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Effect of fractional parameter on plane waves of generalized magneto–thermoelastic diffusion with reference temperature-dependent elastic medium

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ABSTRACT

The present paper is concerned with the investigation of disturbances in a homogeneous, isotropic reference temperature-dependent elastic medium with fractional order generalized thermoelastic diffusion. The formulation is applied to the generalized thermoelasticity based on the fractional time derivatives under the effect of diffusion. The analytical expressions for displacement components, stresses, temperature field, concentration and chemical potential are obtained in the physical domain by using the normal mode analysis technique. These expressions are calculated numerically for a copper-like material and depicted graphically. Effect of fractional parameter and presence of diffusion is analyzed theoretically and numerically. Comparisons are made with the results predicted by the fractional and without fractional order in the presence and absence of diffusion.

1. Introduction

Recently, some interesting models have been proposed successfully by applying the fractional calculus to study the physical processes, particularly in the area of mechanics of solids, control theory, electricity, heat conduction, diffusion problems and viscoelasticity etc. It has been verified/examined that the use of fractional order derivatives/integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There are some materials (e.g., porous materials, man-made and biological materials/polymers and colloids, glassy etc.) and physical situations (like low-temperature, amorphous media and transient loading etc.) where the CTE theory based on the classical Fourier's law is unsuitable. In such cases, it is better to use a generalized thermoelasticity (and more generally thermo–viscoelasticity) theory based on an anomalous heat conduction theory involving *fractional time-derivatives* (see [1]). Abel is the first author, who applied fractional calculus to obtain the solution of an integral equation arising in the formulation of the *tautochrone problem*. After Abel's study, great attention has been devoted to the major study of fractional calculus by Liouville. Fractional order derivatives have been employed for the description of viscoelastic materials by Caputo and Mainardi [2,3] and Caputo [4] and they have established the connection between fractional derivatives and the linear theory of viscoelasticity. They also obtained a very good agreement with the experimental results successfully. In [5,6], one can find many applications of fractional calculus to various problems of mechanics of solids. A considerable research effort has been extended to study anomalous diffusion that is characterized by the time-fractional diffusion wave equation introduced by Kimmich [7].

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During recent years, fractional calculus has also been introduced in the field of thermoelasticity. Povstenko [8] has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional order time derivative. He used the Caputo fractional derivative [9] and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. Povstenko [10] also studied factional Cattaneo-type equations and generalized thermoelasticity. Ezzat and Fayik [11] constructed a model in generalized thermoelastic diffusion by using *fractional time-derivatives*.

Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibrium is reached. It occurs as a result of the second law of thermodynamics which states that the entropy or disorder of any system must always increase with time. Diffusion is important in many life processes. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce dopants in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction. The phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermoelastic diffusion for more efficient extraction of oil from oil deposits. The thermodiffusion process also helps the investigation in the field associated with the advent of semiconductor devices and the advancement of microelectronics.

Thermodiffusion in the solids is one of the transport processes that has great practical importance. Most of the research associated with the presence of concentration and temperature gradients has been made with metals and alloys. The first critical review was published in the work of Oriani [12]. With the advancement of nuclear energetics, the interest in thermodiffusion has returned to metallic oxides that often heats up in an inhomogeneous temperature field [13] in connection with technological conditions.

Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion and that of strain. Heat and mass exchange with the environment during the process of thermodiffusion in an elastic solid. The concept of thermodiffusion is used to describe the processes of thermomechanical treatment of metals (carbonizing, nitriding steel, etc.) and these processes are thermally activated, their diffusing substances being, e.g., nitrogen, carbon etc. They are accompanied by deformations of the solid. Nowacki [14–16] and Podstrigach [17] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Sherief et al. [18] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. The reflection phenomena of *P* and SV waves from the free surface of an elastic solid with thermodiffusion was considered by Singh [19]. Sherief and Saleh [20] worked on a problem of a thermoelastic half-space with a permeating substance in contact with the bounding plane in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Recently, Othman et al. [21] studied the effect of diffusion on the two-dimensional problem of generalized thermoelasticity with Green and Naghdi theory. Owing to the mathematical difficulties encountered in two-dimensional multi-field coupled generalized heat conduction problems, the problems become too complicated to obtain an analytical solution. Instead of analytical methods, several authors applied numerical techniques such as finite difference method, finite element method, boundary value method etc., for solving such kind of problems. In recent years, normal mode analysis method has been applied to study various problem of generalized thermoelasticity [21–24].

The present study is motivated by the importance of thermoelastic diffusion process in the field of oil extraction. The theory of thermodiffusion is also applied in the description of thermo-mechanical treatment of porous media of sintered powder metals. Thermodiffusion methods have been successfully applied in the last few years in improving the mechanical properties of product made of powder metals.

The present paper is concerned with the investigation of disturbances in a homogeneous, isotropic temperature-dependent elastic medium with fractional order generalized thermodiffusion. The formulation is applied to the generalized thermoelasticity based on the fractional time derivatives under the effect of diffusion. The analytical expressions for displacement components, stresses, temperature field, concentration and chemical potential are obtained in the physical domain by using the normal mode analysis techniques. These expressions are calculated numerically for a copper-like material and depicted graphically. Effect of fractional parameter and presence of diffusion is analyzed theoretically and numerically. Comparisons are made with the results predicted by the fractional and without fractional order in the presence and absence of diffusion.

2. Basic equations and formulation of the problem

Let us consider an isotropic, homogeneous, thermally and perfectly conducting elastic medium with temperature-dependent modulus of elasticity. We consider an orthogonal Cartesian coordinate system *oxyz* having originated on the surface $z = 0$ and *oz* being a line drawn vertically downwards. The medium is subjected to a initial magnetic field $\mathbf{H} = (0, 0, H_0)$ which is parallel to *y*-axis. Maxwell's equations for homogeneous isotropic perfectly conducting material (Strictly speaking, when the material is subjected to magnetic fields and thermal field, the material will not remain homogeneous and isotropic; this variation is ignored in this investigation) are given by [25]

$$\nabla \wedge \mathbf{h} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

$$\nabla \wedge \mathbf{E} = -\mu_e \frac{\partial \mathbf{h}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{h} = 0, \tag{3}$$

$$\mathbf{E} = -\mu_e \frac{\partial \mathbf{u}}{\partial t} \wedge \mathbf{H} \tag{4}$$

where $\mathbf{E}, \mathbf{h}, \mathbf{J}, \mu_e, \sigma, T, t$ and ε are the electric intensity, the magnetic intensity, the current density vector, the magnetic permeability, the electric conductivity, the absolute temperature, time and the electric permeability respectively.

Following [11], the governing equations for an isotropic, homogeneous temperature-dependent elastic solid with generalized thermodiffusion at uniform temperature T_0 in the undisturbed state, in the absence of external body forces and heat sources are:

(1) the equation of motion

$$\rho \ddot{u}_i = \sigma_{ij,j} + F_i, \tag{5}$$

where ρ is the density, $\mathbf{u}$ is the displacement vector, σ_{ij} are the components of the stress tensor and F_i is the Lorentz force given by

$$\mathbf{F} = (F_x, F_y, F_z) = \mu_e (\mathbf{J} \wedge \mathbf{H})_i. \tag{6}$$

(2) The strain–displacement relation

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{7}$$

(3) the constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + [\lambda e_{kk} - \nu(T - T_0) - \beta C]\,\delta_{ij}, \tag{8}$$

$$P = -\beta e_{kk} + bC - a(T - T_0), \tag{9}$$

where e_{ij} are the components of strain tensor, T is the absolute temperature, C is the concentration of the diffusive material in the elastic medium, λ, μ are Lame's constant, ν and β are the material constants given by

$$\nu = (3\lambda + 2\mu)\alpha_t \quad \text{and} \quad \beta = (3\lambda + 2\mu)\alpha_c,$$

α_t is the coefficient of linear thermal expansion, α_c is the coefficient of linear diffusion expansion, P is the chemical potential, a is the measure of thermodiffusion effect and b is the measure of diffusive effect.

(4) the energy equation with fractional order time derivatives [11]

$$K\nabla^2 T = \frac{\partial}{\partial t}\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E T + \nu T_0 e + a T_0 C), \quad 0 < \alpha \le 1, \tag{10}$$

where K is the thermal conductivity, c_E is the specific heat at constant strain, T_0 is the temperature of the medium in its natural state assumed to be such that $\left|\frac{T-T_0}{T_0}\right| \ll 1$, e is the cubical dilatation given by $e = \nabla \cdot \mathbf{u}$, τ_0 is the thermal relaxation time and

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = \begin{cases} f(x,t) - f(x,0) & \text{when } \alpha \to 0, \\ I^{1-\alpha}\dfrac{\partial f(x,t)}{\partial t} & \text{when } 0 < \alpha < 1, \\ \dfrac{\partial f(x,t)}{\partial t} & \text{when} \alpha = 1. \end{cases}$$

In the above definition, the *Riemann–Liouville* fractional integral operator I^α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{11}$$

where $\Gamma(\alpha)$ is the well-known Gamma function. When $\alpha = 0.5$, we say that it is *weak conductivity* and when $\alpha = 1.0$, we say that it is *normal conductivity*, see [26–28] for details.

(5) the generalized diffusion equation

$$d\beta e_{kk,ii} + daT_{,ii} + \frac{\partial}{\partial t}\left(1 + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right) C - dbC_{,ii} = 0, \quad 0 < \alpha \le 1, \tag{12}$$

where d is the diffusion coefficient and τ is the *diffusion relaxation time*. Also note that, in the above equations, a *comma followed by a suffix* denotes material derivative and a *superposed dot* denotes the derivative with respect to time t.

We consider, all quantities are functions of the coordinates x, z and t. The displacement components thus have the following form

$$u_x = u(x, z, t), \qquad u_y = 0, \qquad u_z = w(x, z, t). \tag{13}$$

Now, we assume that

$$\lambda = \lambda_0(1 - \alpha^* T_0), \qquad \mu = \mu_0(1 - \alpha^* T_0), \qquad \nu = \nu_0(1 - \alpha^* T_0), \qquad \beta = \beta_0(1 - \alpha^* T_0), \tag{14}$$

where λ_0, μ_0, ν_0 and β_0 are constants and α^* is the linear temperature coefficient. In the case of the modulus elasticity is temperature independent, $\alpha^* = 0$.

By using Eqs. (1)–(4) and (6), we get

$$\mathbf{E} = \mu_e H_0(\dot{w}, 0, -\dot{u}), \qquad \mathbf{h} = -H_0(0, e, 0), \tag{15}$$

$$\mathbf{F} = -\mu_e H_0 \left[\left(\frac{\partial h}{\partial x} + \mu_e \varepsilon \ddot{u} \right),\ 0,\ \left(\frac{\partial h}{\partial z} + \mu_e \varepsilon \ddot{w} \right) \right]. \tag{16}$$

By substituting from Eq. (14) in Eqs. (8) and (9), we obtain

$$\alpha_1 \sigma_{xx} = (\lambda_0 + 2\mu_0) e_{xx} + \lambda_0 e_{zz} - \nu_0 (T - T_0) - \beta_0 C, \tag{17}$$

$$\alpha_1 \sigma_{zz} = (\lambda_0 + 2\mu_0) e_{zz} + \lambda_0 e_{xx} - \nu_0 (T - T_0) - \beta_0 C, \tag{18}$$

$$\alpha_1 \sigma_{xz} = 2\mu_0 e_{xz}, \tag{19}$$

$$P = -\frac{\beta_0}{\alpha_1} e_{kk} + bC - a(T - T_0), \tag{20}$$

where

$$\alpha_1 = \frac{1}{1 - \alpha^* T_0}. \tag{21}$$

By using Eqs. (15)–(19) in Eq. (5), we get

$$\alpha_1 \rho \ddot{u} = (\lambda_0 + \mu_0) e_{,x} + \mu_0 \nabla^2 u - \nu_0 T_{,x} - \beta_0 C_{,x} + \alpha_1 \mu_e H_0^2 \left(e_{,x} - \mu_e \varepsilon \ddot{u} \right), \tag{22}$$

$$\alpha_1 \rho \ddot{w} = (\lambda_0 + \mu_0) e_{,z} + \mu_0 \nabla^2 w - \nu_0 T_{,z} - \beta_0 C_{,z} + \alpha_1 \mu_e H_0^2 \left(e_{,z} - \mu_e \varepsilon \ddot{w} \right). \tag{23}$$

By substituting from Eq. (14) in Eqs. (10) and (12), we obtain

$$K \nabla^2 T = \frac{\partial}{\partial t} \left(1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left(\rho C_E T + \frac{\nu_0 T_0}{\alpha_1} e + a T_0 C \right). \tag{24}$$

$$\frac{d\beta_0}{\alpha_1} \nabla^2 e + da \nabla^2 \theta + \frac{\partial}{\partial t} \left(1 + \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) C - db \nabla^2 C = 0. \tag{25}$$

To transform the above equations in non-dimensional forms, we will use the following non-dimensional variables

$$(x', z') = \frac{\widetilde{\omega}}{c_1}(x, z), \qquad (u', w') = \frac{\widetilde{\omega}}{c_1}(u, w), \qquad (t', \tau_0', \tau') = \widetilde{\omega}(t, \tau_0, \tau),$$

$$\sigma_{ij}' = \frac{\sigma_{ij}}{\rho c_1^2}, \qquad h' = \frac{h}{H_0}, \qquad C' = \frac{\beta_0}{\rho c_1^2} C, \qquad \theta = \frac{\nu_0 (T - T_0)}{\rho c_1^2}, \qquad P' = \frac{P}{\beta_0},$$

where $\widetilde{\omega} = \frac{\rho C_E c_1^2}{K}$ and $c_1^2 = \frac{\lambda_0 + 2\mu_0}{\rho}$.

Using these non-dimensional variables, equations take the following form (omitting the primes for convenience)

$$\ddot{u} = \frac{1}{\alpha_1} \left[\beta_1 e_{,x} + (1 - \beta_1) \nabla^2 u - \theta_{,x} - C_{,x} \right] + A_0 e_{,x} - B_0 \ddot{u}, \tag{26}$$

$$\ddot{w} = \frac{1}{\alpha_1} \left[\beta_1 e_{,z} + (1 - \beta_1) \nabla^2 w - \theta_{,z} - C_{,z} \right] + A_0 e_{,z} - B_0 \ddot{w}, \tag{27}$$

$$\nabla^2 \theta = \frac{\partial}{\partial t} \left(1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left(\theta + \frac{\delta \delta_0}{\alpha_1} e + a_1 \delta_0 C \right). \tag{28}$$

$$\nabla^2 e + \alpha_1 \alpha_2 \nabla^2 \theta + \alpha_1 \alpha_3 \frac{\partial}{\partial t} \left(1 + \frac{\tau^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) C - \alpha_1 \alpha_4 \nabla^2 C = 0. \tag{29}$$

$$\sigma_{xx} = \frac{1}{\alpha_1} \left[u_{,x} + (2\beta_1 - 1) w_{,z} - \theta - C \right], \tag{30}$$

$$\sigma_{zz} = \frac{1}{\alpha_1}\left[w_{,z} + (2\beta_1 - 1)u_{,x} - \theta - C\right], \tag{31}$$

$$\sigma_{xz} = \frac{1-\beta_1}{\alpha_1}\left[u_{,z} + w_{,x}\right], \tag{32}$$

$$P = -\frac{1}{\alpha_1}\left(u_{,x} + w_{,z}\right) + \alpha_4 C - \alpha_2\theta, \tag{33}$$

where

$$A_0 = \frac{\mu_e H_0^2}{\rho c_1^2}, \qquad B_0 = \frac{\varepsilon\mu_e^2 H_0^2}{\rho}, \qquad \delta = \frac{\nu_0}{\rho C_E}, \qquad \delta_0 = \frac{\nu_0 T_0}{\rho c_1^2}, \qquad a_1 = \frac{a c_1^2}{\beta_0 C_E},$$

$$\alpha_2 = \frac{a\rho c_1^2}{\beta_0\nu_0}, \qquad \alpha_3 = \frac{K c_1^2}{d\beta_0^2 C_E}, \qquad \alpha_4 = \frac{b\rho c_1^2}{\beta_0^2}, \qquad \beta_1 = \frac{(\lambda_0 + \mu_0)}{\rho c_1^2}.$$

Introducing the potential functions $\phi(x, z, t)$ and $\psi(x, z, t)$ defined by the relations in the non-dimensional form:

$$u = (\phi_{,x} - \psi_{,z}), \qquad w = (\phi_{,z} + \psi_{,x}). \tag{34}$$

By substituting Eq. (34) in Eqs. (26)–(29), we get

$$\left[\gamma_1\nabla^2 - \alpha_1\gamma_2\frac{\partial^2}{\partial t^2}\right]\phi = \theta + C, \tag{35}$$

$$\left[(1-\beta_1)\nabla^2 - \alpha_1\gamma_2\frac{\partial^2}{\partial t^2}\right]\psi = 0, \tag{36}$$

$$\left[\nabla^2 - \frac{\partial}{\partial t}\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\right]\theta = \delta_0\frac{\partial}{\partial t}\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\frac{\delta}{\alpha_1}\nabla^2\phi + a_1 C\right), \tag{37}$$

$$\nabla^4\phi + \alpha_1\alpha_2\nabla^2\theta + \left[\alpha_1\alpha_3\frac{\partial}{\partial t}\left(1 + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right) - \alpha_1\alpha_4\nabla^2\right]C = 0, \tag{38}$$

where $\gamma_1 = 1 + \alpha_1 A_0$ and $\gamma_2 = 1 + B_0$.

3. Normal mode analysis

The solution of the physical quantities can be decomposed in terms of *normal modes* in the following form:

$$\left[C, u, w, e, \phi, \psi, \theta, \sigma_{ij}\right](x, z, t) = \left[C^*, u^*, w^*, e^*, \phi^*, \psi^*\theta^*, \sigma_{ij}^*\right](z)\exp(\omega t + imx), \tag{39}$$

where $C^*(z)$ etc. is the amplitude of the function $C(x, z, t)$ etc., i is the imaginary unit, ω (complex) is the frequency and m is the wave number in the x-direction.

Using Eq. (39), then Eqs. (35)–(38) take the following forms

$$\gamma_1\left[D^2 - g_1\right]\phi^*(z) - \theta^*(z) - C^*(z) = 0, \tag{40}$$

$$\left[(1-\beta_1)\left(D^2 - m^2\right) - \alpha_1\gamma_2\omega^2\right]\psi^*(z) = 0, \tag{41}$$

$$\alpha_1\gamma_1\left[D^2 - g_3\right]\theta^*(z) - g_2\phi^*(z) - g_4 C^*(z) = 0, \tag{42}$$

$$g_8\left[D^2 - g_9\right]C^*(z) - g_5\phi^*(z) - g_6\theta^*(z) = 0, \tag{43}$$

where

$$g_1 = \left[m^2 + \frac{\alpha_1\gamma_2\omega^2}{\gamma_1}\right],$$

$$g_2 = \left[\alpha_1\gamma_2\delta\delta_0\omega^2\omega_1\right],$$

$$g_3 = \left[m^2 + \omega_1 + \frac{\delta\delta_0\omega_1}{\alpha_1\gamma_1}\right],$$

$$g_4 = \left[\delta_0\omega_1(\delta + \alpha_1\gamma_1 a_1)\right],$$

$$g_5 = \left[\alpha_1^2\gamma_2^2\omega^4 + \frac{g_2(1 + \alpha_1\alpha_2\gamma_1)}{\alpha_1}\right],$$

$$g_6 = \left[\gamma_1(1 + \alpha_1\alpha_2\gamma_1)(g_3 - m^2) + \alpha_1\gamma_2\omega^2\right],$$

$$g_7 = \left[\frac{g_4(1+\alpha_1\alpha_2\gamma_1)}{\alpha_1} + \alpha_1\gamma_2\omega^2 + \alpha_1\alpha_3\gamma_1^2\omega_1\right],$$

$$g_8 = [\gamma_1(1-\alpha_1\alpha_4\gamma_1)], \qquad g_9 = \left[m^2 - \frac{g_7}{g_8}\right]$$

and

$$D \equiv \frac{d}{dz}, \qquad \omega_1 = \left[\omega + \frac{\omega^{\alpha+1}\tau_0^\alpha}{\alpha!}\right].$$

Eliminating $\theta^*(z)$ and $C^*(z)$ between Eqs. (40), (42) and (43), we get after some simple computations the following sixth-order ordinary differential equation satisfied by $\phi^*(z)$

$$\left(D^6 - l_1D^4 + l_2D^2 - l_3\right)\phi^*(z) = 0, \tag{44}$$

$$l_1 = \frac{g_{11}}{g_{10}}, \qquad l_2 = \frac{g_{12}}{g_{10}}, \qquad l_3 = \frac{g_{13}}{g_{10}},$$

$$g_{10} = \alpha_1\gamma_1^2 g_8, \qquad g_{11} = \alpha_1\gamma_1^2 g_8\left[g_1 + g_3 + g_9\right],$$

$$g_{12} = \left[\alpha_1\gamma_1^2 g_8(g_1g_3 + g_3g_9 + g_9g_1) - \gamma_1g_4g_6 - \alpha_1\gamma_1g_5 - g_2g_8\right],$$

$$g_{13} = \left[\alpha_1\gamma_1^2 g_1g_3g_8g_9 - \alpha_1\gamma_1g_3g_5 - \gamma_1g_1g_4g_6 + g_4g_5 + g_2g_6 - g_2g_8g_9\right].$$

In a similar manner, we can show that $\theta^*(z)$ and $C^*(z)$ satisfy the following equations

$$\left(D^6 - l_1D^4 + l_2D^2 - l_3\right)\{\theta^*(z), C^*(z)\} = 0. \tag{45}$$

The general solution of Eq. (44) which is regular at $z \to \infty$ can be written as

$$\phi^*(z) = \sum_{j=1}^{3} R_j(m,\omega)e^{-k_jz}, \tag{46}$$

where k_j $(j = 1, 2, 3)$ are the eigenvalues (roots) of the following characteristics equations

$$k^6 - l_1k^4 + l_2k^2 - l_3 = 0, \tag{47}$$

given by

$$k_1^2 = \frac{1}{3}(2p\sin q + l_1),$$

$$k_2^2 = \frac{-1}{3}\left(p[\sqrt{3}\cos q + \sin q] - l_1\right),$$

$$k_3^2 = \frac{1}{3}\left(p[\sqrt{3}\cos q - \sin q] + l_1\right),$$

and

$$p = \sqrt{l_1^2 - 3l_2}, \qquad q = \frac{\sin^{-1} r}{3}, \qquad r = \frac{9l_1l_2 - 2l_1^3 - 27l_3}{2p^3}.$$

Following the same process, we obtain the solution for $\theta^*(z)$ and $C^*(z)$ as follows

$$\theta^*(z) = \sum_{j=1}^{3} R'_j(m,\omega)e^{-k_jz}, \tag{48}$$

$$C^*(z) = \sum_{j=1}^{3} R''_j(m,\omega)e^{-k_jz}, \tag{49}$$

where $R_j(m,\omega)$, $R'_j(m,\omega)$ and $R''_j(m,\omega)$ are arbitrary constants, depending on m and ω to be determined by the boundary conditions of the problem. Substituting from Eqs. (46), (48) and (49) into the Eqs. (40), (42) and (43), we can easily obtain

$$R'_j(m,\omega) = M_{1j}R_j(m,\omega), \quad j = 1, 2, 3 \tag{50}$$

$$R''_j(m,\omega) = M_{2j}R_j(m,\omega), \quad j = 1, 2, 3. \tag{51}$$

We thus have

$$\theta^*(z) = \sum_{j=1}^{3} M_{1j}R_j(m,\omega)e^{-k_jz}, \tag{52}$$

$$C^*(z) = \sum_{j=1}^{3} M_{2j}R_j(m,\omega)e^{-k_jz}, \tag{53}$$

where

$$M_{1j} = \frac{\gamma_1 g_4(k_j^2 - g_1) + g_2}{\alpha_1\gamma_1(k_j^2 - g_3) + g_4}, \tag{54}$$

$$M_{2j} = \frac{\alpha_1\gamma_1^2(k_j^2 - g_1)(k_j^2 - g_3) - g_2}{\alpha_1\gamma_1(k_j^2 - g_3) + g_4}. \tag{55}$$

The solution of Eq. (41) can be written as

$$\psi^*(z) = R_4(m, \omega)e^{-k_4 z}, \tag{56}$$

where

$$k_4 = \sqrt{m^2 + \frac{\alpha_1\gamma_2\omega^2}{1-\beta_1}}.$$

In order to obtain the displacement components u and w, using Eq. (39), Eq. (34), becomes

$$u^*(z) = im\phi^*(z) - D\psi^*(z), \tag{57}$$

$$w^*(z) = D\phi^*(z) + im\psi^*(z) \tag{58}$$

which give on using Eqs. (46) and (56)

$$u^*(z) = k_4 R_4(m, \omega)e^{-k_4 z} + im\sum_{j=1}^{3} R_j(m, \omega)e^{-k_j z}, \tag{59}$$

$$w^*(z) = imR_4(m, \omega)e^{-k_4 z} - \sum_{j=1}^{3} k_j R_j(m, \omega)e^{-k_j z}. \tag{60}$$

Substitution of Eqs. (39), (53), (59) and (60) into Eqs. (30)–(33), we get

$$\sigma_{xx}^*(z) = M_1 R_4(m, \omega)e^{-k_4 z} + \sum_{j=1}^{3} M_{3j} R_j(m, \omega)e^{-k_j z}, \tag{61}$$

$$\sigma_{zz}^*(z) = -M_1 R_4(m, \omega)e^{-k_4 z} + \sum_{j=1}^{3} M_{4j} R_j(m, \omega)e^{-k_j z}, \tag{62}$$

$$\sigma_{xz}^*(z) = -M_2 R_4(m, \omega)e^{-k_4 z} - \sum_{j=1}^{3} M_{5j} R_j(m, \omega)e^{-k_j z}, \tag{63}$$

$$P^*(z) = \sum_{j=1}^{3} M_{6j} R_j(m, \omega)e^{-k_j z} \tag{64}$$

where

$$\begin{aligned}
M_1 &= \frac{2imk_4(1-\beta_1)}{\alpha_1},\\
M_2 &= \frac{(1-\beta_1)(k_4^2 + m^2)}{\alpha_1},\\
M_{3j} &= \frac{1}{\alpha_1}\left[(2\beta_1 - 1)k_j^2 - m^2 - M_{1j} - M_{2j}\right],\\
M_{4j} &= \frac{1}{\alpha_1}\left[k_j^2 - (2\beta_1 - 1)m^2 - M_{1j} - M_{2j}\right],\\
M_{5j} &= \frac{2im(1-\beta_1)k_j}{\alpha_1},\\
M_{6j} &= \frac{1}{\alpha_1}\left[m^2 - k_j^2 - \alpha_1\alpha_2 M_{1j} + \alpha_1\alpha_4 M_{2j}\right],
\end{aligned} \tag{65}$$

4. Application

The non-dimensional boundary conditions on the surface $z = 0$ are:
(1) the concentrated load is suddenly applied normal to the free surface:

$$\sigma_{zz} = -F_0 \exp(\omega t + imx), \tag{66}$$

where F_0 is the normal load of intensity per unit length,
(2) the tangential stress component must be vanishing:

$$\sigma_{xz} = 0, \tag{67}$$

(3) there is no variation of concentration and temperature on the surface $z = 0$,

$$\frac{\partial C}{\partial z} = 0, \tag{68}$$

$$\frac{\partial \theta}{\partial z} = 0. \tag{69}$$

Substituting the expressions of the variables considered into the above boundary conditions, we can obtain the following equations satisfied by the parameters R_j $(j = 1, 2, 3, 4)$

$$\begin{aligned}
&\sum_{j=1}^{3} M_{4j} R_j - M_1 R_4 = -F_0, \\
&\sum_{j=1}^{3} M_{5j} R_j - M_2 R_4 = 0, \\
&\sum_{j=1}^{3} M_{2j} k_j R_j = 0, \\
&\sum_{j=1}^{3} M_{1j} k_j R_j = 0.
\end{aligned} \tag{70}$$

Solving the system of Eq. (70), we get the parameters R_j $(j = 1, 2, 3, 4)$, defined as follows:

$$R_j = \frac{\Delta_j}{\Delta}, \quad (j = 1, 2, 3, 4), \tag{71}$$

where

$$\begin{aligned}
\Delta_1 &= F_0 k_2 k_3 M_2 \left(M_{13} M_{22} - M_{12} M_{23}\right), \\
\Delta_2 &= F_0 k_1 k_3 M_2 \left(M_{11} M_{23} - M_{13} M_{21}\right), \\
\Delta_3 &= F_0 k_1 k_2 M_2 \left(M_{12} M_{21} - M_{11} M_{22}\right), \\
\Delta_4 &= F_0 k_1 k_2 M_{53} \left(M_{12} M_{21} - M_{11} M_{22}\right) + F_0 k_2 k_3 M_{51} \left(M_{13} M_{22} - M_{12} M_{23}\right) \\
&\quad + F_0 k_1 k_3 M_{52} \left(M_{11} M_{23} - M_{13} M_{21}\right), \\
\Delta &= k_1 k_3 M_2 M_{42} \left(M_{13} M_{21} - M_{11} M_{23}\right) + k_1 k_3 M_1 M_{52} \left(M_{11} M_{23} - M_{13} M_{21}\right) \\
&\quad + k_2 k_3 M_2 M_{41} \left(M_{12} M_{23} - M_{13} M_{22}\right) + k_2 k_3 M_1 M_{51} \left(M_{13} M_{22} - M_{12} M_{23}\right) \\
&\quad + k_1 k_2 M_2 M_{43} \left(M_{11} M_{22} - M_{12} M_{21}\right) + k_1 k_2 M_1 M_{53} \left(M_{12} M_{21} - M_{11} M_{22}\right).
\end{aligned} \tag{72}$$

5. Particular case

By putting $C = 0,\ a = 0,\ b = 0,\ \beta = 0$, we get the equations for the displacements component, the stresses and the temperature without the effect of diffusion. In this case, we obtain:

$$\gamma_1 \left[D^2 - g_1\right] \phi^*(z) - \theta^*(z) = 0, \tag{73}$$

$$\left(D^2 - k_4^2\right) \psi^*(z) = 0, \tag{74}$$

$$\alpha_1 \gamma_1 \left[D^2 - g_3\right] \theta^*(z) - g_2 \phi^*(z) = 0, \tag{75}$$

where

$$k_4 = \sqrt{m^2 + \frac{\alpha_1 \gamma_2 \omega^2}{1 - \beta_1}}.$$

Eliminating $\phi^*(z)$ and $\theta^*(z)$ between Eqs. (80) and (82), we get fourth-order ordinary differential equation satisfied with $\phi^*(z)$ and $\theta^*(z)$ given by

$$\left(D^4 - g_{14}D^2 + g_{15}\right)\{\phi^*(z), \theta^*(z)\} = 0, \tag{76}$$

$$g_{14} = (g_1 + g_3), \qquad g_{15} = \left[g_1 g_3 - \frac{g_2}{\alpha_1 \gamma_1^2}\right].$$

The solution of Eq. (83) is given by

$$\phi^*(z) = \sum_{j=1}^{2} S_j(m, \omega) e^{-\lambda_j z}, \tag{77}$$

where λ_j $(j = 1, 2)$ are the eigenvalues (roots) of the following characteristic equations

$$\lambda^4 - g_{14}\lambda^2 + g_{15} = 0. \tag{78}$$

Similarly

$$\theta^*(z) = \sum_{j=1}^{2} S'_j(m, \omega) e^{-\lambda_j z}, \tag{79}$$

where $S_j(m, \omega)$ and $S'_j(m, \omega)$, are parameters depending on m and ω.

Substituting Eqs. (84) and (86) into Eqs. (80)–(82), we get

$$S'_j(m, \omega) = N_{1j} S_j(m, \omega), \quad j = 1, 2. \tag{80}$$

We thus have

$$\theta^*(z) = \sum_{j=1}^{2} N_{1j} S_j(m, \omega) e^{-\lambda_j z}, \tag{81}$$

where

$$N_{1j} = \left[\gamma_1(\lambda_j^2 - m^2)\right]. \tag{82}$$

The solution of Eq. (86) is the same in Eq. (56) and

$$u^*(z) = k_4 R_4(m, \omega) e^{-k_4 z} + im \sum_{j=1}^{2} S_j(m, \omega) e^{-\lambda_j z}, \tag{83}$$

$$w^*(z) = imR_4(m, \omega) e^{-k_4 z} - \sum_{j=1}^{2} \lambda_j S_j(m, \omega) e^{-\lambda_j z}. \tag{84}$$

$$\sigma^*_{xx}(z) = M_1 R_4(m, \omega) e^{-k_4 z} + \sum_{j=1}^{2} N_{2j} S_j(m, \omega) e^{-\lambda_j z}, \tag{85}$$

$$\sigma^*_{zz}(z) = -M_1 R_4(m, \omega) e^{-k_4 z} + \sum_{j=1}^{2} N_{3j} S_j(m, \omega) e^{-\lambda_j z}, \tag{86}$$

$$\sigma^*_{xz}(z) = -M_2 R_4(m, \omega) e^{-k_4 z} - \sum_{j=1}^{2} N_{4j} S_j(m, \omega) e^{-\lambda_j z}, \tag{87}$$

$$P^*(z) = \sum_{j=1}^{2} N_{5j} S_j(m, \omega) e^{-\lambda_j z}, \tag{88}$$

where

$$N_{2j} = \frac{\left[(2\beta_1 - 1)\lambda_j^2 - m^2 - N_{1j}\right]}{\alpha_1},$$
$$N_{3j} = \frac{\left[\lambda_j^2 - (2\beta_1 - 1)m^2 - N_{1j}\right]}{\alpha_1},$$
$$N_{4j} = \frac{2im(1-\beta_1)\lambda_j}{\alpha_1}, \tag{89}$$
$$N_{5j} = \frac{\left(m^2 - \lambda_j^2\right)}{\alpha_1}, \quad j = 1, 2.$$

In this case, the non-dimensional boundary conditions on the surface $z = 0$ are:

$$\sigma_{zz} = -F_0 \exp(\omega t + imx), \qquad \sigma_{xz} = 0, \qquad \frac{\partial \theta}{\partial z} = 0 \quad \text{on } z = 0. \tag{90}$$

Substituting the expressions of the variables considered into the above boundary conditions, we can obtain the following equations satisfied by the parameters S_j $(j = 1, 2)$ and R_4

$$\sum_{j=1}^{2} N_{3j} S_j - M_1 R_4 = -F_0,$$
$$\sum_{j=1}^{2} N_{4j} S_j - M_2 R_4 = 0, \tag{91}$$
$$\sum_{j=1}^{2} N_{1j} \lambda_j S_j = 0.$$

Solving the above system of Eq. (91), we get the parameters S_j $(j = 1, 2)$ and R_4 in the following forms respectively:

$$S_j = \frac{\Delta_j^*}{\Delta^*}, \quad j = 1, 2,$$
$$R_4 = \frac{\Delta_3^*}{\Delta^*}, \tag{92}$$

where

$$\Delta_1^* = -F_0 \lambda_2 M_2 N_{12},$$
$$\Delta_2^* = F_0 \lambda_1 M_2 N_{11},$$
$$\Delta_3^* = F_0(\lambda_1 N_{11} N_{42} - \lambda_2 N_{12} N_{41}), \tag{93}$$
$$\Delta^* = \lambda_1 N_{11}(M_1 N_{42} - M_2 N_{32}) + \lambda_2 N_{12}(M_2 N_{31} - M_1 N_{41}).$$

6. Numerical results and discussions

The copper material was chosen for the purpose of numerical example. Since, we have $\omega = \omega_0 + i\zeta$, where i is the imaginary unit, $e^{\omega t} = e^{\omega_0 t}(\cos \zeta t + i \sin \zeta t)$ and for small values of time, we can take $\omega = \omega_0$ (real).

The numerical constants of the problems were taken as (in SI unit):

$\lambda_0 = 0.5 \times 10^{11}$, $\mu_0 = 3.86 \times 10^{11}$, $\beta_0 = 0.1$, $\nu_0 = 0.3 \times 10^{-2}$, $\alpha_t = 1.78 \times 10^5$,
$\alpha_c = 1.98 \times 10^{-4}$, $a = 1.2 \times 10^4$, $b = 0.9 \times 10^6$, $d = 0.85 \times 10^{-8}$, $\rho = 8954$, $\mu_e = 0.02$,
$\varepsilon = 0.1$, $T_0 = 293$, $C_E = 383.1$, $\tau_0 = 0.03$, $\alpha^* = 0.002$, $\omega_0 = 2.5$,
$m = 1.8$, $F_0 = 0.1$.

In the present work, numerical calculations are carried out in three different cases. In the first case, we are investigating how the non-dimensional displacement components, the temperature, the stress components and the chemical potential vary with different values of the fractional parameter α against z in the presence and absence of diffusion when the time and the magnetic field remain constant. In the second case, we will show, how the non-dimensional displacement components, the temperature, the stress components and the chemical potential vary with different values of α_1 against z when the time instant and the magnetic field remain constant. The third case is investigating how the non-dimensional displacement components, the temperature, the stress components and the chemical potential vary with the presence and absence of the

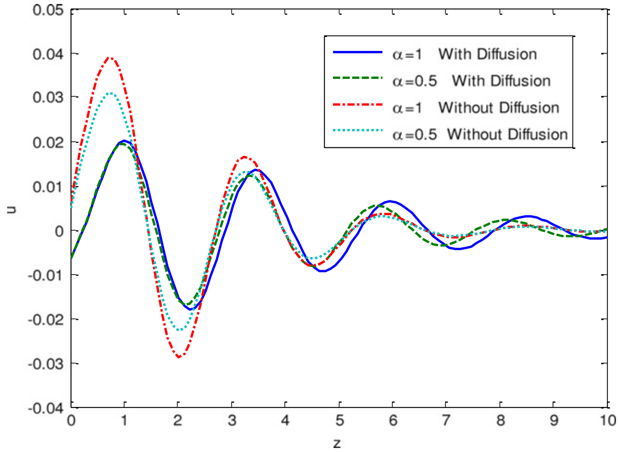

Fig. 1. Variation of displacement distribution u at $H_0 = 10^4$, $\alpha_1 = 1.6$.

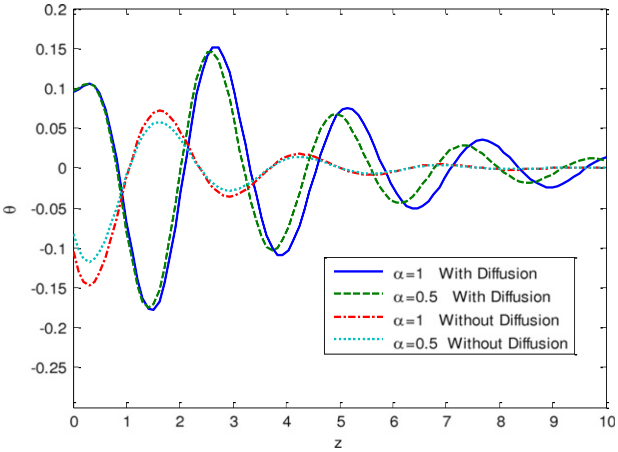

Fig. 2. Variation of temperature distribution θ at $H_0 = 10^4$, $\alpha_1 = 1.6$.

magnetic field. The computations are carried out at $x = 3.5$ for the time instant $t = 0.1$ in the range $0 \leq z \leq 10$. The numerical results of the real parts of all the physical quantities are obtained and presented graphically in Figs. 1–12 of the above three different cases.

Figs. 1–4 depict the variety of the displacement component (u), the temperature (θ), the stress component (σ_{zz}) and the chemical potential (P) for two different values of the fractional parameter (α), namely for $\alpha = 1.0$ and $\alpha = 0.5$ in the presence and absence of the diffusion effect. Fig. 1 shows that for all the cases, u remains close to the zero value in the considered domain of the distance z far from the origin, except near the vicinity of the load where slight variations are noticed. It is also clearly depicted from Fig. 1 that, the value of u is maximum in the thermoelastic medium without diffusion effect for $\alpha = 1.0$. Figs. 2 and 4 clearly show that the range of magnitude of the temperature (θ) and the chemical potential (P) is greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Fig. 5 shows that, the value of the stress σ_{zz} is maximum in the thermoelastic medium with diffusion effect for $\alpha = 1.0$.

Figs. 5–8 exhibit the variations of the displacement component (u), the temperature (θ), the stress component (σ_{zz}) and the chemical potential (P) for two different values of the magnetic field H_0, namely for $H_0 = 0.0$ (without magnetic effect) and $H_0 = 10^4$ (with magnetic effect) in the presence and absence of diffusion. From Fig. 5, we can notice that, as the value of z increases, the values of the displacement function for all the cases approach to zero value, except near the origin where significant difference can be noticed in the magnitude of u for all the considered cases. It is also clearly depicted that, the value of u is maximum in the thermoelastic medium without diffusion effect in the absence of the magnetic field. We noticed from Figs. 6 and 8 that, the range of the magnitudes of θ and P are greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Also these values approach rapidly to the zero value with distance z in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Fig. 7 depicts that the value of the stresses σ_{zz} is maximum in the thermoelastic medium with diffusion effect for $H_0 = 0.0$ and as the distance z increases, the values of the stress function approach to zero, rapidly in the case of the presence of a diffusion effect than in the case of the absence of a diffusion effect.

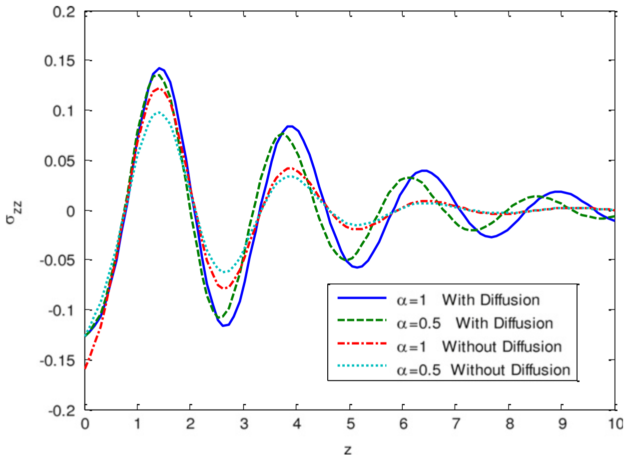

Fig. 3. Stress distribution σ_{zz} at $H_0 = 10^4$, $\alpha_1 = 1.6$.

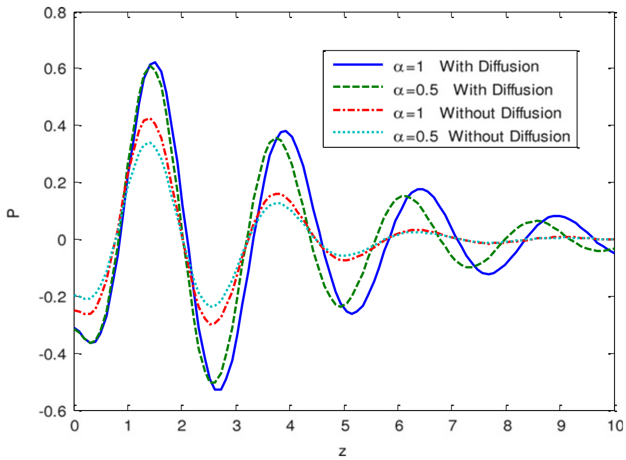

Fig. 4. Variation of chemical potential P at $H_0 = 10^4$, $\alpha_1 = 1.6$.

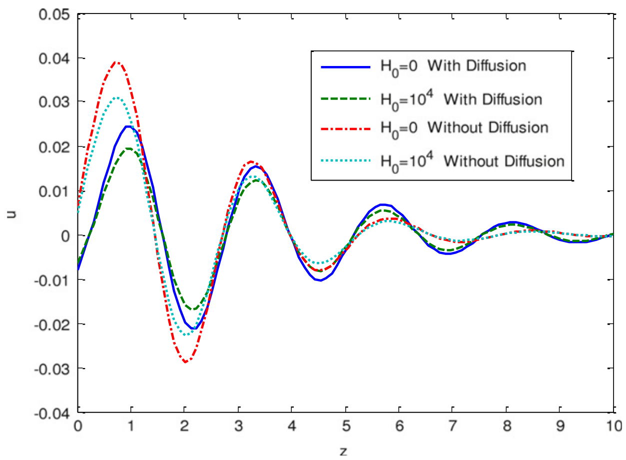

Fig. 5. Variation of displacement distribution u at $\alpha_1 = 1.6$, $\alpha = 0.5$.

Figs. 9–12 display the distribution of the displacement function (u), the temperature (θ), the stress function (σ_{zz}) and the chemical potential (P) for two different values of the parameter α_1, namely for $\alpha_1 = 1.0$ (temperature independent modulus of elasticity) and $\alpha_1 = 1.6$ (temperature dependent modulus of elasticity) in the presence and absence of the diffusion effect. Fig. 9 exhibits that, the value of u is maximum in the thermoelastic medium without diffusion effect when the modulus of

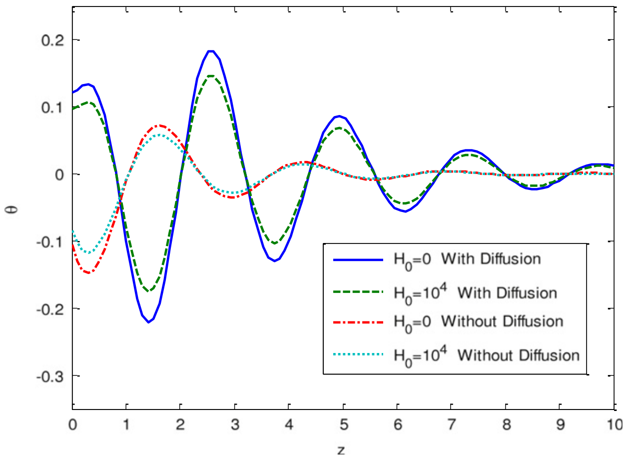

Fig. 6. Variation of temperature distribution θ at $\alpha_1 = 1.6$, $\alpha = 0.5$.

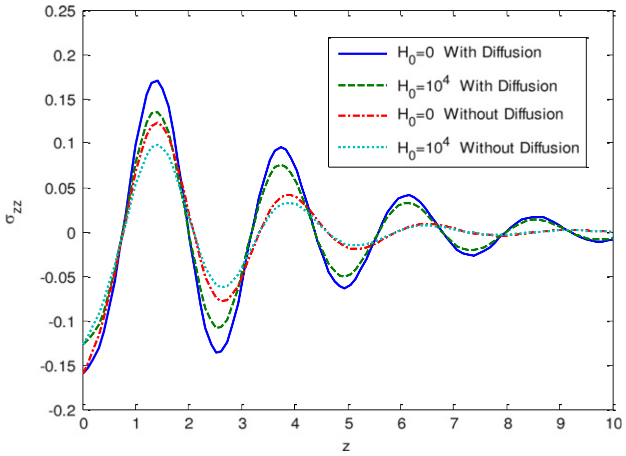

Fig. 7. Stress distribution σ_{zz} at $\alpha_1 = 1.6$, $\alpha = 0.5$.

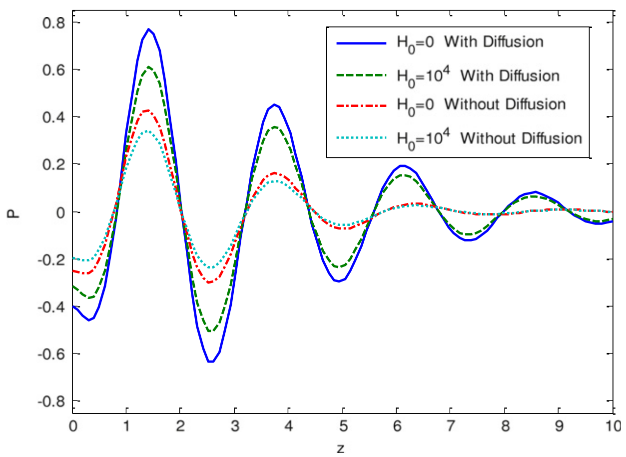

Fig. 8. Variation of chemical potential P at $\alpha_1 = 1.6$, $\alpha = 0.5$.

elasticity is temperature dependent. The values of the displacement function u vanishes after $z > 10$ (approximately). Figs. 10 and 12 show that, the range of magnitudes of θ and P are greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. It is also clearly depicted from Figs. 10 and 12 that the values of θ and P are maximum in the thermoelastic medium with diffusion effect when the modulus of elasticity is temperature dependent.

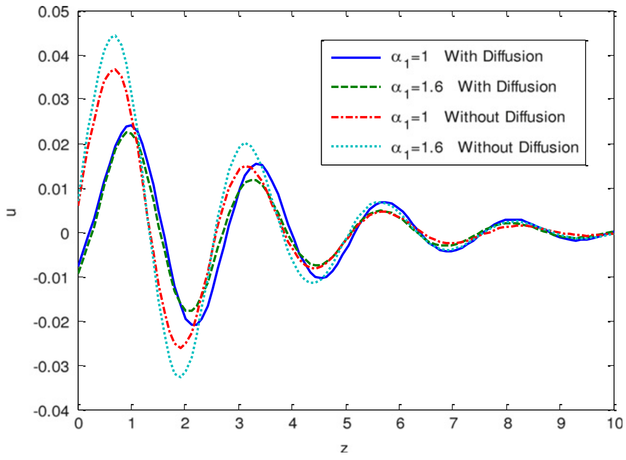

Fig. 9. Variation of displacement distribution u at $H_0 = 10^4$, $\alpha = 0.5$.

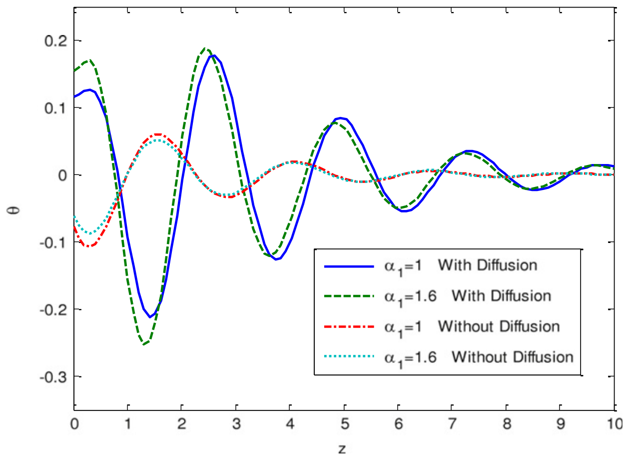

Fig. 10. Variation of temperature distribution θ at $H_0 = 10^4$, $\alpha = 0.5$.

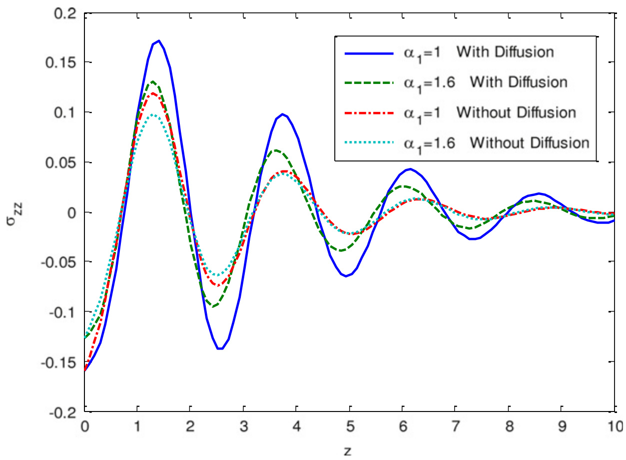

Fig. 11. Stress distribution σ_{zz} at $H_0 = 10^4$, $\alpha = 0.5$.

Also these values approach to the zero value rapidly with distance z in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect. Fig. 11 displays that, the value of the stresses σ_{zz} is maximum in the thermoelastic medium with diffusion effect for when the modulus of elasticity of the medium is temperature independent.

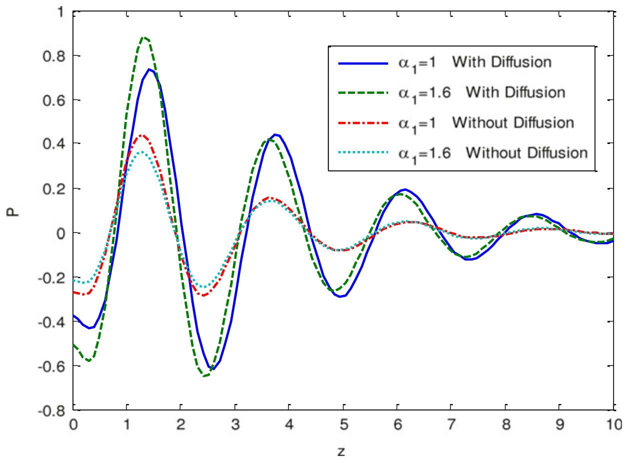

Fig. 12. Variation of chemical potential P at $H_0 = 10^4$, $\alpha = 0.5$.

Also it can be noticed that, the values of the stress function approach to zero rapidly in the case of the presence of a diffusion effect than in the case of the absence of a diffusion effect with the distance z increases.

7. Concluding remarks

According to the analysis above and from the numerical results presented in Figs. 1–12, we can conclude the following important points:

(i) The presence of diffusion plays a significant role in all the quantities and has an important effect on the vertical and normal components of displacement, the temperature, the stress components and the chemical potential.

(ii) The magnetic field also has a significant effect on all the physical quantities. For both the thermoelastic medium with and without diffusion, the magnetic field acts to decrease the absolute values of the temperature field. For thermoelastic medium with diffusion, the temperature distribution starts with positive value at $z = 0$ and then decreases due to the effect of the magnetic field whereas for the thermoelastic medium without diffusion, the temperature distribution starts with negative value at $z = 0$ and then decreases due to the effect of the magnetic field. The magnetic field thus shows its *damping effects* on the temperature field.

(iii) It was observed that, the dependence of the modulus of elasticity on the reference temperature (α_1) plays a significant role in the thermal interactions, while the presence of the modulus of elasticity on reference temperature has a significant effect in all the physical quantities. The important point of this work is the consideration that, the temperature dependence on the material properties, while in other works these material properties were assumed to be constant. This study is very important for *microscale problems*, because in these cases the material parameters are temperature dependent.

(iv) The method used in the present article is applicable to a wide range of problems in thermoelasticity.

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