundness is established. Sections 3 and 4 deal with the relationship among fuzzy geometric logic, fuzzy topological system and fuzzy topological space. In Section 5, fuzzy geometric logic with graded consequence is defined. Section 6 defines fuzzy topological space with graded inclusion, graded frame and graded fuzzy topological systems. Finally in Section 7 the interrelation among graded fuzzy topological system, fuzzy topological space with graded inclusion and fuzzy geometric logic with graded consequence has been discussed.

2. Fuzzy geometric logic

The **alphabet** of the language $\mathscr{L}$ of fuzzy geometric logic comprises of the connectives $\wedge$, $\bigvee$, the existential quantifier $\exists$, parentheses) and (as well as:

- countably many individual constants $c_1, c_2, \ldots$;
- denumerably many individual variables $x_1, x_2, \ldots$;
- propositional constants $\top$, $\bot$;
- for each $i > 0$, countably many i-place predicate symbols p_j^i's, including at least the 2-place symbol "=" for identity;
- for each $i > 0$, countably many i-place function symbols f_j^i's.

Definition 2.1 *(Term).* **Terms** are recursively defined in the usual way.

- every constant symbol c_i is a term;
- every variable x_i is a term;
- if f_j is an i-place function symbol, and $t_1, t_2, \ldots, t_i$ are terms then $f^i_j t_1 t_2 \ldots t_i$ is a term;
- nothing else is a term.

Definition 2.2 *(Geometric formula).* **Geometric formulae** are recursively defined as follows:

- $\top$, $\bot$ are geometric formulae;
- if p_j is an i-place predicate symbol, and $t_1, t_2, \ldots, t_i$ are terms then $p^i_j t_1 t_2 \ldots t_i$ is a geometric formula;
- if t_i, t_j are terms then $(t_i = t_j)$ is a geometric formula;
- if ϕ and ψ are geometric formulae then $(\phi \wedge \psi)$ is a geometric formula;
- if ϕ_i's $(i \in I)$ are geometric formulae then $\bigvee\{\phi_i\}_{i\in I}$ is a geometric formula, when $I = \{1, 2\}$ then the above formula is written as $\phi_1 \vee \phi_2$;
- if ϕ is a geometric formula and x_i is a variable then $\exists x_i \phi$ is a geometric formula;
- nothing else is a geometric formula.

$t[t'/x]$ and $\phi[t/x]$ are standard notations for replacing a variable x by a term t' in a term t and a free variable x by an admissible term t in a wff ϕ respectively.

Definition 2.3 *(Interpretation).* An **interpretation** I consists of

- a set D, called the domain of interpretation;
- an element $I(c_i) \in D$ for each constant c_i;
- a function $I(f^i_j) : D^i \longrightarrow D$ for each function symbol f^i_j;
- a fuzzy relation $I(p^i_j) : D^i \longrightarrow [0, 1]$ for each predicate symbol p^i_j i.e. it is a fuzzy subset of D^i.

Definition 2.4 *(Graded Satisfiability).* Let s be a sequence over D. Let $s = (s_1, s_2, \ldots)$ be a sequence over D where $s_1, s_2, \ldots$ are all elements of D. Let d be an element of D. Then $s(d/x_i)$ is the result of replacing i'th coordinate of s by d i.e., $s(d/x_i) = (s_1, s_2, \ldots, s_{i-1}, d, s_{i+1}, \ldots)$. Let t be a term. Then s assigns an element $s(t)$ of D as follows:

- if t is the constant symbol c_i then $s(c_i) = I(c_i)$;
- if t is the variable x_i then $s(x_i) = s_i$;
- if t is the function symbol $f^i_j t_1 t_2 \ldots t_i$ then $s(f^i_j t_1 t_2 \ldots t_i) = I(f^i_j)(s(t_1), s(t_2), \ldots, s(t_i))$.

Now we define grade of satisfiability of ϕ by s written as $gr(s\ sat\ \phi)$, where ϕ is a geometric formula, as follows:

- $gr(s\ sat\ p^i_j t_1 t_2 \ldots t_i) = I(p^i_j)(s(t_1), s(t_2), \ldots, s(t_i))$;
- $gr(s\ sat\ \top) = 1$;
- $gr(s\ sat\ \bot) = 0$;
- $gr(s\ sat\ t_i = t_j) = \begin{cases} 1 & \text{if } s(t_i) = s(t_j) \\ 0 & \text{otherwise}; \end{cases}$
- $gr(s\ sat\ \phi_1 \wedge \phi_2) = gr(s\ sat\ \phi_1) \wedge gr(s\ sat\ \phi_2)$;
- $gr(s\ sat\ \phi_1 \vee \phi_2) = gr(s\ sat\ \phi_1) \vee gr(s\ sat\ \phi_2)$;
- $gr(s\ sat\ \bigvee\{\phi_i\}_{i\in I}) = sup\{gr(s\ sat\ \phi_i) \mid i \in I\}$;
- $gr(s\ sat\ \exists x_i \phi) = sup\{gr(s(d/x_i)\ sat\ \phi) \mid d \in D\}$.

In $[0, 1]$, we have used $\wedge$ and $\vee$ to mean min and max respectively – a convention that will be followed throughout. The expression $\phi \vdash \psi$, where ϕ, ψ are wffs, is called a sequent. We now define satisfiability of a sequent.

Definition 2.5. 1. $s\ sat\ \phi \vdash \psi$ if and only if $gr(s\ sat\ \phi) \leq gr(s\ sat\ \psi)$.
2. $\phi \vdash \psi$ is valid in I if and only if $s\ sat\ \phi \vdash \psi$ for all s in the domain of I.
3. $\phi \vdash \psi$ is universally valid if and only if it is valid in all interpretations.

Theorem 2.6 *(Substitution Theorem). Let D be the domain of interpretation I:*

1. *Let t and t' be terms. For every sequence s over D,* $s(t[t'/x_k]) = s(s(t')/x_k)(t)$.

2. *Let ϕ be a geometric formula and t be a term. For every sequence s over D, $gr(s\ sat\ \phi[t/x_k]) = gr(s(s(t)/x_k)\ sat\ \phi)$.*

Proof. By induction on t and ϕ respectively. □

2.1. Rules of inference

In this subsection the rules of inference for fuzzy geometric logic are given. A rule of inference for fuzzy geometric logic is of the form $\dfrac{\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_i}{\mathscr{S}}$, where each of the $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_i$ and $\mathscr{S}$ is a sequent. The sequents $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_i$ are known as premises and the sequent $\mathscr{S}$ is called the conclusion. It should be noted that for a rule of inference the set of premises can be empty also.

The rules of inference for fuzzy geometric logic are as follows.

1. $\phi \vdash \phi$,
2. $\dfrac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi}$,
3. (i) $\phi \vdash \top$, (ii) $\phi \wedge \psi \vdash \phi$, (iii) $\phi \wedge \psi \vdash \psi$, (iv) $\dfrac{\phi \vdash \psi \qquad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi}$,
4. (i) $\phi \vdash \bigvee S$ $(\phi \in S)$, for any S, (ii) $\dfrac{\phi \vdash \psi \qquad \text{all } \phi \in S}{\bigvee S \vdash \psi}$, for any S,
5. $\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}$,
6. $\top \vdash (x = x)$,
7. $((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)]$,
8. (i) $\dfrac{\phi \vdash \psi[x/y]}{\phi \vdash \exists y \psi}$, (ii) $\dfrac{\exists y \phi \vdash \psi}{\phi[x/y] \vdash \psi}$,
9. $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$.

2.2. Soundness

The soundness of a rule means that if all the premises are valid in an interpretation I then the conclusion must also be valid in the same interpretation I. Satisfaction relation being many-valued, the validity of a sequent has a meaning different from that in the classical geometric logic. In this subsection we will show the soundness of the above rules of inference.

Theorem 2.7. *The rules of inference for fuzzy geometric logic are universally valid.*

Proof.

1. $gr(s \text{ sat } \phi) = gr(s \text{ sat } \phi)$, for any s. Hence $\phi \vdash \phi$ is valid.
2. Given $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \psi) \le gr(s \text{ sat } \chi)$ for any s. Therefore $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \chi)$ for any s. Hence $\phi \vdash \chi$ is valid when $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid.
3. (i) $gr(s \text{ sat } \phi) \le 1 = gr(s \text{ sat } \top)$ for any s. Hence $\phi \vdash \top$ is valid.
   (ii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \le gr(s \text{ sat } \phi)$ for any s. Hence $\phi \wedge \psi \vdash \phi$ is valid.
   (iii) $gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \le gr(s \text{ sat } \psi)$ for any s. Hence $\phi \wedge \psi \vdash \psi$ is valid.
   (iv) Given $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid. So $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \psi)$ and $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \chi)$ for any s. So $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \psi) \wedge gr(s \text{ sat } \chi) = gr(s \text{ sat } \psi \wedge \chi)$ for any s. Hence $\phi \vdash \psi \wedge \chi$ is valid when $\phi \vdash \psi$ and $\phi \vdash \chi$ are valid.
4. (i) $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \bigvee S(\phi \in S))$ for any s. Hence $\phi \vdash \bigvee S(\phi \in S)$ is valid. (ii) Given $\phi \vdash \psi$ is valid for all $\phi \in S$. So $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \psi)$ for all $\phi \in S$ and any s. So, $sup_{\phi \in S}\{gr(s \text{ sat } \phi)\} \le gr(s \text{ sat } \psi)$ for any s. Hence $gr(s \text{ sat } \bigvee S) \le gr(s \text{ sat } \psi)$ for any s. So, $\bigvee S \vdash \psi$ is valid when $\phi \vdash \psi$ is valid for all $\phi \in S$.
5. Let us show the soundness of the 5th rule.
   $gr(s \text{ sat } \phi \wedge \bigvee S) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S)$
   $= gr(s \text{ sat } \phi) \wedge sup_{\psi \in S}\{gr(s \text{ sat } \psi)\} = sup_{\psi \in S}\{gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi)\} =$
   $sup\{gr(s \text{ sat } \phi \wedge \psi) \mid \psi \in S\}$, for any s. Hence $\phi \wedge \bigvee S \vdash sup\{\phi \wedge \psi \mid \psi \in S\}$ is valid.
6. $gr(s \text{ sat } \top) = 1 = gr(s \text{ sat } x = x)$, for any s. Hence $\top \vdash x = x$ is valid.
7. $gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi)$
   $= gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n))) \wedge gr(s \text{ sat } \phi)$.
   Now $gr(s \text{ sat } \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)])$
   $= gr(s(s((y_1, \ldots, y_n))/(x_1, \ldots, x_n)) \text{ sat } \phi)$.
   When $s((y_1, \ldots, y_n)) = s((x_1, \ldots, x_n))$
   then $gr(s(s((y_1, \ldots, y_n))/(x_1, \ldots, x_n)) \text{ sat } \phi) = gr(s \text{ sat } \phi)$.

Hence, $gr(s \text{ sat } ((x_1,\dots,x_n) = (y_1,\dots,y_n)) \wedge \phi)$
$\le gr(s \text{ sat } \phi[(y_1,\dots,y_n)/(x_1,\dots,x_n)])$, for any s.
So, $((x_1,\dots,x_n) = (y_1,\dots,y_n)) \wedge \phi \vdash \phi[(y_1,\dots,y_n)/(x_1,\dots,x_n)]$ is valid.

8. (i) $\phi \vdash \psi[x/y]$ is valid so, $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \psi[x/y])$, for any s. Using Theorem 2.6(2) $gr(s \text{ sat } \phi) \le gr(s(s(x)/y) \text{ sat } \psi)$, for any s, which implies that $gr(s \text{ sat } \phi) \le sup\{gr(s(d/y) \text{ sat } \psi) \mid d \in D\}$, for any s. So, $gr(s \text{ sat } \phi) \le gr(s \text{ sat } \exists y\psi)$ and hence $\phi \vdash \exists y\psi$ is valid.
(ii) $\exists y\phi \vdash \psi$ is valid if and only if $gr(s \text{ sat } \exists y\phi) \le gr(s \text{ sat } \psi)$, for any s. Hence $sup\{gr(s(d/y) \text{ sat } \phi) \mid d \in D\} \le gr(s \text{ sat } \psi)$, for any s. So, $gr(s(s(x)/y) \text{ sat } \phi) \le gr(s \text{ sat } \psi)$, for any s, using Theorem 2.6(2). Therefore $gr(s \text{ sat } \phi[x/y]) \le gr(s \text{ sat } \psi)$, for any s and hence $\phi[x/y] \vdash \psi$ is valid provided $\exists y\phi \vdash \psi$ is valid.
9. $gr(s \text{ sat } \phi \wedge (\exists y)\psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \exists y\psi) = gr(s \text{ sat } \phi) \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\} = sup_{d\in D}\{gr(s \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\} \le sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\} = sup_{d\in D}\{gr(s \text{ sat } \phi \wedge \psi)\} = gr(s \text{ sat } (\exists y)\phi \wedge \psi)$, for any s. Hence $\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)$ is valid. □

Fuzzy geometric logic is sound in the sense that in every interpretation whenever the premise set of sequents is valid the conclusion sequent is also valid.

3. Fuzzy topological system via fuzzy geometric logic

Definition 3.1 *(Fuzzy topological systems).* [12] A fuzzy topological system is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is a fuzzy relation from X to A such that

1. if S is a finite subset of A, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) \mid s \in S\}$;
2. if S is any subset of A, then $gr(x \models \bigvee S) = sup\{gr(x \models s) \mid s \in S\}$.

It is to be noted that $\bigwedge S$ is either $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ if $S = \{a_1, a_2, \dots, a_n\}$ and is $\top$ if $S = \emptyset$.

Let us consider the triplet $(X, \models, A)$ where X is the non-empty set of assignments s, A is the set of geometric formulae and $\models$ defined as $gr(s \models \phi) = gr(s \text{ sat } \phi)$.

Theorem 3.2. *(i) $gr(s \models \phi \wedge \psi) = gr(s \models \phi) \wedge gr(s \models \psi)$.*
(ii) $gr(s \models \bigvee\{\phi_i\}_{i\in I}) = sup_{i\in I}\{gr(x \models \phi_i)\}$.

Proof. (i) $gr(s \models \phi \wedge \psi) = gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) = gr(s \models \phi) \wedge gr(s \models \psi)$. (ii) $gr(s \models \bigvee\{\phi_i\}_{i\in I}) = gr(s \text{ sat } \bigvee\{\phi_i\}_{i\in I}) = sup_{i\in I}\{gr(s \text{ sat } \phi_i)\} = sup_{i\in I}\{gr(s \models \phi_i)\}$. □

Definition 3.3. $\phi \approx \psi$ iff $gr(s \models \phi) = gr(s \models \psi)$ for any $s \in X$ and $\phi, \psi \in A$.

The above defined "$\approx$" is an equivalence relation. Thus we get $A/_\approx$.

Theorem 3.4. $(X, \models', A/_\approx)$ *is a fuzzy topological system, where $\models'$ is defined by $gr(s \models' [\phi]) = gr(s \models \phi)$.*

Proof. X is a non-empty set of assignments s. Let us first prove that $A/_\approx$ is a frame in the following way. Here we define $[\phi] \le [\psi]$ as follows: $[\phi] \le [\psi]$ iff $gr(s \models \phi) \le gr(s \models \psi)$ for any s i.e., $\phi \vdash \psi$ is valid. Now in fuzzy geometric logic $\phi \vdash \phi$ is valid and if $\phi \vdash \psi$ and $\psi \vdash \chi$ are valid then $\phi \vdash \chi$ is valid. Thus $\le$ is reflexive and transitive. If $[\phi] \le [\psi]$ and $[\psi] \le [\phi]$ then $gr(s \models \phi) \le gr(s \models \psi)$ and $gr(s \models \psi) \le gr(s \models \phi)$ for any s. Therefore $gr(s \models \phi) = gr(s \models \psi)$ for any s. So $\phi \approx \psi$. Consequently $[\phi] = [\psi]$. Hence $A/_\approx$ is a poset. Now if $\phi, \psi \in A$ then $\phi \wedge \psi \in A$ (by Theorem 3.2). So $[\phi], [\psi] \in A/_\approx$ and $[\phi \wedge \psi] \in A/_\approx$ i.e., $[\phi] \wedge [\psi] \in A/_\approx$. Similarly arbitrary join exists in $A/_\approx$. $[\phi] \wedge \bigvee\{[\psi_i]\}_{i\in I} = [\phi] \wedge [\bigvee\{\psi_i\}_{i\in I}] = [\phi \wedge \bigvee\{\psi_i\}_{i\in I}]$ Now we have $\phi \wedge \bigvee\{\psi_i\}_{i\in I} \vdash sup_{i\in I}\{\phi \wedge \psi_i\}$ is valid. Hence $gr(s \text{ sat } \phi \wedge \bigvee\{\psi_i\}_{i\in I}) \le gr(s \text{ sat } \bigvee\{\phi \wedge \psi_i\}_{i\in I})$ for any s. $sup_{i\in I}\{\phi \wedge \psi_i\} \vdash \phi \wedge \bigvee\{\psi_i\}_{i\in I}$ is derivable, so $gr(s \text{ sat } \bigvee\{\phi \wedge \psi_i\}_{i\in I}) \le gr(s \text{ sat } \phi \wedge \bigvee\{\psi_i\}_{i\in I})$ for any s. Therefore $gr(s \text{ sat } \phi \wedge \bigvee\{\psi_i\}_{i\in I}) = gr(s \text{ sat } \bigvee\{\phi \wedge \psi_i\}_{i\in I})$ for any s. So, $[\phi \wedge \bigvee\{\psi_i\}_{i\in I}] = [\bigvee\{\phi \wedge \psi_i\}_{i\in I}]$. Hence $[\phi] \wedge \bigvee\{[\psi_i]\}_{i\in I} = [\bigvee\{\phi \wedge \psi_i\}_{i\in I}] = \bigvee\{[\phi \wedge \psi_i]\}_{i\in I} = \bigvee\{([\phi] \wedge [\psi_i])\}_{i\in I}$. Hence $A/_\approx$ is a frame.

Now it is left to show that (a) $gr(s \models' [\phi] \wedge [\psi]) = gr(s \models' [\phi]) \wedge gr(s \models' [\psi])$ and (b) $gr(s \models' \bigvee\{[\phi_i]\}_{i\in I}) = sup_{i\in I}\{gr(s \models' [\phi_i])\}$.

Proof of the above follows easily using Theorem 3.2. Hence $(X, \models', A/_\approx)$ is a fuzzy topological system. □

Proposition 3.1. *In the fuzzy topological system $(X, \models', A/_\approx)$, defined above, for all $s \in X$, $gr(s \models' [\phi]) = gr(s \models' [\psi])$ implies $[\phi] = [\psi]$.*

Proof. As $gr(s \models' [\phi]) = gr(s \models' [\psi])$, for any s, we have $gr(s \models \phi) = gr(s \models \psi)$, for any s. Hence $\phi \approx \psi$ and consequently $[\phi] = [\psi]$. □

4. Fuzzy topology via fuzzy geometric logic

We first construct the fuzzy topological system $(X, \models', A/_\approx)$ from fuzzy geometric logic. Then $(X, ext(A/_\approx))$ is constructed as follows:

$ext(A/_\approx) = \{ext([\phi])\}_{[\phi]\in A/_\approx}$ where $ext([\phi]) : X \longrightarrow [0,1]$ is such that, for each $[\phi] \in A/_\approx$, $ext([\phi])(s) = gr(s \models' [\phi]) = gr(s \models \phi)$.

It can be shown that $ext(A/_\approx)$ forms a fuzzy topology on X as follows.

Let $ext([\phi]), ext([\psi]) \in ext(A/_\approx)$. Then $(ext([\phi]) \cap ext([\psi]))(s) = (ext([\phi]))(s) \wedge (ext([\psi]))(s) = gr(s \models' [\phi]) \wedge gr(s \models' [\psi]) = gr(s \models \phi) \wedge gr(s \models \psi) = gr(s \models \phi \wedge \psi) = gr(s \models' [\phi \wedge \psi]) = (ext([\phi \wedge \psi]))(s)$. Hence $ext([\phi]) \cap ext([\psi]) = ext([\phi \wedge \psi]) \in ext(A/_\approx)$. Similarly it can be shown that $ext(A/_\approx)$ is closed under arbitrary union. Hence $(X, ext(A/_\approx))$ is a fuzzy topological space obtained via fuzzy geometric logic.

Note: In [12], the way to construct a fuzzy topological space from a given fuzzy topological system is explained. Here we obtain the space $(X, ext(A/_\approx))$ from the system $(X, \models', A/_\approx)$ following the same method as in [12].

5. Fuzzy geometric logic with graded consequence

Here the alphabet, terms, formulae, interpretation and satisfiability of a formula are the same as in fuzzy geometric logic. The difference lies in the definition of satisfiability of a sequent. This notion is also graded now.

Definition 5.1. 1. $gr(s\ sat\ \phi \vdash \psi)$=$gr(s\ sat\ \phi) \rightarrow gr(s\ sat\ \psi)$, where $\rightarrow : [0,1] \times [0,1] \longrightarrow [0,1]$ is the Gödel arrow defined as follows:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b, \end{cases}$$

for $a, b \in [0,1]$.
2. $gr(\phi \vdash \psi) = inf_s\{gr(s\ sat\ \phi \vdash \psi)\}$, where s ranges over all sequences over the domain D of interpretation.

Before proceeding on to the rules of inference let us enlist below some properties of Gödel arrow [16] that would be used in the sequel.

5.1. Properties of Gödel arrow

In this subsection some required properties of Gödel arrow are listed. Most of the properties can be verified by routine check and hence the detailed verifications are omitted.

1. $a \rightarrow a = 1$, for any $a \in [0,1]$.
2. $(a \rightarrow b) \wedge (b \rightarrow c) \leq (a \rightarrow c)$, for any $a,\ b,\ c \in [0,1]$.
3. $a \leq b$ implies $(a \rightarrow x) \geq (b \rightarrow x)$, for any $a,\ b,\ x \in [0,1]$.
4. $a \leq b$ implies $(x \rightarrow a) \leq (x \rightarrow b)$, for any $a,\ b,\ x \in [0,1]$.
5. $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$, for any $a,\ b,\ c \in [0,1]$.
6. $inf_i\{(a_i \rightarrow b)\} = sup_i\{a_i\} \rightarrow b$, for any $a_i,\ b \in [0,1]$.
7. $a \leq b$ iff $a \rightarrow b = 1$.
8. $a \wedge (a \rightarrow b) \leq b$.

Theorem 5.2. *Graded sequents satisfy the following properties*

1. $gr(\phi \vdash \phi) = 1$,
2. $gr(\phi \vdash \psi) \wedge gr(\psi \vdash \chi) \leq gr(\phi \vdash \chi)$,
3. *(i)* $gr(\phi \vdash \top) = 1$, *(ii)* $gr(\phi \wedge \psi \vdash \phi) = 1$,
   (iii) $gr(\phi \wedge \psi \vdash \psi) = 1$, *(iv)* $gr(\phi \vdash \psi) \wedge gr(\phi \vdash \chi) = gr(\phi \vdash \psi \wedge \chi)$,
4. *(i)* $gr(\phi \vdash \bigvee S) = 1$ *if* $\phi \in S$,
   (ii) $inf_{\phi \in S}\{gr(\phi \vdash \psi)\} \leq gr(\bigvee S \vdash \psi)$,
5. $gr(\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}) = 1$,
6. $gr(\top \vdash (x = x)) = 1$,
7. $gr(((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)]) = 1$,
8. *(i)* $gr(\phi \vdash \psi[x/y]) \leq gr(\phi \vdash \exists y \psi)$, *(ii)* $gr(\exists y \phi \vdash \psi) \leq gr(\phi[x/y] \vdash \psi)$,
9. $gr(\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)) = 1$.

Proof.

1. $gr(s \text{ sat } \phi \vdash \phi) = gr(s \text{ sat } \phi) \to gr(s \text{ sat } \phi) = 1$, by Property 5.1(1).
2. $gr(s \text{ sat } \phi \vdash \psi) \wedge gr(s \text{ sat } \psi \vdash \chi)$
$= (gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)) \wedge (gr(s \text{ sat } \psi) \to gr(s \text{ sat } \chi))$
$\leq gr(s \text{ sat } \phi) \to gr(s \text{ sat } \chi)$ [by Property 5.1(2)]
$= gr(s \text{ sat } \phi \vdash \chi)$.
3. (i) $gr(s \text{ sat } \phi \vdash \top) = gr(s \text{ sat } \phi) \to gr(s \text{ sat } \top) = gr(s \text{ sat } \phi) \to 1 = 1$, as $gr(s \text{ sat } \phi) \leq 1$.
(ii) $gr(s \text{ sat } \phi \wedge \psi \vdash \phi) = gr(s \text{ sat } \phi \wedge \psi) \to gr(s \text{ sat } \phi) = (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi)) \to gr(s \text{ sat } \phi) = 1$, as $gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \phi)$.
(iii) $gr(s \text{ sat } \phi \wedge \psi \vdash \psi) = gr(s \text{ sat } \phi \wedge \psi) \to gr(s \text{ sat } \psi) = (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi)) \to gr(s \text{ sat } \psi) = 1$, as $gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) \leq gr(s \text{ sat } \psi)$.
(iv) $gr(s \text{ sat } \phi \vdash \psi) \wedge gr(s \text{ sat}\phi \vdash \chi)$
$= (gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)) \wedge (gr(s \text{ sat } \phi) \to gr(s \text{ sat } \chi))$
$= gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi \wedge \chi)$ [by Property 5.1(5)]
$= gr(s \text{ sat } \phi \vdash \psi \wedge \chi)$.
4. (i) $gr(s \text{ sat } \phi \vdash \bigvee S \ (\phi \in S)) = gr(s \text{ sat } \phi) \to gr(s \text{ sat } \bigvee S \ (\phi \in S)) = gr(s \text{ sat } \phi) \to sup_{\phi \in S}\{gr(s \text{ sat } \phi)\} = 1$, as $gr(s \text{ sat } \phi) \leq sup_{\phi \in S}\{gr(s \text{ sat } \phi)\}$.
(ii) $inf_{\phi \in S}\{gr(s \text{ sat } \phi \vdash \psi)\}$
$= inf_{\phi \in S}\{gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)\}$
$= sup_{\phi \in S}\{gr(s \text{ sat } \phi)\} \to gr(s \text{ sat } \psi)$ [by Property 5.1(6)]
$= gr(s \text{ sat } \bigvee S \ (\phi \in S)) \to gr(s \text{ sat } \psi)$
$= gr(s \text{ sat } \bigvee S \vdash \psi \ (\phi \in S))$.
5. $gr(s \text{ sat } \phi \wedge \bigvee S \vdash sup\{\phi \wedge \psi \mid \psi \in S\})$
$= gr(s \text{ sat } \phi \wedge \bigvee S) \to gr(s \text{ sat } \bigvee\{\phi \wedge \psi \mid \psi \in S\})$
$= (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S)) \to sup_{\psi \in S}\{gr(s \text{ sat } \phi \wedge \psi)\}$
$= (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S)) \to (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \bigvee S))$
$= 1$ [by Property 5.1(1)].
6. $gr(s \text{ sat } \top \vdash x = x) = gr(s \text{ sat } \top) \to gr(s \text{ sat } x = x) = 1$.
7. $gr(s \text{ sat } ((x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)) \wedge \phi)$
$= gr(s \text{ sat } ((x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n))) \wedge gr(s \text{ sat } \phi)$.
Now $gr(s \text{ sat } \phi[(y_1, y_2, \ldots, y_n)/(x_1, x_2, \ldots, x_n)])$
$= gr(s(s((y_1, y_2, \ldots, y_n))/(x_1, x_2, \ldots, x_n)) \text{ sat } \phi)$.
When $s((y_1, y_2, \ldots, y_n)) = s((x_1, x_2, \ldots, x_n))$
then $gr(s(s((y_1, y_2, \ldots, y_n))/(x_1, x_2, \ldots, x_n)) \text{ sat } \phi) = gr(s \text{ sat } \phi)$. So,
$gr(s \text{ sat } ((x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)) \wedge \phi)$
$\leq gr(s \text{ sat } \phi[(y_1, y_2, \ldots, y_n)/(x_1, x_2, \ldots, x_n)])$, for any s.
Hence $gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi)$
$\to gr(s \text{ sat } \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)]) = 1$. So,
$gr(s \text{ sat } ((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)]) = 1$.
8. (i) $gr(s \text{ sat } \phi \vdash \psi[x/y]) = gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi[x/y]) = gr(s \text{ sat } \phi) \to gr(s(s(x)/y) \text{ sat } \psi)$ and $gr(s \text{ sat } \phi \vdash \exists y\psi) = gr(s \text{ sat } \phi) \to gr(s \text{ sat } \exists y\psi)$
$= gr(s \text{ sat } \phi) \to sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}$.
Now $gr(s(s(x)/y) \text{ sat } \psi) \leq sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}$, as $s(x) \in D$. So, by Property 5.1(4)
$gr(s \text{ sat } \phi) \to gr(s(s(x)/y) \text{ sat } \psi)$
$\leq gr(s \text{ sat } \phi) \to sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}$ and consequently $gr(s \text{ sat } \phi \vdash \psi[x/y]) \leq gr(s \text{ sat } \phi \vdash \exists y\psi)$.
(ii) $gr(s \text{ sat } \exists y\phi \vdash \psi) = gr(s \text{ sat } \exists y\phi) \to gr(s \text{ sat } \psi)$
$= sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\} \to gr(s \text{ sat } \psi)$.
$gr(s \text{ sat } \phi[x/y] \vdash \psi) = gr(s \text{ sat } \phi[x/y]) \to gr(s \text{ sat } \psi)$
$= gr(s(s(x)/y) \text{ sat } \phi) \to gr(s \text{ sat } \psi)$.
Now $gr(s(s(x)/y) \text{ sat } \phi) \leq sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\}$, as $s(x) \in D$. So, by Property 5.1(3)
$sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\} \to gr(s \text{ sat } \psi) \leq gr(s(s(x)/y) \text{ sat } \phi) \to gr(s \text{ sat } \psi)$ and consequently $gr(s \text{ sat } \exists y\phi \vdash \psi) \leq gr(s \text{ sat } \phi[x/y] \vdash \psi)$.
9. $gr(s \text{ sat } \phi \wedge (\exists y)\psi \vdash \exists y(\phi \wedge \psi))$
$= gr(s \text{ sat } \phi \wedge \exists y\psi) \to gr(s \text{ sat } \exists y(\phi \wedge \psi))$
$= (gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \exists y\psi)) \to sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi \wedge \psi)\}$
$= (gr(s \text{ sat } \phi) \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}) \to (sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi) \wedge gr(s(d/y) \text{ sat } \psi)\})$
$= (gr(s \text{ sat } \phi) \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}) \to$
$(sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\} \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\})$.
Now $gr(s \text{ sat } \phi) \leq sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\}$.

So, $(gr(s \text{ sat } \phi) \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\})$
$\leq (sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\} \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\})$.
Hence $(gr(s \text{ sat } \phi) \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}) \rightarrow$
$(sup_{d\in D}\{gr(s(d/y) \text{ sat } \phi)\} \wedge sup_{d\in D}\{gr(s(d/y) \text{ sat } \psi)\}) = 1$ and consequently $gr(s \text{ sat } \phi \wedge (\exists y)\psi \vdash \exists y(\phi \wedge \psi)) = 1$. □

These properties may be considered as the counterparts of the rules of inference in the previous section. In fact in the derivation of a sequent from other sequents the same rules as in the previous section shall be employed but now the sequents are tagged with grades. Fuzzy geometric logic with graded consequence deals with the following questions:

1. If some sequents are given with grades then it can be investigated which sequents are derivable from them and they are sequents of what grade in the least.
2. From a set of graded sequents whether a given graded sequent is derivable or not by the rules of inference.

Fuzzy geometric logic with graded consequence is sound in the following senses. Let, as premise, a finite number of sequents $\phi_i \vdash \psi_i$ with grades a_i, $i = 1, 2, \ldots, n$ be taken. By Theorem 5.2, any sequent derived from this premise by rules will be of a grade at least equal to $min\{a_1, a_2, \ldots, a_n\}$. This is also a consequence of Corollary 5.3.4 [10]. This answers question (1). To answer question (2) we need to check whether the sequent considered is derivable by syntactic rules from the sequents of the premise and whether the grade of the sequent is greater than or equal to the minimum of the grades of the premise sequents. In particular, in every interpretation whenever the sequents of the premise are of grade 1, the conclusion sequent is also of grade 1.

Note: A graded sequent $\phi \vdash \psi$ may be read as ψ is a consequence of ϕ and there is a grade (or strength) of the consequence relation. A graded consequence relation is a fuzzy relation [31] between the power set of wffs and the set of wffs if and only if the following conditions are satisfied:

- if $\alpha \in \Gamma$ then $gr(\Gamma \vdash \alpha) = 1$ (overlap);
- If $\Gamma \subseteq \Delta$ then $gr(\Gamma \vdash \alpha) \leq gr(\Delta \vdash \alpha)$ (monotonicity);
- $inf_{\delta\in\Delta} gr(\Gamma \vdash \delta) * gr(\Delta \vdash \alpha) \leq gr(\Gamma \vdash \alpha)$ (cut);

where Γ, Δ are sets of wffs, α, δ are wffs, the grade of relatedness denoted by $gr(\Gamma \vdash \alpha)$ is an element of a residuated lattice and $*$ is the product operation relative to which the residuum is constructed [1,4]. In particular the residuated lattice can be taken as the unit interval $[0, 1]$ and the product as the minimum ($\wedge$). In the present work this particular case has been taken. It may be remarked that this particular product operation and its corresponding residuum have been used in Definition 5.1. Secondly, in this work the set Γ is always a singleton set. So overlap condition takes the form $gr(\alpha \vdash \alpha) = 1$ and cut condition becomes $gr(\beta \vdash \delta) \wedge gr(\delta \vdash \alpha) \leq gr(\beta \vdash \alpha)$. A special case of monotonicity condition is taken here in which the set $\{\beta, \delta\}$ to the left of $\vdash$ is equated with the single wff $\beta \wedge \delta$. So the condition becomes $gr(\beta \vdash \alpha) \leq gr(\beta \wedge \delta \vdash \alpha)$ for all α, β, δ. It can be shown that this inequality is derivable from the above rules of inference.

In fuzzy geometric logic with graded consequence a sequent with grade is derived, from a given set of sequents with grades.

6. Fuzzy topological space with graded inclusion, graded frame and graded fuzzy topological system

Definition 6.1. Let X be a set, τ be a collection of fuzzy subsets of X s.t.

1. $\tilde{\emptyset}$, $\tilde{X} \in \tau$, where $\tilde{\emptyset}(x) = 0$ and $\tilde{X}(x) = 1$, for all $x \in X$;
2. $\tilde{T}_i \in \tau$ for $i \in I$ imply $\bigcup_{i\in I} \tilde{T}_i \in \tau$, where $\bigcup_{i\in I} \tilde{T}_i(x) = sup_{i\in I}\{\tilde{T}_i(x)\}$;
3. $\tilde{T}_1$, $\tilde{T}_2 \in \tau$ imply $\tilde{T}_1 \cap \tilde{T}_2 \in \tau$, where $(\tilde{T}_1 \cap \tilde{T}_2)(x) = \tilde{T}_1(x) \wedge \tilde{T}_2(x)$,

and $\subseteq$ be a fuzzy inclusion relation for fuzzy sets defined as $gr(\tilde{T}_1 \subseteq \tilde{T}_2) = inf_{x\in X}\{\tilde{T}_1(x) \rightarrow \tilde{T}_2(x)\}$, where $\tilde{T}_1, \tilde{T}_2$ are fuzzy subsets of X and $\rightarrow$ is the Gödel arrow.

Then $(X, \tau, \subseteq)$ is called a **fuzzy topological space with graded inclusion**. $(\tau, \subseteq)$ is called a **fuzzy topology with graded inclusion** over X.

It is to be noted that we preferred to use the traditional notation $\tilde{A}$ to denote a fuzzy set [2]. We list the properties of the members of fuzzy topology with graded inclusion, as propositions, that would be used subsequently. By routine check all the propositions can be verified. Some of the proof of the propositions are provided here.

Proposition 6.1. $gr(\tilde{T} \subseteq \tilde{T}) = 1$.

Proof. $gr(\tilde{T} \subseteq \tilde{T}) = inf_x\{\tilde{T}(x) \rightarrow \tilde{T}(x)\} = 1$. □

Proposition 6.2. $gr(\tilde{T_1} \subseteq \tilde{T_2}) = 1 = gr(\tilde{T_2} \subseteq \tilde{T_1}) \Rightarrow \tilde{T_1} = \tilde{T_2}$.

Proposition 6.3. $gr(\tilde{T_1} \subseteq \tilde{T_2}) \wedge gr(\tilde{T_2} \subseteq \tilde{T_3}) \leq gr(\tilde{T_1} \subseteq \tilde{T_3})$.

Proposition 6.4. $gr(\tilde{T_1} \cap \tilde{T_2} \subseteq \tilde{T_1}) = 1 = gr(\tilde{T_1} \cap \tilde{T_2} \subseteq \tilde{T_2})$.

Proposition 6.5. $gr(\tilde{T} \subseteq \tilde{X}) = 1$.

Proof. $gr(\tilde{T} \subseteq \tilde{X}) = inf_x\{\tilde{T}(x) \to \tilde{X}(x)\} = inf_x\{\tilde{T}(x) \to 1\} = 1$. □

Proposition 6.6. $gr(\tilde{T_1} \subseteq \tilde{T_2}) \wedge gr(\tilde{T_1} \subseteq \tilde{T_3}) = gr(\tilde{T_1} \subseteq \tilde{T_2} \cap \tilde{T_3})$.

Proposition 6.7. $gr(\tilde{T_i} \subseteq \bigcup_i \tilde{T_i}) = 1$.

Proof. $gr(\tilde{T_i} \subseteq \bigcup_i \tilde{T_i}) = inf_x\{\tilde{T_i}(x) \to \bigvee_i\{\tilde{T_i}(x)\}\} = 1$. □

Proposition 6.8. $inf_{\tilde{T_i} \in \tilde{S}}\{gr(\tilde{T_i} \subseteq \tilde{T})\} = gr(\bigcup \tilde{S} \subseteq \tilde{T})$.

Proposition 6.9. $gr(\tilde{T} \cap \bigcup_i \tilde{T_i} \subseteq \bigcup_i (\tilde{T} \cap \tilde{T_i})) = 1$.

Proposition 6.10. $\tilde{T_1}(x) \wedge gr(\tilde{T_1} \subseteq \tilde{T_2}) \leq \tilde{T_2}(x)$, *for each x.*

Note 1: In [8] a notion called localic preordered topological space has been defined which is a 4-tuple (X, L, τ, P) where L is a frame, (X, L, τ) is an L-valued fuzzy topological space (localic topological space), P is an L-valued fuzzy preorder (reflexive and transitive) on X and $P(x, y) \wedge \tilde{T}(y) \leq \tilde{T}(x)$, for any $x,\ y \in X$ and $\tilde{T} \in \tau$. In our case, we have taken L as $[0, 1]$ and usual fuzzy topological space with the exception that the inclusion relation in τ is defined in terms of a fuzzy implication (Gödel). That is, the set X has no ordering here but the topology τ is endowed with a fuzzy ordering relation namely fuzzy inclusion.

Definition 6.2 *(Graded frame).* A **graded frame** is a 5-tuple $(A, \top, \wedge, \bigvee, R)$, where A is a non-empty set $\top \in A$, $\wedge$ is a binary operation, $\bigvee$ is an operation on arbitrary subset of A, R is a $[0, 1]$-valued fuzzy binary relation on A satisfying the following conditions:

1. $R(a, a) = 1$;
2. $R(a, b) = 1 = R(b, a) \Rightarrow a = b$;
3. $R(a, b) \wedge R(b, c) \leq R(a, c)$;
4. $R(a \wedge b, a) = 1 = R(a \wedge b, b)$;
5. $R(a, \top) = 1$;
6. $R(a, b) \wedge R(a, c) = R(a, b \wedge c)$;
7. $R(a, \bigvee S) = 1$ if $a \in S$;
8. $inf\{R(a, b) \mid a \in S\} = R(\bigvee S, b)$;
9. $R(a \wedge \bigvee S, \bigvee\{a \wedge b \mid b \in S\}) = 1$;

for any $a,\ b,\ c \in A$ and $S \subseteq A$. We will denote graded frame by (A, R).

Defining $a \prec b$ if and only if $R(a, b) = 1$, it can be checked that $(A, \prec)$ is a partial order relation with an upper bound $\top$ and with respect to this partial ordering, $a \wedge b$ is the greatest lower bound (g.l.b.) of two element set $\{a, b\}$, $\bigvee S$ is the least upper bound (l.u.b.) of S. Consequently $(A, \prec)$ forms a frame.

Proposition 6.11. $R(\bigvee \emptyset, a) = 1$, *for any* $a \in A$.

Proof. $R(\bigvee \emptyset, a) = inf\{R(b, a) \mid b \in \emptyset\} = inf\ \ \emptyset = 1$. □

It should be noted that $\bigvee \emptyset$ plays the role of the bottom element of A and hence can be denoted by the symbol $\perp$. Hence by Proposition 6.11 $R(\perp, a) = 1$ for any $a \in A$.

Proposition 6.12. $R(a, b) = R(a \wedge b, a) \wedge R(a, a \wedge b) = R(a \vee b, b) \wedge R(b, a \vee b)$.

Proof. Propositions 6.2(4), 6.2(6) and 6.2(1) give the following.
$R(a \wedge b, a) \wedge R(a, a \wedge b) = 1 \wedge R(a, a \wedge b) = R(a, a \wedge b) = R(a, a) \wedge R(a, b) = 1 \wedge R(a, b) = R(a, b)$.
Similarly using Propositions 6.2(7), 6.2(8) and 6.2(1) it can be shown that $R(a \vee b, b) \wedge R(b, a \vee b) = R(a, b)$. □

It is to note that as a special case of the above definition A with operations $\wedge$, $\bigvee$, and a relation $R : A \times A \longrightarrow \{0, 1\}$ satisfying the conditions 1–9 forms a frame where $a \leq b$ iff $R(a, b) = 1$.

Example 6.1. If $(X, \tau, \subseteq)$ is a fuzzy topological space with graded inclusion then $(\tau, \subseteq)$ with operations $\cap$, $\bigcup$ forms a graded frame as Propositions 6.1–6.9 hold. In this graded frame the top element is X and the bottom is $\emptyset$.

Note 2: A graded frame is a localic preordered set [8] with the value set $[0, 1]$ having some extra conditions namely 4–9 of Definition 6.2.

Definition 6.3. A **Graded fuzzy topological system** is a quadruple $(X, \models, A, R)$ consisting of a nonempty set X, a graded frame (A, R) and a fuzzy relation $\models$ from X to A such that

1. $gr(x \models a) \wedge R(a, b) \leq gr(x \models b)$;
2. for any finite subset including null set, S, of A, $gr(x \models \bigwedge S) =$ $inf\{gr(x \models a) \mid a \in S\}$;
3. for any subset S of A, $gr(x \models \bigvee S) = sup\{gr(x \models a) \mid a \in S\}$.

Note 3: The localic preordered topological system defined in [8] has a departure from the above notion in that the fuzzy order relation of localic preordered topological system is defined on the set X while in Definition 6.3 it is defined on the set A. That means in [8] the fuzzy preorder relation is defined on the object set X while in our case the fuzzy preorder is imposed on the set of properties A, additionally we need more conditions on the preorder of the property set to connect the fuzzy topological system with fuzzy geometric logic.

Theorem 6.4. *For any fuzzy topological space with graded inclusion $(X, \tau, \subseteq)$, $(X, \in, \tau, \subseteq)$ forms a graded fuzzy topological system, where $gr(x \in \tilde{T}) = \tilde{T}(x)$ for any $x \in X$ and $\tilde{T} \in \tau$.*

Proof. As $(X, \tau, \subseteq)$ is a fuzzy topological space with graded inclusion so X is a nonempty set, $(\tau, \subseteq)$ is a graded frame. Now $gr(x \in \tilde{T_1}) \wedge gr(\tilde{T_1} \subseteq \tilde{T_2}) \leq gr(x \in \tilde{T_2})$, as $gr(x \in \tilde{T}) = \tilde{T}(x)$ by Proposition 6.10. The rest of the conditions, viz. $gr(x \in \tilde{T_1} \cap \tilde{T_2}) = gr(x \in \tilde{T_1}) \wedge gr(x \in \tilde{T_2})$ and for any $S \subseteq \tau$, $gr(x \in \bigcup S) = sup\{gr(x \in \tilde{T_i}) \mid \tilde{T_i} \in \tau\}$, follow from the definition of fuzzy topological space with graded inclusion. Hence $(X, \in, \tau, \subseteq)$ is a graded fuzzy topological system. □

Definition 6.5. For any graded fuzzy topological system $(X, \models, A, R)$, the **extent** of each $a \in A$ denoted by $ext(a)$ is a mapping from X to $[0, 1]$ such that $ext(a)(x) = gr(x \models a)$, $x \in X$. Let us denote the set $\{ext(a)\}_{a \in A}$ by $ext(A)$.

Theorem 6.6. *For any graded fuzzy topological system $(X, \models, A, R)$, $(X, ext(A), \subseteq)$ forms a fuzzy topological space with graded inclusion, where $\subseteq$ is the fuzzy inclusion relation.*

Proof. To show that $(ext(A), \subseteq)$ forms a graded fuzzy topology on X let us proceed as follows.

1. For any $x \in X$, $ext(\perp)(x) = gr(x \models \perp) = gr(x \models \bigvee \emptyset) = sup\{gr(x \models a) \mid a \in \emptyset\} = sup\{\emptyset\} = 0$. Similarly $ext(\top)(x) = 1$, for any $x \in X$.
2. Taking $ext(a_i)$'s from $ext(A)$, we get $\bigcup_i ext(a_i)(x) = sup_i\{ext(a_i)(x)\} = sup_i\{gr(x \models a_i)\} = gr(x \models \bigvee\{a_i\}_i) = ext(\bigvee\{a_i\}_i)(x)$.
   Hence $\bigcup_i ext(a_i) = ext(\bigvee\{a_i\}_i) \in ext(A)$, as $\bigvee\{a_i\}_i \in A$ for a_i's $\in A$.
3. Similarly $ext(a_1), ext(a_2) \in ext(A)$ imply $ext(a_1) \cap ext(a_2) \in ext(A)$. □

Definition 6.7. Let $(X, \models, A, R)$ be a graded fuzzy topological system. Define $a \approx b$ iff $gr(x \models a) = gr(x \models b)$ for any $x \in X$ and $a,\ b \in A$.

Proposition 6.13. *$\approx$ is an equivalence relation.*

So we get the quotient $A/_{\approx}$. The operations $\wedge$, $\bigvee$ can be lifted in the following way so that they will become independent of choice:
$[a] \wedge [b] =^{def} [a \wedge b]$, $[a] \vee [b] =^{def} [a \vee b]$, $\bigvee\{[a_i]\}_i =^{def} [\bigvee\{a_i\}_i]$.

Definition 6.8. Let us define the relation $\models'$ between X and $A/_{\approx}$ in the following way : $gr(x \models' [a]) = gr(x \models a)$.

Proposition 6.14. $\forall x \in X,\ (gr(x \models' [a]) = gr(x \models' [b])) \Rightarrow [a] = [b]$.

Proof. $\forall x \in X,\ (gr(x \models' [a]) = gr(x \models' [b]))$ if and only if $\forall x \in X,\ (gr(x \models a) = gr(x \models b))$. Hence $a \approx b$, and so $[a] = [b]$. □

Theorem 6.9. *$(X, \models', A/_{\approx}, R)$ is a graded fuzzy topological system.*

Proof. X is a nonempty set.

Let us show that $(A/_{\approx}, R)$ be a graded frame where R is defined as $R([a],[b]) = inf_x\{(gr(x \models' [a]) \to gr(x \models' [b]))\}$, and $\to$ is the Gödel arrow.

1. $(gr(x \models' [a]) \to gr(x \models' [a])) = 1$ iff $R([a],[a]) = 1$.
2. Let $R([a],[b]) = 1 = R([b],[a])$ is given. Now $R([a],[b]) = 1$ if and only if $inf_x\{(gr(x \models' [a]) \to gr(x \models' [b]))\} = 1$. Hence $gr(x \models' [a]) \to gr(x \models' [b]) = 1$ for any x. Using Definition 6.8 $gr(x \models a) \to gr(x \models b) = 1$ for any x. So by Property 5.1(7) $gr(x \models a) \le gr(x \models b)$ for any x. Similarly $R([b],[a]) = 1$ if and only if $gr(x \models b) \le gr(x \models a)$ for any x. Hence $gr(x \models a) = gr(x \models b)$ for any x iff $a \approx b$ iff $[a] = [b]$.
3. $R([a],[b]) \wedge R([b],[c]) = inf_x\{gr(x \models' [a]) \to gr(x \models' [b])\} \wedge inf_x\{gr(x \models' [b]) \to gr(x \models' [c])\} = inf_x\{(gr(x \models' [a]) \to gr(x \models' [b])) \wedge (gr(x \models' [b]) \to gr(x \models' [c]))\} \le inf_x\{gr(x \models' [a]) \to gr(x \models' [c])\} = R([a],[c])$.
4. $R([a] \wedge [b],[a]) = inf_x\{gr(x \models' [a] \wedge [b]) \to gr(x \models' [a])\} = inf_x\{gr(x \models' [a \wedge b]) \to gr(x \models' [a])\} = inf_x\{gr(x \models a \wedge b) \to gr(x \models a)\} = inf_x\{(gr(x \models a) \wedge gr(x \models b)) \to gr(x \models a)\} = 1$. Similarly $R([a] \wedge [b],[b]) = 1$.
5. $R([a],[\top]) = inf_x\{gr(x \models' [a]) \to gr(x \models' [\top])\} = inf_x\{gr(x \models a) \to gr(x \models \top)\} = inf_x\{gr(x \models a) \to 1\} = 1$.
6. $R([a],[b]) \wedge R([a],[c]) = inf_x\{gr(x \models' [a]) \to gr(x \models' [b])\} \wedge inf_x\{gr(x \models' [a]) \to gr(x \models' [c])\} = inf_x\{(gr(x \models a) \to gr(x \models b)) \wedge (gr(x \models a) \to gr(x \models c))\} = inf_x\{gr(x \models a) \to (gr(x \models b) \wedge gr(x \models c))\} = inf_x\{gr(x \models' [a]) \to (gr(x \models' [b]) \wedge gr(x \models' [c]))\} = inf_x\{gr(x \models' [a]) \to (gr(x \models' [b] \wedge [c]))\} = inf_x\{gr(x \models' [a]) \to (gr(x \models' [b \wedge c]))\} = R([a],[b \wedge c]) = R([a], [b] \wedge [c])$.
7. Let $a \in S$ and $S \subseteq A$.
   $R([a], \bigvee[S]) = R([a], [\bigvee S]) = inf_x\{gr(x \models' [a]) \to gr(x \models' [\bigvee S])\} = inf_x\{gr(x \models a) \to gr(x \models \bigvee S)\} = 1$.
8. $inf_{a \in S}\{R([a],[b])\} = inf_{a \in S}\{inf_x\{gr(x \models' [a]) \to gr(x \models' [b])\}\} = inf_x\{sup_{a \in S}\{gr(x \models a)\} \to gr(x \models b)\} = inf_x\{gr(x \models \bigvee S) \to gr(x \models b)\} = inf_x\{gr(x \models' [\bigvee S]) \to gr(x \models' [b])\} = R(\bigvee[S],[b])$.
9. $R([a] \wedge \bigvee[S], \bigvee\{[a] \wedge [b]\}_{b \in S}) = R([a \wedge \bigvee S], [\bigvee\{a \wedge b\}_{b \in S}]) = inf_x\{gr(x \models' [a \wedge \bigvee S]) \to gr(x \models' [\bigvee\{a \wedge b\}_{b \in S}])\} = inf_x\{gr(x \models a \wedge \bigvee S) \to sup_{b \in S}\{gr(x \models a \wedge b\}\} = inf_x\{gr(x \models a \wedge \bigvee S) \to (gr(x \models a) \wedge sup_{b \in S}\{gr(x \models b)\})\} = inf_x\{gr(x \models a \wedge \bigvee S) \to gr(x \models a \wedge \bigvee S)\} = 1$.

It is now to show that the three conditions of Definition 6.3 hold.

1. $gr(x \models' [a]) \wedge R([a],[b]) = gr(x \models' [a]) \wedge inf_x\{gr(x \models' [a]) \to gr(x \models' [b])\}$
   $= inf_x\{gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b]))\}$. Now if $gr(x \models' [a]) \le gr(x \models' [b])$, then
   $gr(x \models' [a]) \to gr(x \models' [b]) = 1$.
   Hence in this case
   $gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b]))$
   $= gr(x \models' [a]) \wedge 1$
   $= gr(x \models' [a]) \le gr(x \models' [b])$.
   Now when $gr(x \models' [a]) > gr(x \models' [b])$, we get $gr(x \models' [a]) \to gr(x \models' [b]) = gr(x \models' [b])$.
   Hence in this case
   $gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b])) = gr(x \models' [a]) \wedge gr(x \models' [b]) = gr(x \models' [b])$.
   So for any x, $gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b])) \le gr(x \models' [b])$.
   Therefore $gr(x \models' [a]) \wedge R([a],[b]) \le gr(x \models' [b])$, as $inf_x\{gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b]))\} \le gr(x \models' [a]) \wedge (gr(x \models' [a]) \to gr(x \models' [b]))$.
2. $gr(x \models' \bigwedge[S]) = gr(x \models' [\bigwedge S]) = gr(x \models \bigwedge S) = inf_{a \in S}\{gr(x \models a)\} = inf_{a \in S}\{gr(x \models' [a])\}$.
3. $gr(x \models' \bigvee[S]) = gr(x \models' [\bigvee S]) = gr(x \models \bigvee S) = sup_{a \in S}\{gr(x \models a)\} = sup_{a \in S}\{gr(x \models' [a])\}$. □

7. Interrelations

7.1. Fuzzy geometric logic with graded consequence to graded fuzzy topological system

Let us consider the quadruple $(X, \models, A, R)$ where X be a non-empty set of assignments s, A be the set of geometric formulae, $\models$ defined as $gr(s \models \phi) = gr(s \text{ sat } \phi)$ and $R(\phi, \psi) = gr(\phi \vdash \psi) = inf_s\{gr(s \text{ sat } \phi \vdash \psi)\}$.

Theorem 7.1. *(i)* $gr(s \models \phi) \wedge R(\phi, \psi) \le gr(s \models \psi)$;

(ii) $gr(s \models \phi \wedge \psi) = gr(s \models \phi) \wedge gr(s \models \psi)$;
(iii) $gr(s \models \bigvee\{\phi_i\}_{i\in I}) = sup_{i\in I}\{gr(x \models \phi_i)\}$.

Proof. (i) For any s, $gr(s \models \phi) \wedge R(\phi, \psi) = gr(s \text{ sat } \phi) \wedge gr(\phi \vdash \psi) = gr(s \text{ sat } \phi) \wedge inf_s\{gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)\} = inf_s\{gr(s \text{ sat } \phi) \wedge (gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi))\} \leq gr(s \text{ sat } \phi) \wedge (gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)) \leq gr(s \text{ sat } \psi) = gr(s \models \psi)$.

(ii) $gr(s \models \phi \wedge \psi) = gr(s \text{ sat } \phi \wedge \psi) = gr(s \text{ sat } \phi) \wedge gr(s \text{ sat } \psi) = gr(s \models \phi) \wedge gr(s \models \psi)$.

(iii) $gr(s \models \bigvee\{\phi_i\}_{i\in I}) = gr(s \text{ sat } \bigvee\{\phi_i\}_{i\in I}) = sup_{i\in I}\{gr(s \text{ sat } \phi_i)\} = sup_{i\in I}\{gr(s \models \phi_i)\}$. □

It is to note that (A, R) is not a graded frame since R does not satisfy antisymmetry viz. condition 2, Definition 6.2 and hence $(X, \models, A, R)$ is not a graded fuzzy topological system. Let us construct a graded fuzzy topological system using $(X, \models, A, R)$ in the following way.

Definition 7.2. $\phi \approx \psi$ iff $gr(s \models \phi) = gr(s \models \psi)$ for any $s \in X$ and $\phi,\ \psi \in A$.

It can be shown that the above defined "≈" is an equivalence relation. Thus we get $A/_{\approx}$.

Theorem 7.3. *$(X, \models', A/_{\approx}, R)$ is a graded fuzzy topological system, where $\models'$ is defined by $gr(s \models' [\phi]) = gr(s \models \phi)$ and $R([\phi], [\psi]) = inf_s\{gr(s \models' [\phi]) \to gr(s \models' [\psi])\}$.*

Proof. X is a non-empty set of assignments s. Using the fact $R([\phi], [\psi]) = inf_s\{gr(s \models' [\phi]) \to gr(s \models' [\psi])\} = inf_s\{gr(s \models \phi) \to gr(s \models \psi)\} = inf_s\{gr(s \text{ sat } \phi) \to gr(s \text{ sat } \psi)\} = inf_s\{gr(s \text{ sat } \phi \vdash \psi)\} = gr(\phi \vdash \psi)$, it can be shown that $(A/_{\approx}, R)$ is a graded frame (by routine check).

Now it is left to show that (a) $gr(s \models' [\phi]) \wedge R([\phi], [\psi]) \leq gr(s \models' [\psi])$, (b) $gr(s \models' [\phi] \wedge [\psi]) = gr(s \models' [\phi]) \wedge gr(s \models' [\psi])$ and (c) $gr(s \models' \bigvee\{[\phi_i]\}_{i\in I}) = sup_{i\in I}\{gr(s \models' [\phi_i])\}$.

Proof of the above statement follows easily using Theorem 7.1. Hence $(X, \models', A/_{\approx}, R)$ is a graded fuzzy topological system. □

Proposition 7.1. *If $(X, \models', A/_{\approx}, R)$ is a graded fuzzy topological system defined as above then for all $s \in X$, $(gr(s \models' [\phi]) = gr(s \models' [\psi])) \Rightarrow ([\phi] = [\psi])$.*

Proof. $gr(s \models' [\phi]) = gr(s \models' [\psi])$, for any s if and only if $gr(s \models \phi) = gr(s \models \psi)$, for any s. Thereby $\phi \approx \psi$. So, $[\phi] = [\psi]$. □

7.2. Fuzzy topological space with graded inclusion and graded fuzzy topological system

Using Theorem 6.6 a fuzzy topological space with graded inclusion, $(X, ext(A/_{\approx}), \subseteq)$, can be obtained from the graded fuzzy topological system $(X, \models', A/_{\approx}, R)$.

Using Theorem 6.4 it can be shown that if $(X, ext(A/_{\approx}), \subseteq)$ is a fuzzy topological space with graded inclusion then, $(X, \in, ext(A/_{\approx}), \subseteq)$ forms a graded fuzzy topological system.

7.3. Fuzzy topological space with graded inclusion to fuzzy geometric logic with graded consequence

Let $(X, \tau, \subseteq)$ be a fuzzy topological space with graded inclusion. For each $\tilde{T}_i \in \tau$ let $P_{\tilde{T}_i}$ be a propositional variable and let the following axioms for sequents $P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}$ be assumed. These axioms are taken keeping in view the intended interpretation.

1. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}) = gr(\tilde{T}_i \subseteq \tilde{T}_j)$;
2. $gr(P_{\tilde{T}_i} \wedge P_{\tilde{T}_j} \vdash P_{\tilde{T}_i \cap \tilde{T}_j}) = 1 = gr(P_{\tilde{T}_i \cap \tilde{T}_j} \vdash P_{\tilde{T}_i} \wedge P_{\tilde{T}_j})$, for any $\tilde{T}_i, \tilde{T}_j \in \tau$;
3. $gr(P_{\bigcup_i \tilde{T}_i} \vdash \bigvee\{P_{\tilde{T}_i}\}_i) = 1 = gr(\bigvee\{P_{\tilde{T}_i}\}_i \vdash P_{\bigcup_i \tilde{T}_i})$, for any $\tilde{T}_i$'s$\in \tau$.

We adopt binary meet ($\wedge$) and arbitrary join ($\bigvee$) to form other formulae.

Definition 7.4. Let for wffs α, β the relation $\equiv$ be defined by $\alpha \equiv \beta$ if and only if $gr(\alpha \vdash \beta) = 1 = gr(\beta \vdash \alpha)$.

From (2) and (3) it is clear that for each formula α there exists a propositional variable $P_{\tilde{T}_\alpha}$ such that $\alpha \equiv P_{\tilde{T}_\alpha}$.

We now extend the definition of assignment of grades to an arbitrary sequent by $gr(\alpha \vdash \beta) = gr(P_{\tilde{T}_\alpha} \vdash P_{\tilde{T}_\beta}) = gr(\tilde{T}_\alpha \subseteq \tilde{T}_\beta)$. Because of the above remark this extension is justified.

The following lemmas may be established.

Lemma 7.1. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_i}) = 1$.

Proof. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_i}) = gr(\tilde{T}_i \subseteq \tilde{T}_i) = 1$, using Proposition 6.1. □

Lemma 7.2. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}) \wedge gr(P_{\tilde{T}_j} \vdash P_{\tilde{T}_k}) \leq gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_k})$.

Proof. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}) \wedge gr(P_{\tilde{T}_j} \vdash P_{\tilde{T}_k}) = gr(\tilde{T}_i \subseteq \tilde{T}_j) \wedge gr(\tilde{T}_j \subseteq \tilde{T}_k) \leq gr(\tilde{T}_i \subseteq \tilde{T}_k) = gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_k})$, using Proposition 6.3. □

Lemma 7.3. $gr(P_{\tilde{T}_i} \vdash \tilde{X}) = 1$.

Proof. $gr(P_{\tilde{T}_i} \vdash \tilde{X}) = gr(\tilde{T}_i \subseteq \tilde{X}) = 1$, using Proposition 6.5. □

Lemma 7.4. $gr(P_{\tilde{T}_i} \wedge P_{\tilde{T}_j} \vdash P_{\tilde{T}_i}) = 1$.

Proof. $gr(P_{\tilde{T}_i} \wedge P_{\tilde{T}_j} \vdash P_{\tilde{T}_i}) = gr(P_{\tilde{T}_i \cap \tilde{T}_j} \vdash P_{\tilde{T}_i}) = gr(\tilde{T}_i \cap \tilde{T}_j \subseteq \tilde{T}_i) = 1$, using Definition 7.4 and Proposition 6.4. □

Lemma 7.5. $gr(P_{\tilde{T}_i} \wedge P_{\tilde{T}_j} \vdash P_{\tilde{T}_j}) = 1$.

Proof. $gr(P_{\tilde{T}_i} \wedge P_{\tilde{T}_j} \vdash P_{\tilde{T}_j}) = gr(P_{\tilde{T}_i \cap \tilde{T}_j} \vdash P_{\tilde{T}_j}) = gr(\tilde{T}_i \cap \tilde{T}_j \subseteq \tilde{T}_j) = 1$, using Definition 7.4 and Proposition 6.4. □

Lemma 7.6. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}) \wedge gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_k}) = gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j} \wedge P_{\tilde{T}_k})$.

Proof. $gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j}) \wedge gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_k}) = gr(\tilde{T}_i \subseteq \tilde{T}_j) \wedge gr(\tilde{T}_i \subseteq \tilde{T}_k) = gr(\tilde{T}_i \subseteq \tilde{T}_j \cap \tilde{T}_k) = gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j \cap \tilde{T}_k}) = gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}_j} \wedge P_{\tilde{T}_k})$, using Proposition 6.6 and Definition 7.4. □

Lemma 7.7. $gr(P_{\tilde{T}_i} \vdash \bigvee\{P_{\tilde{T}_i}\}_i) = 1$.

Proof. $gr(P_{\tilde{T}_i} \vdash \bigvee\{P_{\tilde{T}_i}\}_i) = gr(P_{\tilde{T}_i} \vdash P_{\bigcup_i \tilde{T}_i}) = gr(\tilde{T}_i \subseteq \bigcup_i \tilde{T}_i) = 1$, using Definition 7.4 and Proposition 6.7. □

Lemma 7.8. $inf_i\{gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}})\} \leq gr(\bigvee\{P_{\tilde{T}_i}\}_i \vdash P_{\tilde{T}})$.

Proof. $inf_i\{gr(P_{\tilde{T}_i} \vdash P_{\tilde{T}})\} = inf_i\{gr(\tilde{T}_i \subseteq \tilde{T})\} \leq gr(\bigcup_i \tilde{T}_i \subseteq \tilde{T}) = gr(P_{\bigcup_i \tilde{T}_i} \vdash P_{\tilde{T}}) = gr(\bigvee\{P_{\tilde{T}_i}\}_i \vdash P_{\tilde{T}})$, using Proposition 6.8 and Definition 7.4. □

Lemma 7.9. $gr(P_{\tilde{T}} \wedge \bigvee\{P_{\tilde{T}_i}\}_i \vdash \bigvee\{P_{\tilde{T}} \wedge P_{\tilde{T}_i}\}_i) = 1$.

Proof. $gr(P_{\tilde{T}} \wedge \bigvee\{P_{\tilde{T}_i}\}_i \vdash \bigvee\{P_{\tilde{T}} \wedge P_{\tilde{T}_i}\}_i) = gr(P_{\tilde{T} \cap \bigcup_i \tilde{T}_i} \vdash P_{\bigcup_i (\tilde{T} \cap \tilde{T}_i)}) = gr(\tilde{T} \cap \bigcup_i \tilde{T}_i \subseteq \bigcup_i (\tilde{T} \cap \tilde{T}_i)) = 1$, using Definition 7.4 and Proposition 6.9. □

With these lemmas and the extended definition of grade assignment to sequents of formulae the following theorem is obtained.

Theorem 7.5. *For any fuzzy topological space with graded inclusion $(X, \tau, \subseteq)$, the logic defined as above forms a propositional fuzzy geometric logic with graded consequence.*

Subsections 7.1, 7.2 and 7.3 show that there exists a closed relationship between the three notions.

8. Concluding remarks

The logic that is developed here is not many-sorted. It is clear that the present formalism can be extended to include multiple sorts. An important issue, however, is to place things into categorical setup. It is well known that the research-culture in geometric logic is deeply involved with category theory. In the opinion of Vickers, of this field “the full mathematical insights come only through category theory” [26]. The categorical recast of the notions included in this paper has already been developed by us which will be presented in our subsequent papers.

In summary, the chief contribution here lies in the definition and treatment of the notions fuzzy geometric logic, fuzzy geometric logic with graded consequence, graded fuzzy topological system, fuzzy topology with graded inclusion and graded frame. It seems that the last two notions may be interesting in themselves. The study of fuzzy topology may have an additional dimension by the incorporation of graded inclusion of fuzzy sets instead of usual crisp inclusion as is in vogue. The way in which graded frame has been defined may be used to define graded counterparts of other algebraic structures. The consequence is yet to be seen.

It may be marked that Gödel arrow has been used while defining graded inclusion of fuzzy sets as well as fuzzy geometric logic with graded consequence. This fuzzy implication operator has been essential at some steps of proof. In our future papers we shall endeavor to get out of this restriction which is a bit unsatisfactory and see how much could be achieved by other important implications.

We have not raised the point of completeness of fuzzy geometric logic and fuzzy geometric logic with graded consequence. In both cases the logics would be incomplete since their restrictions to the crisp case, that is taking the values either 0 or 1, reduce to ordinary geometric logic and it is incomplete [28]. But it may be an interesting question that under which sufficient and necessary conditions the following holds

$$gr(\phi \vdash \psi) = inf_s\{gr(s\ sat\ \phi) \rightarrow gr(s\ sat\ \psi)\},\ \text{if}\ 0 < gr(\phi \vdash \psi) < 1.$$

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