

Dispersion and suppression of sound near QCD critical point

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We have used second order relativistic hydrodynamics equipped with equation of state which includes the critical point to study the propagation of perturbation in a relativistic QCD fluid. Dispersion relation for the sound wave has been derived to ascertain the fate of the perturbation in the fluid near the QCD critical end point (CEP). We observe that the threshold value of the wavelength of the sound in the fluid diverges at the CEP, implying that all the modes of the perturbations are dissipated at this point. Some consequences of the suppression of sound near the critical point have been discussed.

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I. INTRODUCTION

Relativistic heavy ion collision experiments (RHIC-E) are carried out to create a new state of strongly interacting matter, called Quark-Gluon Plasma (QGP)[1, 2], where color degrees of freedom are deconfined from their parent hadrons and its properties are governed by the colored quarks and gluons. The study of the transition from QGP to hadron phase is one of the main goals of RHIC-E. For last several years, a lot of works have been done to explore the QCD phase diagram in the $T - \mu$ plane where T and μ denote temperature and baryonic chemical potential respectively. Lattice QCD simulations shows that, at vanishing baryon chemical potential ($\mu = 0$), the transition from hadron to QGP is a crossover [3–7] whereas, at large μ , the transition from hadronic matter to QGP is found to be first order [6, 8]. Therefore, it is expected that the first order phase transition ends at some point in the $\mu - T$ plane which is called the Critical End Point (CEP). The existence of CEP was suggested theoretically in Refs.[5, 9–11] and predicted later in lattice simulation [12–14]. The experimental search for the CEP has been taken up through the beam energy scan (BES) programme at Relativistic Heavy Ion Collider (RHIC). The search will continue in future experiments at Facility for Anti-proton and Ion Research (GSI-FAIR) and Nuclotron-based Ion Collider Facility (JINR-NICA) rigorously [15].

A major issue in the exploration of the phase diagram of QCD is to find out the location of the CEP. The exact location of the CEP is not known theoretically because of the difficulties associated with the sign problem of Dirac fermion [16–18] in Lattice QCD calculation. Some of the QCD based effective models such as NJL, PNJL predict the location of the CEP[19] with uncertainties ranging from 266-504 MeV in μ_c and 115-162 MeV in T_c . Therefore, location of CEP in QCD phase diagram remains as a big challenging task. It is one of the main aim of RHIC-BES programme [20, 21] to find the CEP by the creating systems with different μ and T by tuning the colliding energy, ($\sqrt{s_{NN}}$) of the nuclei. At the CEP the correlation length diverges [22–24] resulting in divergences in several thermodynamic quantities which may affect signals of QGP. The chances of detecting such effects become greater if the freeze out curve in $\mu - T$ plane is sufficiently close to the CEP.

In the present work, however, we are not into the search of the location of CEP. Rather, we want to examine its effects on the fate of the sound wave propagating through the fluid in presence of CEP. Here the location of the CEP is taken at: $(T_c, \mu_c) = (154\text{MeV}, 367\text{MeV})$ [25]. It is expected that a system conducive to study the effects of CEP may be realised through nuclear collisions at GSI-FAIR, NICA BES-RHIC. The QGP produce in such collisions will expand rapidly along a trajectory with s/n constant (s and n stand for the entropy density and baryon number density respectively) and cools down consequently. It is assumed that the isentropic trajectory followed by QGP in the $\mu - T$ plane will pass through trajectories which are very close to the CEP.

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The space time evolution of the QGP can be modelled by the relativistic viscous hydrodynamics. The first order theory of relativistic viscous hydrodynamics governed by Navier-Stokes (NS) equations depends on the first order in dissipative fluxes which is known to violate causality and gives unstable solutions [26]. Therefore, making it unsuitable for the description of QGP. These problems were cured by Muller [27] and Grad [28] after including quantities in second order dissipative flux and therefore, these theories are called the 'second order hydrodynamics'. The relativistic generalization is due to Israel and Stewart [29] which can be used to describe the space-time evolution of QGP. The response of the QGP fluid to the perturbation is dictated by the relevant transport coefficients (shear and bulk viscosities, thermal conductivity, etc.) of the fluid. The effects of thermal conductivity (κ) and the shear viscosity (η) have been considered here to investigate the propagation of acoustic wave when the system passes through the CEP. The effects of CEP in the hydrodynamic evolution enters through the Equation of State (EoS). The EoS is constructed based on the hypothesis that the transition from QGP to hadrons belongs to the same universality class as that of the 3D Ising model. The behaviour of thermodynamic quantities near CEP is governed by the critical exponents. Dispersion relation *i.e.* the functional dependence of the frequency (ω) on the wave vector (k) will be set up to study the effects of CEP on the propagation of the sound wave in the fluid.

The present work is organized as follows. In section II we will discuss the EoS which includes the CEP. In section III formulation for the propagation of the acoustic wave is presented. The dispersion relation is discussed in section IV. Results are presented in section V and section VI is devoted to summary and discussions. The space like Minkowski metric $g^{\mu\nu} = (-, +, +, +)$ and the natural unit *i.e.*, $c = \hbar = k_B = 1$ have been used in this work.

II. EQUATION OF STATE

The CEP in QGP-hadron transition belongs to the same universality class as that of the 3D Ising model, thus a mapping onto QCD phase diagram from the Ising model calculation can be performed. It can be shown [25, 30, 31] that the critical entropy density (s_c) in QCD is analogous to the magnetization (M) in 3D Ising model. The parameter plane in 3D Ising model are: $r = \frac{T-T_c}{T_c}$ (reduced temperature) and the strength of the magnetic field (\mathcal{H}). The CEP in 3D Ising model is located at $(r, \mathcal{H}) = (0, 0)$. Thus $r < 0$ represents first order phase transition and $r > 0$ signifies crossover transition. A critical region is being assumed with linear mapping from the (r, \mathcal{H}) to (μ, T) plane. The mapping is implemented through the relation:

$$r = \frac{\mu - \mu_c}{\Delta\mu_c}; \quad \mathcal{H} = \frac{T - T_c}{\Delta T_c} \quad (1)$$

where (T_c, μ_c) is the location of the CEP as mentioned above. ΔT_c and $\Delta\mu_c$ are chosen as elongations of the critical region along T and μ axis respectively. The critical entropy density can be written as

$$s_c = \frac{M(r, \mathcal{H})}{\Delta T_c} = M\left(\frac{T - T_c}{\Delta T_c}, \frac{\mu - \mu_c}{\Delta\mu_c}\right) \frac{1}{\Delta T_c} \quad (2)$$

Firstly, a dimensionless entropy density is constructed as

$$S_c = A(\Delta T_c, \Delta\mu_c) s_c(T, \mu) \quad (3)$$

where A is defined as

$$A(\Delta T_c, \Delta\mu_c) = B\sqrt{\Delta T_c^2 + \Delta\mu_c^2} \quad (4)$$

and B is a dimensionless quantity, represents the spread of the critical region. In this work we have used $(T_c, \mu_c) = (154\text{MeV}, 367\text{MeV})$ with $(\Delta T_c, \Delta\mu_c, B) = (0.1\text{ GeV}, 0.2\text{ GeV}, 2)$. Using S_c as a switching function, the full entropy density is constructed by making a bridge between the entropy density of QGP (s_Q) and the hadron (s_H) phases. The result reads as:

$$S(T, \mu) = \frac{1}{2}[1 - \tanh S_c(T, \mu)]s_Q(T, \mu) + \frac{1}{2}[1 + \tanh S_c(T, \mu)]s_H(T, \mu) \quad (5)$$

s_Q is calculated by [30, 32]

$$s_Q(T, \mu) = \frac{32 + 21N_f}{45}\pi^2 T^3 + \frac{N_f}{9}\mu^2 T \quad (6)$$

where N_f is the number of flavour of quarks.
 s_H can be estimated from the following expression [33],

$$s_H(T, \mu_B) = \pm \sum_i \frac{g_i}{2\pi^2} \int_0^\infty p^2 \left[\ln \left(1 \pm \{ \exp(E_i - \mu_i)/T \} \right) \pm \frac{E_i - \mu_i}{T \{ \exp(E_i - \mu_i)/T \pm 1 \}} \right] \quad (7)$$

where the sum is taken over all hadrons with mass up to 2.5 GeV [34], g_i is the statistical degeneracy and $E_i = \sqrt{p_i^2 + m_i^2}$ is the energy of the i^{th} hadrons.

Once entropy density is known the thermodynamic quantities such as baryon number density, pressure and energy density can be evaluated as follows. The net baryon number density (n) is given by:

$$n(T, \mu) = \int_0^T \frac{\partial S(T', \mu)}{\partial \mu} dT' \quad (8)$$

To get the first order phase boundary, we need to take into account the discontinuity in the entropy density along the transition line. We add the following term to the above equation to take this possibility into account (for $T > T_c$):

$$\left| \frac{\partial T_c(\mu)}{\partial \mu} \right| \left[S(T_c + \delta, \mu) - S(T_c - \delta, \mu) \right] \quad (9)$$

where $\left| \frac{\partial T_c}{\partial \mu} \right| = \tan \theta_c$ is the tangent at the T_c and δ is the small temperature deviation from T_c . The pressure can be calculated as:

$$p(T, \mu) = \int_0^T S(T', \mu) dT' \quad (10)$$

Finally, the energy density is given by,

$$\epsilon(T, \mu) = T s(T, \mu) - p(T, \mu) + \mu n \quad (11)$$

III. PROPAGATION OF THE PERTURBATION IN VISCOUS FLUID

IS second order hydrodynamics is appropriate to study the relativistic fluid nature of QGP as the first order theory (relativistic NS) violates causality and introduce instability in the solution. Therefore, in this section we study the propagation of perturbations through viscous fluid by using second order causal hydrodynamics.

One of the major difference between relativistic and non-relativistic fluid originates from the definition of chemical potential. In non-relativistic case chemical potential constraint the total number of particles in the system. But in a relativistic system, the total number of particles does not remain constant due annihilation and creation of particles within the fluid. However, through the annihilation and creation processes the conservation of certain quantum numbers remain intact. For example in strong interaction net (baryon-antibaryon) baryon number, net electric charge, net strangeness remain conserved (although strangeness is not conserved in weak interaction). The present study is concerned with the strong interaction. Accordingly the net baryon number will remain conserved throughout the evolution of the QGP. Therefore, in the discussion below the net charge density stands for net baryon number density.

The relativistic energy-momentum tensor ($T^{\lambda\mu}$) in the Israel-Stewart second order hydrodynamics is given by [29]

$$T^{\lambda\mu} = \epsilon u^\lambda u^\mu + P \Delta^{\lambda\mu} + 2h^{(\lambda} u^{\mu)} + \tau^{\lambda\mu} \quad (12)$$

where u^μ is the hydrodynamic four velocity subjected to the normalization condition $u^\mu u_\mu = -1$, P is thermodynamic pressure. The dissipative viscous stress $\tau^{\lambda\mu} = \Pi \Delta^{\lambda\mu} + \pi^{\lambda\mu}$, where Π is the bulk viscous pressure, $\Delta^{\lambda\mu} = g^{\lambda\mu} + u^\lambda u^\mu$ is the spatial projection tensor orthogonal to u^λ and $\pi^{\lambda\mu}$ is the shear viscous stress with $\pi^\lambda_\lambda = h^\lambda u_\lambda = \tau^{\lambda\mu} u_\lambda = 0$. The heat flux four vector is defined as $q^\lambda = h^\lambda - n^\lambda (\epsilon + P)/n$, where n is the net baryon number density. The particle four flow is defined as,

$$N^\lambda = n u^\lambda + n^\lambda \quad (13)$$

where, n^λ is called the particle diffusion current, with $n^\lambda u_\lambda = 0$. The symmetric tensor, $h^{(\lambda} u^{\mu)}$ is defined as $h^{(\lambda} u^{\mu)} = \frac{1}{2}(h^\lambda u^\mu + h^\mu u^\lambda)$.

The definition of fluid four velocity in Eq.(12) can be fixed by choosing a suitable reference frame attached to the fluid element according to Landau-Lifshitz (LL)[35] or Eckart[36]. The Eckart frame represents a Local Rest Frame (LRF) for which the net charge dissipation is zero but the net energy dissipation is non zero and the LL frame represents a local rest frame where the energy dissipation is zero but the net charge dissipation is non-zero. We consider LL frame here to study a system having non-zero net baryon number density.

In LL frame: $h^\mu = 0$, $n^\mu = -nq^\mu/(\epsilon + P)$ and the different viscous fluxes are given by [29],

$$\begin{aligned}\Pi &= -\frac{1}{3}\zeta(\partial_\mu u^\mu + \beta_0 D\Pi - \alpha_0 \partial_\mu q^\mu) \\ \pi^{\lambda\mu} &= -2\eta\Delta^{\lambda\mu\alpha\beta}\left[\partial_\alpha u_\beta + \beta_2 D\pi_{\alpha\beta} - \alpha_1 \partial_\alpha q_\beta\right] \\ q^\lambda &= \kappa T\Delta^{\lambda\mu}\left[\frac{nT}{\epsilon + P}(\partial_\mu \alpha) - \beta_1 Dq_\mu + \alpha_0 \partial_\mu \Pi + \alpha_1 \partial_\nu \pi_\mu^\nu\right]\end{aligned}\quad (14)$$

where $D \equiv u^\mu \partial_\mu$, is known as co-moving derivative and in LRF, $D\Pi = \dot{\Pi}$ represents the time derivative. The double symmetric traceless projection operator is defined by $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}]$, $\Delta^{\mu\nu}\partial_\nu = \nabla^\mu$ and $\Delta^{\mu\nu}u_\mu = 0$. The quantity $\alpha = \mu/T$ appearing in Eq.(14) is known as thermal potential and η , ζ , κ are the coefficients of shear viscosity, bulk viscosity and thermal conductivity respectively, $\beta_0, \beta_1, \beta_2$ are relaxation coefficients, α_0 and α_1 are coupling coefficients. The relaxation times for the bulk pressure (τ_Π), the heat flux (τ_q) and the shear tensor (τ_π) are defined as [37]

$$\tau_\Pi = \zeta\beta_0, \quad \tau_q = T\beta_1, \quad \tau_\pi = 2\eta\beta_2 \quad (15)$$

The relaxation lengths which couple to heat flux and bulk pressure ($l_{\Pi q}, l_{q\Pi}$), the heat flux and shear tensor ($l_{q\pi}, l_{\pi q}$) are defined as,

$$l_{\Pi q} = \zeta\alpha_0, \quad l_{q\Pi} = k_B T\alpha_0, \quad l_{q\pi} = k_B T\alpha_1, \quad l_{\pi q} = 2\eta\alpha_1 \quad (16)$$

At the ultra-relativistic limit, $\beta = m/T \rightarrow 0$ where m is the mass of the particle and we have the following relations [29],

$$\alpha_0 \approx 6\beta^{-2}P^{-1}, \alpha_1 \approx -\frac{1}{4}P^{-1}, \beta_0 \approx 216\beta^{-4}P^{-1}, \beta_1 \approx \frac{5}{4}P^{-1}, \beta_2 \approx \frac{3}{4}P^{-1} \quad (17)$$

Since in energy frame, $h^\mu = 0$, then the energy-momentum tensor(EMT) reduces to

$$T^{\lambda\mu} = \epsilon u^\lambda u^\mu + P\Delta^{\lambda\mu} + \Pi\Delta^{\lambda\mu} + \pi^{\lambda\mu} \quad (18)$$

Putting the explicit forms of Π, q^λ and $\pi^{\lambda\mu}$ given by Eq.(14) into Eq.(12) and keeping only the terms up to second order in space time derivatives, the EMT becomes [38]

$$\begin{aligned}T^{\lambda\mu} &= \epsilon u^\lambda u^\mu + P\Delta^{\lambda\mu} - \frac{1}{3}\zeta\Delta^{\lambda\mu}\partial_\alpha u^\alpha + \frac{1}{9}\zeta\beta_0\Delta^{\lambda\mu}D(\zeta\partial_\alpha u^\alpha) + \frac{\zeta\alpha_0}{3}\Delta^{\lambda\mu}\partial_\alpha\left\{\frac{n\kappa T^2}{\epsilon + P}\nabla^\alpha(\alpha)\right\} \\ &\quad - 2\eta\Delta^{\lambda\mu\alpha\beta}\partial_\alpha u_\beta + 4\eta\beta_2\Delta^{\lambda\mu\alpha\beta}D(\eta\Delta_{\alpha\beta}^{\rho\sigma}\partial_\rho u_\sigma) + 2\alpha_1\eta\Delta^{\lambda\mu\alpha\beta}\partial_\alpha\left\{\frac{n\kappa T^2}{\epsilon + P}\nabla_\beta(\alpha)\right\}\end{aligned}\quad (19)$$

The solution of IS hydrodynamical equations grants stability and causality. This is achieved by promoting the dissipative currents as independent dynamical variables and introducing relaxation time scales for these currents. In NS theory the dissipative currents instantaneously respond to the hydrodynamical gradients but in IS theory the response of the dissipative currents is governed by the relaxation time scales (see Eq. (15)). The energy-momentum tensor given in Eq. (19) represents second order dissipative hydrodynamics which is equivalent to IS theory for small gradients. The general form of the EMT constrained by the conformal invariance can be found in [46].

The full charge current (up to second-order in velocity gradient) can be written as,

$$\begin{aligned}N^\mu &= nu^\mu - \frac{n\kappa T}{(\epsilon + P)}\left[\frac{nT}{(\epsilon + P)}\nabla^\mu\alpha - \beta_1\Delta^{\mu\nu}D\left\{\frac{n\kappa T^2}{(\epsilon + P)}\nabla_\nu\alpha\right\} - \frac{\alpha_0}{3}\nabla^\mu(\zeta\partial_\alpha u^\alpha)\right. \\ &\quad \left.- 2\alpha_1\Delta^{\mu\nu}\partial^\rho(\eta\Delta_{\rho\nu}^{\alpha\beta}\partial_\alpha u_\beta)\right]\end{aligned}\quad (20)$$

Eqs.(19) and (20) governs the motion of perturbations in the relativistic viscous fluid with one conserved current (baryonic current for the present case).

We impart small perturbations $P_1, \epsilon_1, n_1, T_1, \mu_1$ and u_1^α to P, ϵ, n, T, μ and u^α respectively to study the propagation of acoustic wave in the fluid with $u^\alpha = (1, 0, 0, 0)$ as outlined in Ref.[39]. We set $u_1^0 = 0$ to preserve the normalization condition $u^\alpha u_\alpha = -1$.

A space time dependent perturbation $\sim \exp[-i(kx - \omega t)]$ is imparted to the fluid and its fate is being studied. The equation of motions that dictate the evolution of different components of the perturbations can be obtained from the the conservation of the energy-momentum tensor ($T^{\mu\lambda}$) and net-baryon number (N^μ) of the fluid:

$$\partial_\mu T^{\mu\lambda} = 0, \quad \partial_\mu N^\mu = 0 \quad (21)$$

The equation of motion of various components of energy momentum tensor are given by:

$$\begin{aligned} 0 &= \omega T_1^{i0} - k_j T_1^{ij} \\ &= \omega(\epsilon + P)u_1^i - k^i P_1 + \frac{1}{3}\zeta k^i \left[i(\mathbf{k} \cdot \mathbf{u}_1) + \frac{1}{3}\zeta\beta_0\omega(\mathbf{k} \cdot \mathbf{u}_1) \right] + i\eta \left[k^2 u_1^i + \frac{1}{3}k^i(\mathbf{k} \cdot \mathbf{u}_1) \right] \\ &\quad - 2\eta^2\beta_2\omega \left[k^2 u_1^i + \frac{1}{3}k^i(\mathbf{k} \cdot \mathbf{u}_1) \right] + \frac{nT\kappa}{(\epsilon + P)}(\mu_1 - \alpha T_1) \left[\frac{\alpha_0\zeta k^2}{3}k^i + \frac{4}{3}\alpha_1\eta k^2 k^i \right] \end{aligned} \quad (22)$$

and the other components of the EMT satisfies,

$$\begin{aligned} 0 &= \omega T_1^{00} - k_i T_1^{i0} \\ &= \omega\epsilon_1 - (\epsilon + P)(\mathbf{k} \cdot \mathbf{u}_1) \end{aligned} \quad (23)$$

The number conservation equation gives,

$$0 = \omega n_1 - n(\mathbf{k} \cdot \mathbf{u}_1) - i \frac{n^2\kappa T k^2}{(\epsilon + P)^2}(\mu_1 - \alpha T_1) \left(1 + i\omega\kappa\beta_1 \right) + \frac{1}{3} \frac{n\kappa T k^2}{(\epsilon + P)}(\zeta\alpha_0 + 4\eta\alpha_1)(\mathbf{k} \cdot \mathbf{u}_1) \quad (24)$$

In Eqs. (22), (23) and (24), we considered terms upto first order in perturbations and neglected the higher order terms. Also, we have not perturbed the different transport coefficients, as they are not hydrodynamical variables. In LRF, we take them as constant in space and time, hence their comoving derivative are zero. For simplicity of calculation we only considered shear viscosity (η) and thermal conductivity (κ) and neglected the bulk viscosity (ζ). We decompose the fluid velocity into directions perpendicular and parallel to the direction of wave vector, \mathbf{k} as:

$$\mathbf{u}_1 = \mathbf{u}_{1\perp} + \mathbf{k}(\mathbf{k} \cdot \mathbf{u}_1)/k^2 \quad (25)$$

The modes propagating along the direction of \mathbf{k} are called longitudinal and those perpendicular to \mathbf{k} are called transverse modes.

The quantities, ϵ_1, P_1 and μ_1 defined above can be expressed in terms of thermodynamic quantities as follows:

$$\begin{aligned} \epsilon_1 &= \left(\frac{\partial\epsilon}{\partial T} \right)_n T_1 + \left(\frac{\partial\epsilon}{\partial n} \right)_T n_1 \\ P_1 &= \left(\frac{\partial P}{\partial T} \right)_n T_1 + \left(\frac{\partial P}{\partial n} \right)_T n_1 \\ \mu_1 &= \left[- \left(\frac{\partial n}{\partial T} \right)_n T_1 + n_1 \right] \left(\frac{\partial\mu}{\partial n} \right)_T \end{aligned} \quad (26)$$

IV. DISPERSION RELATIONS

Eqs. (22), (23) and (24) can be used to write down the algebraic equation satisfied by ω as,

$$a\omega^3 + b\omega^2 + c\omega = 0 \quad \rightarrow \quad \omega(a\omega^2 + b\omega + c) = 0 \quad (27)$$

The coefficients a, b and c are determined by solving Eqs. (22), (23) and (24) simultaneously. The solutions of this equation which provide a relation between ω and k is called the dispersion relation. The equation, (27) has three roots, one real which is $\omega = 0$ and two complex roots with real ($\omega_{\Re e}$) and imaginary ($\omega_{\Im m}$) parts given below by Eqs. (28) and (31) respectively. The real part of ω can be expressed as:

$$\omega_{\Re e} = \sqrt{\frac{a_0 k^2 - a_1 k^3 + a_2 k^4}{b_0 - b_1 k^2}} \quad (28)$$

where

$$\begin{aligned}
a_0 &= 9h \left[\left(\frac{\partial P}{\partial T} \right)_n + \alpha_1 n \left\{ \left(\frac{\partial \epsilon}{\partial n} \right)_T \left(\frac{\partial P}{\partial T} \right)_n - \left(\frac{\partial \epsilon}{\partial T} \right)_n \left(\frac{\partial P}{\partial n} \right)_T \right\} \right] \\
a_1 &= \frac{9\alpha\beta_1 n^2 T^2 \kappa^2}{h} + 12\alpha_1 \eta \kappa n T \left[\alpha + \frac{\alpha n}{h} \left(\frac{\partial \epsilon}{\partial n} \right)_T - \frac{T}{h} \left(\frac{\partial P}{\partial T} \right)_n + \frac{T}{h} \left(\frac{\partial \epsilon}{\partial T} \right)_n \right] \\
a_2 &= \frac{9\beta_1 \kappa^2 n^2 T}{h} \left[\left(\frac{\partial n}{\partial T} \right)_\mu \left(\frac{\partial \mu}{\partial n} \right)_T + \left(\frac{\partial P}{\partial T} \right)_n \left(\frac{\partial P}{\partial n} \right)_T \right] \\
&+ \frac{12\alpha_1 \eta \kappa n T}{h} \left[\left(\frac{\partial P}{\partial T} \right)_n + n \left(\frac{\partial n}{\partial T} \right)_\mu \left(\frac{\partial \mu}{\partial n} \right)_T + \frac{n}{h} \left(\frac{\partial P}{\partial T} \right)_n \left(\frac{\partial \epsilon}{\partial n} \right)_T \right] \\
b_0 &= 9h \left(\frac{\partial \epsilon}{\partial T} \right)_n \\
b_1 &= 24\beta_2 \eta^2 \left(\frac{\partial \epsilon}{\partial T} \right)_n + \frac{9\beta_1 \kappa^2 n^2}{h} \left[T \left(\frac{\partial \mu}{\partial n} \right)_T \left(\frac{\partial n}{\partial T} \right)_\mu - T^2 \left(\frac{\partial \epsilon}{\partial n} \right)_T \left(\frac{\partial \mu}{\partial n} \right)_n + \alpha \left(\frac{\partial \epsilon}{\partial n} \right)_T \right]
\end{aligned} \tag{29}$$

and $h = \epsilon + P$ is the enthalpy density. We have kept terms up to quadratic power of transport coefficients in Eq. (29). We have also neglected the higher order terms in $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$. Expanding $\omega_{\mathfrak{R}e}$ in powers of k and keeping terms up to $\mathcal{O}(k^4)$ we obtain,

$$\omega_{\mathfrak{R}e} = \sqrt{\frac{a_0}{b_0}} \left[k - \frac{1}{2} \frac{a_1}{a_0} k^2 + \left(\frac{1}{2} \frac{a_2}{a_0} - \frac{1}{8} a_1 a_0^2 + \frac{b_1}{b_0} \right) k^3 + \left(\frac{1}{4} \frac{a_1 a_2}{a_0^2} + \frac{1}{16} a_1^2 a_0^3 - \frac{1}{2} \frac{a_1 b_1}{a_0 b_0} \right) k^4 \right] \tag{30}$$

Similarly, the expression for the imaginary part of ω reads as:

$$\omega_{\mathfrak{I}m} = \frac{-c_0 k^2 + c_1 k^3 + c_2 k^4}{d_0 + d_1 k^2} \tag{31}$$

where

$$\begin{aligned}
c_0 &= 2\eta h^2 \left(\frac{\partial \epsilon}{\partial T} \right)_n - 3hn^2 \kappa T \beta_1 \left[\alpha \kappa \left(\frac{\partial \epsilon}{\partial n} \right)_T + h \left(\frac{\partial n}{\partial T} \right)_\mu \left(\frac{\partial \mu}{\partial n} \right)_T + \frac{\alpha_1}{\beta_1} \left(\frac{\partial P}{\partial T} \right)_n \left(\frac{\partial \epsilon}{\partial n} \right)_T \right] \\
c_1 &= 2\alpha\beta_1 \eta n^2 T \left(T \kappa^2 + 4h\eta \frac{\beta - 2}{\beta_1} \right) \\
c_2 &= 8\beta_2 \eta \kappa n^2 T \left[\left(\frac{\partial \epsilon}{\partial T} \right)_n - \left(\frac{\partial P}{\partial n} \right)_T \left(\frac{\partial \epsilon}{\partial n} \right)_T \right] \\
d_0 &= 3h^3 \left(\frac{\partial \epsilon}{\partial T} \right)_n \\
d_1 &= 3h\beta_1 n^2 \kappa \left[\alpha \kappa \left(\frac{\partial \epsilon}{\partial n} \right)_T + 4T^2 \kappa \left(\frac{\partial \epsilon}{\partial T} \right)_n \left(\frac{\partial \mu}{\partial n} \right)_T - T \left(\frac{\partial n}{\partial T} \right)_n \left(\frac{\partial \epsilon}{\partial n} \right)_T \right]
\end{aligned} \tag{32}$$

The imaginary part of ω up to $\mathcal{O}(k^4)$ is given by,

$$\omega_{\mathfrak{I}m} = -\frac{c_0}{d_0} \left[k^2 - \frac{c_1}{c_0} k^3 - \left(\frac{d_1}{d_0} + \frac{c_2}{c_0} \right) k^4 \right] \tag{33}$$

The dispersion relation for first order hydrodynamics can be obtained by setting the relaxation coefficients ($\beta_0, \beta_1, \beta_2$) and the coupling coefficients (α_0, α_1) to zero which allows only a_0, b_0, c_0 and d_0 to be non-zero. Therefore, keeping terms up to $\mathcal{O}(k^2)$ in Eqs. (30) and (33) we get (see also [47]),

$$\omega(k) = c_s k - \frac{i}{2} k^2 \frac{\eta}{s} \frac{4/3}{h} \tag{34}$$

where $c_s = \sqrt{\left(\frac{\partial p}{\partial \epsilon} \right)_{s/n}}$ is the speed of sound and η/s is shear viscosity to entropy density (s) ratio. The Eq.(34) is the dispersion relation for NS hydrodynamics.

A. Fluidity near the critical region

The imaginary and real parts of ω provide the information respectively on attenuation and the propagation of the sound wave in the dissipative fluid. Thus, if magnitude of the imaginary part is larger than the real part, the wave will dissipate quickly. The dispersion relation in Eqs.(28) and (31) can be used to determine the upper limit of k of the sound wave that will dissipate in the medium. The threshold value of k , k_{th} can be calculated by using the following condition [40]

$$\left| \frac{\omega_{\Im m}(k)}{\omega_{\Re e}(k)} \right|_{k=k_{th}} = 1 \quad (35)$$

i.e. any wave with wave vector higher than k_{th} will get dissipated in the fluid. Solving the above equation, we get,

$$k_{th} = \sqrt{\frac{\mathcal{P}}{\mathcal{Q}}}$$

where

$$\mathcal{P} = \frac{a_0}{c_0^2} - \frac{a_0 d_1 c_2^2}{b_0 c_0^2 d_0^2} - \frac{a_1 d_0}{c_0^3} + \frac{a_1 c_2^2 d_1}{b_0 c_0^3 d_0} - \frac{b_1^2 c_2^3 d_1}{b_0 c_0^2 d_0^3} + \frac{a_1 d_0}{a_0 b_0 c_0^2} \quad (36)$$

and

$$\mathcal{Q} = \frac{b_0}{d_0^2} - \frac{b_1}{c_0^2} - \frac{b_0 c_2}{c_0^3} + \frac{a_1}{a_0 b_0 d_0} - \frac{a_1 b_1 d_0}{a_0 b_0^2 c_0^2} + \frac{a_1 c_2 d_0}{b_0^2 c_0^3} \quad (37)$$

Expanding \mathcal{P} and \mathcal{Q} and keeping the first term of the series we get,

$$k_{th} = \sqrt{\frac{a_0 d_0^2}{b_0 c_0^2} \left[1 - \frac{1}{2} \left(\frac{d_1 c_2^2}{b_0 d_0^2} + \frac{a_1 d_0}{a_0 c_0} - \frac{a_1 c_2^2 d_1}{a_0 b_0 c_0 d_0} + \frac{b_1^2 c_2^3 d_1}{a_0 b_0 d_0^3} - \frac{a_1 d_0}{a_0^2 b_0} \right) \right]} \left[1 + \frac{1}{2} \left(\frac{b_1 d_0^2}{b_0 c_0^2} + \frac{c_2 d_0^2}{c_0^3} - \frac{a_1 d_0}{a_0 b_0^2} + \frac{a_1 b_1 d_0^3}{a_0 b_0^3 c_0^2} - \frac{a_1 c_2 d_0^3}{b_0^3 c_0^3} \right) \right] \quad (38)$$

The first term of the above expression gives the value of k_{th} in the NS limit, $k_{th} = \sqrt{\frac{a_0 d_0^2}{b_0 c_0^2}} = \frac{3}{2} c_s \frac{\hbar}{s} \frac{1}{\eta/s}$. The subsequent terms arise from the second order hydrodynamical effects as indicated by the presence of coupling and relaxation coefficients appearing through the quantities defined in Eqs. (29) and (32).

The wavelength (λ_{th}) corresponding to k_{th} is given by $\lambda_{th} = 2\pi/k_{th}$. Sound waves with wavelength, $\lambda < \lambda_{th}$ will dissipate in the medium. However, sound wave with $\lambda > \lambda_{th}$ will propagate in the fluid without much dissipative effects. The quantity λ_{th} can be used to define the fluidity of fluids with widely varying particle density and temperature by selecting a length scale (inter-particle separation), $l \sim \rho^{-1/3}$ [40] of the system as:

$$\mathcal{F} \sim \frac{\lambda_{th}}{l}. \quad (39)$$

where ρ is the particle number density of the fluid (for relativistic fluid l can be chosen as $l \sim s^{-1/3}$). The length scale $R_v \sim 1/k_{th}$, called viscous horizon [41] sets the limit for sound with λ smaller than R_v will be dissipated due to viscous and thermal conduction effects. R_v can be used to estimate the value of the highest harmonics $n_v = 2\pi R/R_v$ which will survive the dissipation *i.e.* any harmonics of order higher than n_v will not survive against dissipation. We find that k_{th} is directly proportional to the speed of sound (c_s), which approaches zero near critical point. Therefore, we can argue that k_{th} also vanishes or in other words λ_{th} diverges at the critical point.

V. RESULTS AND DISCUSSIONS

In Fig. 1 the variation of entropy density (left panel) and pressure (right panel) with μ and T have been depicted. The EoS includes the critical point at $(T_c, \mu_c) = (154 \text{ MeV}, 367 \text{ MeV})$. The effects of critical point is clearly visible on the entropy density and pressure. The discontinuity in entropy density at large baryonic chemical potential (μ) is indicating a first order phase transition. (left panel, Fig. 1).

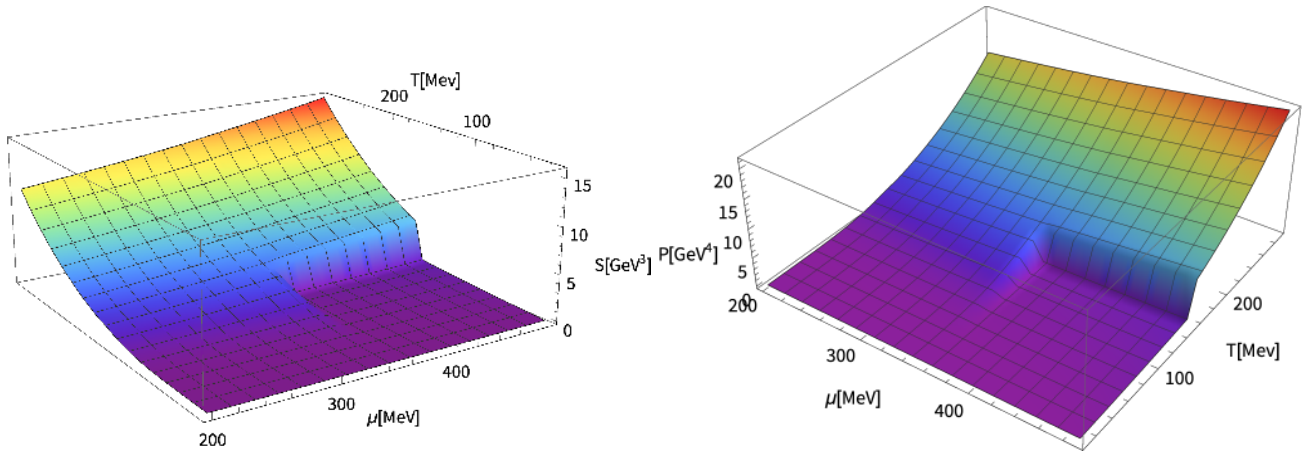


FIG. 1: (color online) Left panel is the constructed entropy density as a function of (T, μ) . Right panel is the pressure as a function of (T, μ) . We consider CEP is at $(T_c, \mu_c) = (154, 367) MeV$.

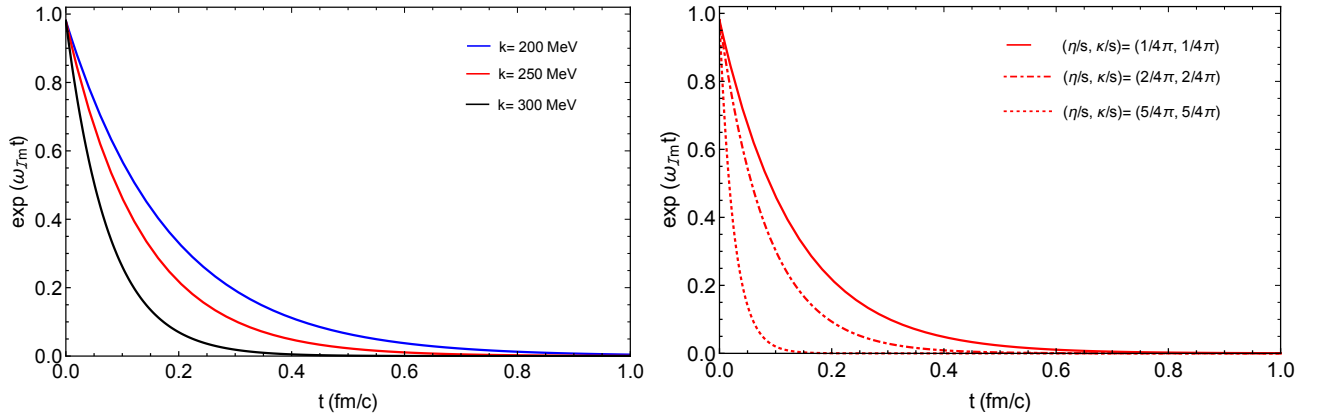


FIG. 2: (color online) a) Dissipation of sound modes with time for different k values at $(T_c, \mu_c) = (154, 367) MeV$ and $(\eta/s = \kappa/s = 1/4\pi)$. b) Damping of the sound waves with different sets of transport coefficients with $k = 250 MeV$.

Now we discuss the dissipation of the perturbation in the fluid when it hits the CEP in the QCD phase diagram. The damping caused by the imaginary part of the frequency of hydrodynamic modes of perturbation at the critical point (T_c, μ_c) is shown in Fig.2. It is clearly seen from the figure that the waves with larger (smaller) values of wavenumber (k) damp faster (slower). The waves in fluid damp faster for larger values of transport coefficients (right panel). Away from the critical point the waves damp slower for high fluid temperature and density as evident from the results displayed in Fig.3 (left panel). The waves in a medium with higher viscosity and thermal conductivity damp faster (Fig. 3, right panel) for obvious reasons.

The variation of λ_{th} with temperature (T/T_c) is shown in Fig.4. The value of the wavelength ($\lambda_{th} = 2\pi/k_{th}$) depends on the transport coefficients (η, κ) as well as on the various response functions appearing through the derivatives, $(\frac{\partial \epsilon}{\partial T})_n, (\frac{\partial \epsilon}{\partial n})_T, (\frac{\partial p}{\partial T})_n, (\frac{\partial p}{\partial n})_T, (\frac{\partial n}{\partial \mu})_T, (\frac{\partial n}{\partial T})_\mu$, relaxation coefficients (β_1, β_2) and the coupling constant (α_1) . The transport coefficients are taken as $\eta/s = \kappa/s = 1/4\pi$. The values of $(\frac{\partial \epsilon}{\partial T})_n, (\frac{\partial \epsilon}{\partial n})_T, (\frac{\partial p}{\partial T})_n, (\frac{\partial p}{\partial n})_T, (\frac{\partial n}{\partial \mu})_T, (\frac{\partial n}{\partial T})_\mu$ are calculated in terms of different response function by using relevant thermodynamic relations (Appendix A). In the left panel of Fig.4 the variation of λ_{th} with T/T_c is depicted. It is observed that at CEP ($T_c = 154 MeV, \mu_c = 367 MeV$) the λ_{th} diverges. As mentioned above λ_{th} is defined as the threshold wavelength *i.e.* waves with wavelengths, $\lambda \geq \lambda_{th}$ are allowed to propagate and others dissipated. It is noted that when we consider $\mu = 347 MeV$ and $387 MeV$ (away from critical point) a finite value of λ_{th} is obtained, that is wave with $\lambda > \lambda_{th}$ will propagate in the medium without substantial dissipation in such cases. At the critical point, however, λ_{th} diverges which imply that waves with any finite wavelength will dissipate strongly in the fluid. We also observe that for lower value of T the value of λ_{th} is smaller. For $\mu = 387 MeV$ the magnitude of λ_{th} is larger compared to $\mu = 347 MeV$. This indicates that for higher values of temperature and chemical potential λ_{th} is higher. The fluidity defined in Eq.(39) is directly proportional

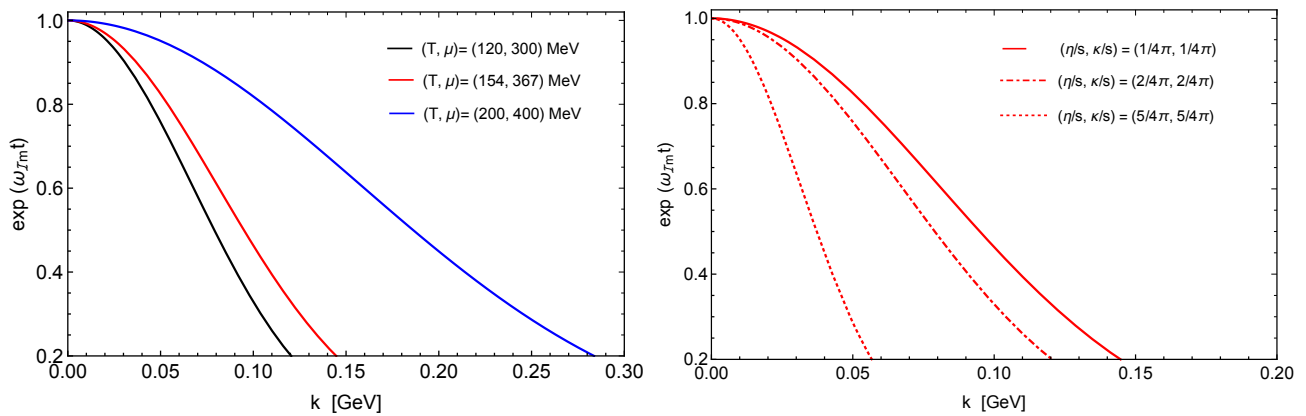


FIG. 3: (color online) a) Dissipation of sound waves with k for different sets of (T, μ) at $t = 0.6 fm/c$ and $(\eta/s = \kappa/s = 1/4\pi)$. b) Damping of the sound waves with different sets of value of transport coefficients with $(T_c, \mu_c) = (154, 367)$ MeV.

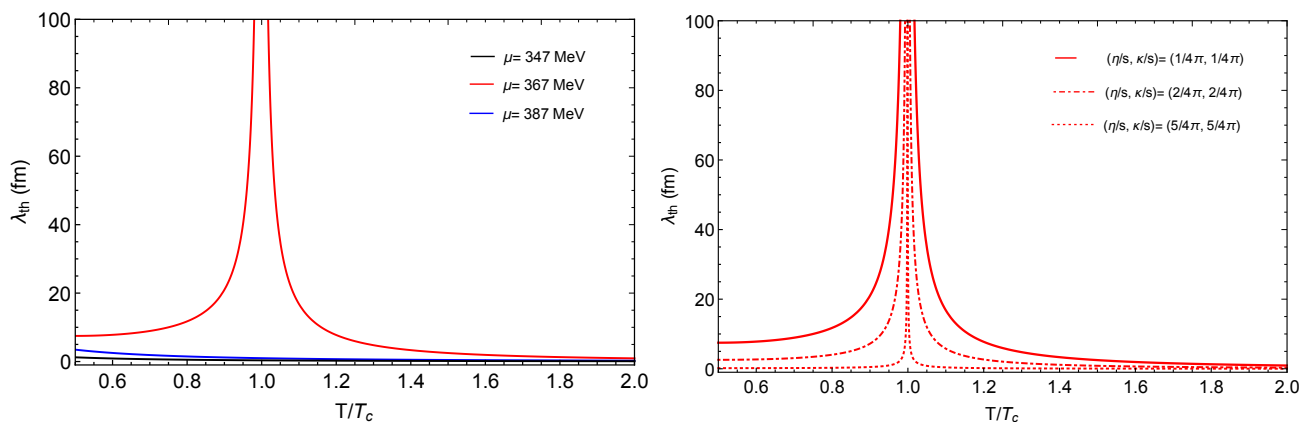


FIG. 4: (color online) a) Left figure is λ_{th} (fm) vs temperature plot. Divergence is observed at $(T_c, \mu_c) = (154, 367)$ MeV. Right plot is λ_{th} (fm) vs T at $\mu = 367 MeV$ for different sets of value transport coefficients.

to λ_{th} which diverges at CEP, implies that fluidity also diverges at the CEP. Away from CEP, the fluidity decreases. The fluidity is larger for $\mu = 387$ MeV compared to $\mu = 347$ MeV.

The viscous damping of perturbation can be understood from the relation: $T_1^{\mu\lambda}(t) = T_1^{\mu\lambda}(0)exp(-\omega_{\gamma m}t)$, where $T_1^{\mu\lambda}(0)$ is the perturbation in EMT at $t = 0$ and $T_1^{\mu\lambda}(t)$ at some later time t which is dissipated as indicated by the exponential term. The spectrum of initial ($t = 0$) perturbations can be associated with the harmonics of the shape deformations and density fluctuations [42]. The dispersion relation for ω provides the value, k_{th} , which can be used to define a length scale, $R_v \sim 1/k_{th}$. For system of size R , R_v can be used to define $n_v = \frac{2\pi R}{R_v}$ which is linked to the value of the highest harmonic n_v (eccentricity-driven) that will effectively survive damping. We have seen that the nature of the plot i.e λ_{th} vs T does not change much with the variation of the shear viscosity (η/s) but changes significantly with the variation of thermal conductivity (κ/s). Right panel of Fig.4 shows the variation of λ_{th} with temperature for higher values of κ/s and η/s . As the magnitude of κ/s increases the gap between the divergences in the two phases gets narrower. It is well-known [43] that the thermal conductivity diverges at critical point. Therefore, the nature of the variation of λ_{th} with T/T_c near the CEP will be essentially governed by the convergence of the thermal conductivity. We have found that with increasing thermal conductivity the width of the divergence gets narrower.

VI. SUMMARY AND CONCLUSION

We have constructed an EoS of a fluid which contain the effects of QCD critical point and used it to study the propagation of sound wave through the medium. A perturbation has been imparted on the relativistic fluid and its

evolution has been studied as it passes through the CEP within the scope of IS like causal hydrodynamics. We have estimated the threshold value of the wavelength, λ_{th} such that any wave with wavelength below λ_{th} is dissipated for given values of transport coefficients and other thermodynamic quantities. Most interestingly we have found that no waves is allowed to propagate if the system hits the CEP *i.e* waves with all wavelength get dissipated at CEP irrespective to the values of transport coefficients. The fluidity of the system diverges at the CEP indicating the fact that the fluid flows without any resistance.

It has been observed experimentally in conventional condensed matter system [44] (see also [45]) that the absorption is maximum due to diffraction of sound from the critical region similar to the scattering of light at the critical point where the opalescence due to critical phenomena is strongest. Therefore, the absorption of sound will indicate the presence of critical point. Near the critical point the correlation length (ξ) becomes very large, therefore, the hydrodynamic limit, $\xi \ll \lambda$ is violated. As a consequence the development of sound wave is prevented. The forbiddance of sound wave will lead to the vanishing of Mach cone (Mach angle, $\alpha = \sin^{-1}(c_s/v)$, v is the fluid velocity). Therefore, the vanishing of Mach angle will indicate the presence of critical point.

Various harmonics of the azimuthal distribution of produced particles in RHIC-E are useful quantities to characterize the matter. For example the triangular flow helps in understanding the initial fluctuations and elliptic flow can be used to comprehend the equation of state of the system. The presence of critical point makes the viscous horizon scale, $R_v \sim 1/k_{th}$ to diverge. Since the highest order of harmonics that survives varies as, $n_v \sim 2\pi R/R_v$, ideally the vanishing harmonics will indicate the presence of critical point. However, the experimentally measure quantities are superpositions of different temperatures and densities from the formation to the freeze-out stage, therefore, even if the system hits the critical point in the $T - \mu$ plane, the harmonics may not vanish, but the critical point may weaken them.

The possibility of the existence and detection of CEP has been studied in [43]. The mode-mode coupling theory has been used to estimate the thermal conductivity at the points near and away from the QCD critical point in the $\mu - T$ plane and shown that the thermal conductivity diverges at the critical point. It has also been demonstrated that the sharp change in thermal conductivity at the critical point is strongly reflected in the two particle correlation of fluctuations in rapidity space, therefore, paving the way to confirm the existence and location of the CEP.

In a realistic scenario the possibility of the trajectories passing through the critical point, *i.e.* the trajectories hitting the (μ_c, T_c) point in the (μ, T) plane is remote, which limits the magnitude of the fluctuations near the critical point. These fluctuations will remain out of equilibrium due to the expansion of the system and critical slowing down [48]. These issues has been considered in [49] while studying the evolution of hydrodynamic fluctuations of the system formed in RHIC-E. The appearance of the Kibble-Zurek length scale and its connection with short range spatial correlations has been discussed. It has also been shown that the non-flow correlations get enhanced in presence of critical point and such correlations should be measured as a function of n/s for detecting the CEP [49].

In a realistic scenario the matter formed in RHIC-E evolves in space and time - from the initial QGP phase to the final hadronic freeze-out state through a phase transition in the intermediate stage. The space time evolution of the locally equilibrated system is described by relativistic viscous hydrodynamics. The experimentally detected signals is the superposition of the yields for all the possible values of temperatures and densities of the system ranging from the initial to freeze-out states. The detection of CEP will require the disentanglement of contributions from the neighbourhood (μ_c, T_c) from all other possible values of μ and T which the system confronts during its evolution history from the initial to the freeze-out stages. In the present work the expansion dynamics has not been taken into consideration, therefore, the results obtained here can not be contrasted with experiments. The effects of the CEP with (3+1) dimensional expansion within the scope of second order viscous hydrodynamics will be published in future [50].

Rigorously speaking hydrodynamics is applicable in the region where $k \ll \xi^{-1}$ is satisfied where k is the wave vector of the sound mode and ξ is correlation length. At the CEP this fundamental assumptions on the application of hydrodynamics becomes invalid as ξ diverges as the system approaches CEP with $T \rightarrow T_c$ and $\mu \rightarrow \mu_c$. However, for a given k there will certainly be a domain in the neighbourhood of (μ_c, T_c) where the predictions of hydrodynamics can be useful. Existence of such region in condensed matter system has been discussed in [31].

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Appendix A

The expressions for $\omega_{\mathfrak{R}e}(k)$ and $\omega_{\mathfrak{I}m}(k)$ contain derivatives of several thermodynamics quantities. In this appendix we recast these derivatives in terms of response functions like: isothermal and adiabatic compressibilities (κ_T and κ_s), volume expansivity α_p , specific heats (c_p and c_v), baryon number susceptibility (χ_B), velocity of sound (c_s), etc. The baryon number density (n) and the entropy density (s) are given by

$$n = \left(\frac{\partial p}{\partial \mu} \right)_T; \quad s = \left(\frac{\partial p}{\partial T} \right)_\mu \quad (40)$$

Baryon number susceptibility (χ_B), isothermal compressibility (κ_T), adiabatic compressibility (κ_s) and volume expansivity (α_p) are given by,

$$\chi_B = \left(\frac{\partial n}{\partial \mu} \right)_T; \kappa_T = \frac{1}{n} \left(\frac{\partial n}{\partial p} \right)_T; \kappa_s = \frac{1}{n} \left(\frac{\partial n}{\partial p} \right)_s; \alpha_p = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p = -\frac{1}{n} \left(\frac{\partial n}{\partial T} \right)_p \quad (41)$$

Specific heats are given by

$$c_p = T \left(\frac{\partial s}{\partial T} \right)_p; c_V = T \left(\frac{\partial s}{\partial T} \right)_V = T \left(\frac{\partial s}{\partial T} \right)_n = \left(\frac{\partial \epsilon}{\partial T} \right)_V = \left(\frac{\partial \epsilon}{\partial T} \right)_n \quad (42)$$

We have to express six quantities such as: $\left(\frac{\partial p}{\partial T} \right)_n$, $\left(\frac{\partial p}{\partial n} \right)_T$, $\left(\frac{\partial \epsilon}{\partial T} \right)_n$, $\left(\frac{\partial n}{\partial T} \right)_\mu$, $\left(\frac{\partial n}{\partial \mu} \right)_T$ and $\left(\frac{\partial \epsilon}{\partial n} \right)_T$.

i) To evaluate: $\left(\frac{\partial p}{\partial T} \right)_n$ we start with

$$\begin{aligned} \left(\frac{\partial p}{\partial T} \right)_n &= \frac{\partial(p, n)}{\partial(T, n)} = \frac{\partial(p, n)}{\partial(T, p)} \frac{\partial(T, p)}{\partial(s, p)} \frac{\partial(s, p)}{\partial(s, \epsilon)} \frac{\partial(s, \epsilon)}{\partial(s, n)} \frac{\partial(s, n)}{\partial(T, n)} \\ &= \left[- \left(\frac{\partial n}{\partial T} \right)_p \right] \left(\frac{\partial T}{\partial s} \right)_p \left(\frac{\partial p}{\partial \epsilon} \right)_s \left(\frac{\partial \epsilon}{\partial n} \right)_s \left(\frac{\partial s}{\partial T} \right)_n \\ &= n \alpha_p \frac{T}{c_p} c_s^2 \left(\frac{\partial \epsilon}{\partial n} \right)_s \frac{c_V}{T} = n c_s^2 \alpha_p \frac{c_V}{c_p} \left(\frac{\partial \epsilon}{\partial n} \right)_s \end{aligned} \quad (43)$$

By using the relation,

$$d\epsilon = T ds + \mu dn \quad \text{and} \quad \mu = \left(\frac{\partial \epsilon}{\partial n} \right)_s \quad (44)$$

we can write

$$\left(\frac{\partial p}{\partial T} \right)_n = \mu n c_s^2 \alpha_p \frac{c_V}{c_p} \quad (45)$$

ii) Now consider $\left(\frac{\partial p}{\partial n} \right)_T$:

$$\left(\frac{\partial p}{\partial n} \right)_T = \frac{1}{n \kappa_T} \quad (46)$$

$$\begin{aligned} \left(\frac{\partial p}{\partial n} \right)_T &= \frac{\partial(p, T)}{\partial(n, T)} = \frac{\partial(p, T)}{\partial(p, s)} \frac{\partial(p, s)}{\partial(\epsilon, s)} \frac{\partial(\epsilon, s)}{\partial(n, s)} \frac{\partial(n, s)}{\partial(n, T)} \\ &= \left(\frac{\partial T}{\partial s} \right)_p \left(\frac{\partial p}{\partial \epsilon} \right)_s \left(\frac{\partial \epsilon}{\partial n} \right)_s \left(\frac{\partial s}{\partial T} \right)_n \\ &= \frac{T}{c_p} c_s^2 \left(\frac{\partial \epsilon}{\partial n} \right)_s \frac{c_V}{T} \end{aligned} \quad (47)$$

$$= \mu c_s^2 \frac{c_V}{c_p} \quad (48)$$

iii) The factor, $\left(\frac{\partial \epsilon}{\partial T} \right)_n$ can be estimates as follows:

$$\left(\frac{\partial \epsilon}{\partial T} \right)_n = c_n \quad (49)$$

For fixed net baryon number, c_n can be written as $c_n = c_V$. Therefore,

$$\left(\frac{\partial \epsilon}{\partial T} \right)_n = c_V \quad (50)$$

iv) $\left(\frac{\partial \epsilon}{\partial n}\right)_T$ can be estimates as:

$$\begin{aligned} \left(\frac{\partial \epsilon}{\partial n}\right)_T &= \frac{\partial(\epsilon, T)}{\partial(n, T)} = \frac{\partial(\epsilon, T)}{\partial(\epsilon, s)} \frac{\partial(\epsilon, s)}{\partial(n, s)} \frac{\partial(n, s)}{\partial(n, T)} \\ &= \left(\frac{\partial T}{\partial s}\right)_\epsilon \left(\frac{\partial \epsilon}{\partial s}\right)_s \left(\frac{\partial s}{\partial T}\right)_n \\ &= \left(\frac{\partial T}{\partial s}\right)_\epsilon \left[\left(\frac{\partial \epsilon}{\partial p}\right)_s \left(\frac{\partial p}{\partial n}\right)_s\right] \frac{c_V}{T} \\ &= \left(\frac{\partial T}{\partial s}\right)_\epsilon \left[\frac{1}{c_s^2} \frac{1}{n\kappa_s}\right] \frac{c_V}{T} = \frac{T}{c_\epsilon} \left[\frac{1}{c_s^2} \frac{1}{n\kappa_s}\right] \frac{c_V}{T} \end{aligned} \quad (51)$$

$$= \frac{c_V}{c_\epsilon} \left[\frac{1}{c_s^2} \frac{1}{n\kappa_s}\right] \quad (52)$$

v) The quantity $\left(\frac{\partial n}{\partial \mu}\right)_T$ can be estimates as

$$\left(\frac{\partial n}{\partial \mu}\right)_T = \chi_B \quad (53)$$

vi) For $\left(\frac{\partial n}{\partial T}\right)_\mu$, we have,

$$\left(\frac{\partial n}{\partial T}\right)_\mu = -\left(\frac{\partial n}{\partial \mu}\right)_T \left(\frac{\partial \mu}{\partial T}\right)_n = -\chi_B \left(\frac{\partial \mu}{\partial T}\right)_n \quad (54)$$

We know that,

$$sdT = dp - nd\mu \rightarrow dp = sdT + nd\mu \quad (55)$$

$$Tds = d\epsilon - \mu dn \rightarrow d\epsilon = Tds + \mu dn \quad (56)$$

$$\left(\frac{\partial p}{\partial \epsilon}\right) = \frac{sdT + nd\mu}{Tds + \mu dn} \quad (57)$$

$$= \frac{s + n\left(\frac{\partial \mu}{\partial T}\right)}{T\left(\frac{\partial s}{\partial T}\right) + \mu\left(\frac{\partial n}{\partial T}\right)} \quad (58)$$

$$n\left(\frac{\partial \mu}{\partial T}\right) = \left(\frac{\partial p}{\partial \epsilon}\right) \left[T\left(\frac{\partial s}{\partial T}\right) + \mu\left(\frac{\partial n}{\partial T}\right)\right] - s \quad (59)$$

Thus

$$\left(\frac{\partial \mu}{\partial T}\right)_n = \frac{c_n^2 C_V - s}{n} \quad \text{where, } c_n^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_n \quad (60)$$

Finally we get,

$$\left(\frac{\partial n}{\partial T}\right)_\mu = \frac{\chi_B}{n} (s - c_n^2 C_V) \quad (61)$$

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