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Demand influenced by enterprises' initiatives — A multi-item EOQ model of deteriorating and ameliorating items

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ABSTRACT

An interesting multi-item EOQ (*Economic Order Quantity*) model is established when the time varying demand is influenced by enterprises' initiatives like advertising media and salesmen' effort. It is developed for deteriorating and ameliorating items with capacity constraint for storage facility. The effect of inflation and time value of money in the profit and cost parameters is also considered. The associated profit function is maximized by Euler–Lagrange's method, and it is illustrated by various time varying demands like quadratic, linear and exponential demand functions.

1. Introduction

It is common to all enterprises that some goods in store undergo *deterioration* and *amelioration*. *Deterioration* is defined as obsolete, pilferage, loss of utility or loss of marginal value of commodity that results in decrease of its usefulness. Most physical goods undergo deterioration and their value reduces with time. Blood, fish, strawberry, alcohol, gasoline, radioactive chemicals and food grains, electronic products and fashionable goods are examples of deteriorating items. It is also observed in the marketplace that the value or utility or quantity of some commodities increases with time. In practice, in wine manufacturing industries, the utility or value of some wine increases with age. Other notable examples occur in farming yards where the amount or quantity of high breed fishes, fast growing animals like broiler, pig, etc., increases with time in the growing phase. The increase of value or quantity or amount of inventory is termed as *amelioration* in the present article.

The first attempt to obtain optimal replenishment policies of perishable items, mainly fashion goods, was made by Whitin [1]. Thereafter, Ghare and Schrader [2] developed a model for an exponentially decaying inventory. Inventory models with a time-dependent rate of deterioration were studied by Covert and Philip [3], Philip [4], Misra [5], Deb and Chaudhuri [6] and others. Numerous inventory models of deteriorating items with various features of the inventory system have been discussed. The models of Shah and Jaiswal [7], Aggarwal [8], Dave and Patel [9], Goswami and Chaudhuri [10,11], Chung and Ting [12], Fujiwara [13], Hariga [14,15], Wee [16], Giri and Chaudhuri [17], Andijani and Al-Dajani [18], Manna and Chaudhuri [19], Skouri and Papachristous [20], Khanra and Chaudhuri [21], Ghosh and Chaudhuri [22] and Roy [23] are worth mentioning in this direction. The first EOQ model of ameliorating items was established by Hwang [24,25]. Recently, Moon, Giri and Byungsung [26] discussed analytically an EOQ model of ameliorating and deteriorating items. They developed models with zero-ending inventory for fixed order interval over a finite planning horizon, considering, linear trend in demand, shortages, effects of inflation and time value of money.

The monetary situation of most of the countries has changed to such an extent due to large scale inflation and consequent sharp decline in the purchasing power of money, that it has not been possible to ignore the effects of inflation and time value

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of money. The first attempt in this direction was by Buzacott [27]. He derived expressions for the optimal order quantity and showed that the choice of the inventory carrying fraction depends on the company's pricing policy, which inversely affects the optimal order quantity. In this connection, the works of Biermann and Thomas [28], Aggarwal [29], Jeya Chandra and Bahner [30], Brahmbhatt [31], Datta and Pal [32], Dohi, Kaio and Osaka [33], Moon and Yun [34], Bose, Goswami and Chaudhuri [35], Ray and Chaudhuri [36], Chung, Liu and Tsai [37], Chen [38], Moon and Lee [39], Chung and Lin [40] and Yang [41] are worth mentioning.

In the proposed model, the author develops an multi-item EOQ model for time varying demand over finite time horizon, taking into account the effect of inflation and time value of money, deterioration and amelioration, salesmen' initiatives and limited storage space for holding inventory. The demand function D_j ($j = 1, 2, \dots, n$) of jth item depends on time and salesmen' initiatives $E_j(t)$ for jth item. Costs $H_j(t)$ of salesmen' initiatives are quadratic functions of $E_j(t)$. Finally, the present worth of integrated profit function under inflation and time value of money is maximized by Euler–Lagrange's variation of calculus method. Moon,Giri and Byungsung [26] studied an single-item inventory model for ameliorating and deteriorating item under discount cash flow. They considered finite horizon with equal cycle lengths for general time varying demand. But, they considered only the linear time varying demand in numerical examples. The new major contributions in the proposed model compared to [26] are (i) multi-item inventory model, (ii) demands are quadratically, linearly and exponentially (both increasing and decreasing type) varying with time, (iii) salesmen' initiatives like advertising and salesmen who sale their items to the customers by traveling door to door to the locality, (iv) the storage space for multi-items is not unlimited. Generally, time varying demand is influenced by salesmen' initiatives like advertising media, salesmen's effort, etc. In oligopolistic competition, salesmen' initiatives accelerate the demand in market. We develop a multi-item EOQ model in this context.

The rest of the paper is organized as follows: Section 2 describes the assumptions and notation of the model. Section 3 formulates the model and its solution procedure by Euler–Lagrange's and Kuhn–Tucker method. Section 4 describes the model for various types of demand pattern with proper illustrations and its discussion is done in this section. Section 5 concludes the paper.

2. Notation and assumptions

The following notation and assumptions are considered to develop the model:

Notation:

- R_j — initial replenishment lot size of jth item;
- $Q_j(t)$ — on-hand inventory at time ($t \geq 0$) of jth item;
- ξ_j $= \xi_{1j}$ and $-\xi_{1j}$ in case of ameliorating and deteriorating items respectively;
- $D_j(t)$ — demand at time t (≥ 0) of jth item;
- $E_j(t)$ — amount or quantity of salesmen' initiatives for jth item;
- $H_j(t)$ — cost of salesmen' initiatives ($E_j(t)$) for jth item;
- C_{pj} — purchasing cost per unit of jth item;
- S_{pj} — selling price per unit of jth item;
- C_{hj} — inventory holding cost per unit of jth item per unit time;
- β_j — storage space (in square units) per unit of jth item;
- δ $= r - i$, where r is the interest per unit currency and i is the inflation per unit currency;
- CAP — storage capacity of the business sector;
- T — the duration of the cycle.

Assumptions:

1. A multi-item inventory model is developed over a finite time horizon.
2. Costs ($H_j(t),\ j = 1, 2, \dots, n$) of salesmen' initiatives are quadratic functions of ($E_j(t),\ j = 1, 2, \dots, n$). Here, $H_j(t) = C_{1j}E_j(t) + C_{2j}E_j^2(t) + C_{3j}$, where C_{1j}, C_{2j} and C_{3j} are positive parameters which are estimated (see Appendix A) from the data of marketing research. It is obvious that the advertising media are spread over world to motivate the customers about good quality and longevity of the merchandize. The salesmen motivate directly the customer to buy more, traveling door-to-door of the locality. For example, some multi-national companies sale their products by the agents/salesmen. In this case, the costs for holding inventory by the agents/salesmen is assumed to be zero which follows JIT (*Just-in-Time*) philosophy. The remuneration of top-class agents/salesmen is too much high compared to the lower-class agents/salesmen. So, in this case, costs of effort ($H_j(t),\ j = 1, 2, \dots, n$) are nonlinear functions of ($E_j(t),\ j = 1, 2, \dots, n$). We consider the costs of effort ($H_j(t),\ j = 1, 2, \dots, n$) to be quadratic functions of ($E_j(t),\ j = 1, 2, \dots, n$).
3. The amounts of ($E_j(t),\ j = 1, 2, \dots, n$) are measured by the total duration of advertising media and working hours of salesmen, and their efficiency. Generally speaking, duration and number of media (like, channels of T.V., Radio, Newspapers and Internet, etc.) and working hours of salesmen accelerate demand of the products. In oligopolistic competition, each management decides to boost sale of their items through its publicity. The costs of all time advertisement media like Internet and Newspapers are constant or discrete throughout a period. These are the sophisticated media and all the customers are not acquainted with these. The media like channels of T.V. and Radio and salesmen motivate the customers directly. Consequently, the demands of the items increase with an increase in the effort made by the advertising media and salesmen. If effort of these media is discrete in nature, then, in oligopolistic

competition, the alternative products of which effort made by advertising are continuous spread over the market that de-accelerate the original purpose. Quite often, the hours per day of advertisement in T.V. channels, Radio and salesmen' effort are increased day by day to sell the products more. More specifically, the purpose of the effort of sales team and advertising is to enhance potential buyers' response to a business organization, providing information and supplying reasons for preferring a particular product or services.

In the effort of sales team and advertising is a continuous time-dependent state variable which motivates the customer to buy more and it is controlled by on-hand inventory at time 't'. Its cost $H_j(t)$ is quadratic function of sales effort that is measured by total volume of messages conveyed by sales team and advertising to the market. This measure is generally used as an indicator of intensity of advertising. As advertising intensity increases the probability of making successful 'hits' (messages noticed) on the population that increases the consumers' demand. When expenditure on advertising decreases across the population, the number of successful 'hits' diminishes accordingly. Consequently, the consumers' demand decrease. A strategy of constant advertising outlays over time would not succeed in maintaining its advertising pressure of other similar products that results in rapid growth in potential customers. From this point of view, the effort by advertising and sales team as well as the budget for advertising are dynamic in nature.

4. The demand rate of jth item is the sum of two terms, i.e., $D_j(t, E_j(t)) = a_j(t) + \gamma_j E_j(t)$, $\gamma_j > 0$. $a_j(t)$ is time dependent, $\gamma_j E_j(t)$ is directly proportional to $E_j(t)$. Generally, the demand is directly proportional to the number of advertising media and number of salesmen.
5. Inflation and time value of money in the profit parameters and cost parameters is considered.
6. Initial replenishment size R_j is finite and its availability in the market is instantaneously infinite.
7. The capacity of storage space for inventory is not unlimited.
8. Time horizon is finite.

3. Formulation of the model

The inventory cycle of jth item ($j = 1, 2, \ldots, n$) starts with initial lot-size R_j ($j = 1, 2, \ldots, n$) at time $t = 0$ and ends with zero inventory at $t = T$. The governing equations of the model are as follows:

$$D_j(t, E_j) = a_j(t) + \gamma_j E_j(t), \quad (j = 1, 2, \ldots, n) \tag{1}$$

where $\gamma_j > 0,\ E_j(t) \geq 0$;

$$H_j(t) = C_{1j}E_j(t) + C_{2j}E_j^2(t) + C_{3j} \tag{2}$$

where C_{1j}, C_{2j} and C_{3j} are positive parameters which are estimated (see Appendix A) from previous knowledge of the businessmen; and

$$\begin{aligned}\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} &= \xi_j Q_j(t) - D_j(t, E_j(t))\\ &= \xi_j Q_j(t) - a_j(t) - \gamma_j E_j(t)\end{aligned} \tag{3}$$

with $Q_j(0) = R_j$ and $Q_j(T) = 0$.

The present worth of total profit during $(0, T)$ is

$$\begin{aligned}\pi(R_1, R_2, \ldots, R_n) &= \int_0^T \sum_{j=1}^n \mathrm{e}^{-\delta t}\{S_{pj}(a_j + \gamma_j E_j(t)) - C_{hj}Q_j(t) - C_{1j}E_j(t) - C_{2j}E_j^2(t) - C_{3j} - C_{pj}R_j\}\mathrm{d}t\\ &= \sum_{j=1}^n \int_0^T \phi_j(t, Q_j(t), \dot{Q}_j(t))\mathrm{d}t,\end{aligned} \tag{4}$$

where

$$\begin{aligned}\phi_j(t, Q_j(t), \dot{Q}_j(t)) &= \mathrm{e}^{-\delta t}\{S_{pj}(a_j + \gamma_j E_j(t)) - C_{hj}Q_j(t) - C_{1j}E_j(t) - C_{2j}E_j^2(t) - C_{3j} - C_{pj}R_j\}\\ &= \mathrm{e}^{-\delta t}\left\{(S_{pj}a_j(t) - C_{3j} - C_{pj}R_j) + S_{pj}(\xi_j Q_j(t) - a_j(t) - \dot{Q}_j(t)) - C_{hj}Q_j(t)\right.\\ &\quad \left. + \frac{C_{1j}}{\gamma_j}(\dot{Q}_j(t) - \xi_j Q_j(t) + a_j(t)) - \frac{C_{2j}}{\gamma_j^2}(-\dot{Q}_j(t) + \xi_j Q_j(t) - a_j(t))^2\right\}, \quad \text{using Eq. (3).}\end{aligned} \tag{5}$$

In practice, the storage space for holding the inventory of all items is not unlimited. Therefore, the required constraint for holding the inventory is

$$\sum_{j=1}^n \beta_j R_j \leq CAP. \tag{6}$$

Now, our objective is to find the optimal replenishment sizes $(R_1^*, R_2^*, \ldots, R_n^*)$ such that the net profit $\pi(R_1, R_2, \ldots, R_n)$ with the constraint $\sum_{j=1}^{n} \beta_j R_j \leq CAP$ is maximized. Our first attempt is to find the optimal paths $Q_j(t)$ and $E_j(t)$, applying *Euler–Lagrange's* method, such that the integral of Eq. (4) is maximized. Next, $\pi(R_1, R_2, \ldots, R_n)$ with the constraint $\sum_{j=1}^{n} \beta_j R_j \leq CAP$ is maximized by *Kuhn–Tucker* method. Here, our objective function π has unique maximum value (see Appendix B). Differentiating the function ϕ_j partially with respect to Q_j and $\dot{Q}_j$, we have

$$\frac{\partial \phi_j}{\partial Q_j} = e^{-\delta t}\left\{S_{pj}\xi_j - C_{hj} - \frac{C_{1j}\xi_j}{\gamma_j} - \frac{2C_{2j}\xi_j}{\gamma_j^2}(-\dot{Q}_j + \xi_j Q_j - a_j)\right\}$$

and

$$\frac{\partial \phi_j}{\partial \dot{Q}_j} = e^{-\delta t}\left\{-S_{pj} + \frac{C_{1j}}{\gamma_j} + \frac{2C_{2j}}{\gamma_j^2}(-\dot{Q}_j + \xi_j Q_j - a_j)\right\}.$$

Again,

$$\frac{d}{dt}\left(\frac{\partial \phi_j}{\partial \dot{Q}_j}\right) = -\delta e^{-\delta t}\left\{-S_{pj} + \frac{C_{1j}}{\gamma_j} + \frac{2C_{2j}}{\gamma_j^2}(-\dot{Q}_j + \xi_j Q_j - a_j)\right\} - \frac{2C_{2j}}{\gamma_j^2}(\ddot{Q}_j - \xi_j\dot{Q}_j + \dot{a}_j)e^{-\delta t}.$$

The *Euler–Lagrange's* necessary conditions for maximum value of π are (see Appendix B)

$$\frac{\partial \phi_j}{\partial Q_j} - \frac{d}{dt}\left(\frac{\partial \phi_j}{\partial \dot{Q}_j}\right) = 0, \quad j = 1, 2, \ldots, n. \tag{7}$$

Using the above partial derivatives in Eq. (7), we obtain

$$\begin{aligned}
&e^{-\delta t}\left\{S_{pj}\xi_j - C_{hj} - \frac{C_{1j}\xi_j}{\gamma_j} - \frac{2C_{2j}\xi_j}{\gamma_j^2}(-\dot{Q}_j + \xi_j Q_j - a_j)\right\} + \delta e^{-\delta t}\left\{-S_{pj} + \frac{C_{1j}}{\gamma_j} + \frac{2C_{2j}}{\gamma_j^2}(-\dot{Q}_j + \xi_j Q_j - a_j)\right\}\\
&\quad + \frac{2C_{2j}}{\gamma_j^2}(\ddot{Q}_j - \xi_j\dot{Q}_j + \dot{a}_j)e^{-\delta t} = 0\\
&\Rightarrow \frac{2C_{2j}}{\gamma_j^2}\ddot{Q}_j - \frac{2C_{2j}}{\gamma_j^2}\delta\dot{Q}_j - \frac{2C_{2j}}{\gamma_j^2}(\xi_j^2 - \xi_j\delta)Q_j\\
&\quad + \left\{S_{pj}\xi_j - C_{hj} - \frac{C_{1j}\xi_j}{\gamma_j} + \frac{2C_{2j}}{\gamma_j^2}\xi_j a_j + \delta\left(-S_{pj} + \frac{C_{1j}}{\gamma_j}\right) - \frac{2C_{2j}}{\gamma_j^2}\delta a_j + \frac{2C_{2j}}{\gamma_j^2}\dot{a}_j\right\} = 0\\
&\Rightarrow \ddot{Q}_j - \delta\dot{Q}_j - (\xi_j^2 - \xi_j\delta)Q_j = -\frac{\gamma_j^2}{2C_{2j}}\left\{S_{pj}(\xi_j - \delta) - \frac{C_{1j}}{\gamma_j}(\xi_j - \delta) - C_{hj}\right\} - (\xi_j - \delta)a_j - \dot{a}_j,\\
&\Rightarrow \ddot{Q}_j - \delta\dot{Q}_j - (\xi_j^2 - \xi_j\delta)Q_j = A_j - (\xi_j - \delta)a_j - \dot{a}_j,
\end{aligned} \tag{8}$$

where $A_j = -\frac{\gamma_j^2}{2C_{2j}}\{S_{pj}(\xi_j - \delta) - \frac{C_{1j}}{\gamma_j}(\xi_j - \delta) - C_{hj}\}$.

Now, the nonhomogeneous second-order linear scalar equation (7) can be written as

$$L(Q_j) \equiv \ddot{Q}_j(t) - \delta\dot{Q}_j(t) - \xi_j(\xi_j - \delta)Q_j(t) = F_j(t) \tag{9}$$

where $F_j(t) = A_j - (\xi_j - \delta)a_j(t) - \dot{a}_j(t)$ is a continuous function of $t\epsilon[0, \infty)$. For convenience, we may use the differential operator notation $L \equiv \theta^2 - \delta\theta - \xi_j(\xi_j - \delta)$, where $\theta = d/dt$ is a differential operator. In the case of homogeneous system, $L(Q_j) = 0$, the auxiliary equations are $\lambda_j^2 - \delta\lambda_j - \xi_j(\xi_j - \delta) = 0$. The roots of these equations are: $(\lambda_{1j} = \delta - \xi_j, \lambda_{2j} = \xi_j, j = 1, 2, \ldots, n)$. Then, the required complementary functions are

$$Q_{jc} = \begin{cases} Y_{1j}e^{-(\xi_j - \delta)t} + Y_{2j}e^{\xi_j t}, & \text{if } \xi_j \neq \delta/2\\ (\hat{Y}_{1j} + \hat{Y}_{2j}t)e^{\delta t/2}, & \text{if } \xi_j = \delta/2\end{cases}$$

where $Y_{1j}, Y_{2j}, \hat{Y}_{1j}$ and $\hat{Y}_{2j}$ are arbitrary constants. For nonhomogeneous system, $L(Q_j) = F_j(t) \neq 0$, we can write the particular solutions as

$$\Psi_j(t) = \int^t G_j(t - s)F_j(s)ds, \tag{10}$$

using the *Green's function (kernel)*:

$$G_j(x) = \begin{cases} (e^{\lambda_{1j}x} - e^{\lambda_{2j}x})/(\lambda_{1j} - \lambda_{2j}), & \text{if } \lambda_{1j} \neq \lambda_{2j} \\ xe^{\lambda_{1j}x}, & \text{if } \lambda_{1j} = \lambda_{2j}. \end{cases}$$

Therefore,

$$G_j(t-s) = \begin{cases} (e^{-(\xi_j-\delta)(t-s)} - e^{\xi_j(t-s)})/(\delta - 2\xi_j), & \text{if } \xi_j \neq \delta/2 \\ (t-s)e^{\delta(t-s)/2}, & \text{if } \xi_j = \delta/2. \end{cases}$$

Therefore, the complete solutions are

$$Q_j(t) = \begin{cases} Y_{1j}e^{-(\xi_j-\delta)t} + Y_{2j}e^{\xi_j t} + \psi_j(t), & \text{if } \xi_j \neq \delta/2 \\ (\hat{Y}_{1j} + \hat{Y}_{2j}t)e^{\delta t/2} + \psi_j(t), & \text{if } \xi_j = \delta/2. \end{cases} \tag{11}$$

Using the boundary conditions $Q_j(0) = R_j$ and $Q_j(T) = 0$, we obtain

$$Y_{1j} + Y_{2j} + \Psi_j(0) = R_j,$$
$$Y_{1j}e^{-(\xi_j-\delta)T} + Y_{2j}e^{\xi_j T} + \Psi_j(T) = 0,$$
$$\hat{Y}_{1j} + \psi_j(0) = R_j,$$
$$(\hat{Y}_{1j} + \hat{Y}_{2j}T)e^{\delta T/2} + \psi_j(T) = 0.$$

Solving these, we get

$$Y_{1j}(R_j) = \frac{\{R_j - \Psi_j(0)\}e^{\xi_j T} + \Psi_j(T)}{e^{\xi_j T} - e^{-(\xi_j-\delta)T}},$$
$$Y_{2j}(R_j) = -\left[\frac{\{R_j - \Psi_j(0)\}e^{-(\xi_j-\delta)T} + \Psi_j(T)}{e^{\xi_j T} - e^{-(\xi_j-\delta)T}}\right],$$
$$\hat{Y}_{1j}(R_j) = R_j - \psi_j(0),$$
$$\hat{Y}_{2j}(R_j) = \frac{\psi_j(0) - R_j - \psi_j(T)e^{-\delta T/2}}{T}.$$

Therefore, the required optimal paths of Q_j $(j = 1, 2, \ldots, n)$ and E_j $(j = 1, 2, \ldots, n)$ are

$$Q_j(t) = \begin{cases} Y_{1j}(R_j)e^{-(\xi_j-\delta)t} + Y_{2j}(R_j)e^{\xi_j t} + \Psi_j(t), & (j = 1, 2, \ldots, n) \text{ if } \xi_j \neq \delta/2 \\ (\hat{Y}_{1j}(R_j) + \hat{Y}_{2j}(R_j)t)e^{\delta/2} + \Psi_j(t), & (j = 1, 2, \ldots, n) \text{ if } \xi_j = \delta/2 \end{cases} \tag{12}$$

and

$$\begin{aligned} E_j(t) &= \frac{1}{\gamma_j}[\xi_j Q_j - \dot{Q}_j - a_j] \\ &= \begin{cases} \dfrac{1}{\gamma_j}[(2\xi_j - \delta)Y_{1j}(R_j)e^{-(\xi_j-\delta)t} + \xi_j\Psi_j(t) - \dot{\Psi}_j(t) - a_j], & \text{if } \xi_j \neq \delta/2 \\ \dfrac{1}{\gamma_j}\left[-\hat{Y}_{2j}(R_j)e^{\delta/2} + \dfrac{\delta}{2}\psi_j(t) - \dot{\psi}_j(t) - a_j\right], & \text{if } \xi_j = \delta/2 \end{cases} \end{aligned} \tag{13}$$

respectively. For ameliorating items, $\xi_j = \xi_{1j}$ must belong to the open interval $(\delta, 1)$ for feasibility of the model. In the case of deteriorating items, $\xi_j = -\xi_{1j}$ where ξ_{1j} must be positive for feasibility of the model as well as real situation. Substituting the values of E_j $(j = 1, 2, \ldots, n)$ and Q_j $(j = 1, 2, \ldots, n)$ for $\xi_j \neq \delta/2$ in the Eq. (4), we have

$$\begin{aligned} \pi(R_1, R_2, \ldots, R_n) &= \sum_{j=1}^{n} \int_0^T e^{-\delta t}\{S_{pj}(a_j + \gamma_j E_j) - C_{hj}Q_j - C_{1j}E_j - C_{2j}E_j^2 - C_{3j} - C_{pj}R_j\}dt \\ &= \sum_{j=1}^{n} \int_0^T e^{-\delta t}\{(S_{pj}a_j - C_{3j} - C_{pj}R_j) + (S_{pj}\gamma_j - C_{1j})E_j - C_{2j}E_j^2 - C_{hj}Q_j\}dt \\ &= \sum_{j=1}^{n} \int_0^T e^{-\delta t}\left[(S_{pj}a_j - C_{3j} - C_{pj}R_j) + (S_{pj}\gamma_j - C_{1j})\frac{1}{\gamma_j}\{(2\xi_j - \delta)Y_{1j}e^{-(\xi_j-\delta)t} + \xi_j\Psi_j - \dot{\Psi}_j - a_j\}\right. \\ &\quad \left. - \frac{C_{2j}}{\gamma_j^2}\{(2\xi_j - \delta)Y_{1j}e^{-(\xi_j-\delta)t} + \xi_j\Psi_j - \dot{\Psi}_j - a_j\}^2 - C_{hj}\{Y_{1j}e^{-(\xi_j-\delta)t} + Y_{2j}e^{\xi_j t} + \Psi_j\}\right]dt \end{aligned}$$

$$= \sum_{j=1}^{n} S_{pj} \int_0^T a_j(t)\mathrm{e}^{-\delta t}\mathrm{d}t - \frac{(1-\mathrm{e}^{-\delta T})}{\delta}\sum_{j=1}^{n}(C_{3j}+C_{pj}R_j) + \sum_{j=1}^{n}(S_{pj}\gamma_j - C_{1j})(2\xi_j-\delta)(1-\mathrm{e}^{-\xi_j T})\frac{Y_{1j}(R_j)}{\xi_j}$$
$$+ \sum_{j=1}^{n}\left(\frac{S_{pj}\gamma_j - C_{1j}}{\gamma_j}\right)\int_0^T(\xi_j\Psi_j(t) - \dot{\Psi}_j(t) - a_j(t))\mathrm{e}^{-\delta t}\mathrm{d}t - \sum_{j=1}^{n} C_{hj}(1-\mathrm{e}^{-\xi_j T})\frac{Y_{1j}(R_j)}{\xi_j}$$
$$- \sum_{j=1}^{n} C_{hj}(\mathrm{e}^{(\xi_j-\delta)T}-1)\frac{Y_{2j}(R_j)}{\xi_j-\delta} - \sum_{j=1}^{n} C_{hj}\int_0^T \Psi_j(t)\mathrm{e}^{-\delta t}\mathrm{d}t - \sum_{j=1}^{n} C_{2j}(2\xi_j-\delta)\left(\frac{Y_{1j}(R_j)}{\gamma_j}\right)^2\{1-\mathrm{e}^{-(2\xi_j-\delta)T}\}$$
$$- \sum_{j=1}^{n}\frac{C_{2j}}{\gamma_j^2}\int_0^T\{\xi_j^2\Psi_j^2(t) + \dot{\Psi}_j^2(t) + a_j^2(t) - 2\xi_j\Psi_j(t)\dot{\Psi}_j(t) - 2\xi_j a_j(t)\Psi_j(t) + 2\dot{\Psi}_j(t)a_j(t)\}\mathrm{e}^{-\delta t}\mathrm{d}t$$
$$-2\sum_{j=1}^{n} C_{2j}(2\xi_j-\delta)\left(\frac{Y_{1j}(R_j)}{\gamma_j^2}\right)\xi_j\int_0^T\Psi_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t + 2\sum_{j=1}^{n} C_{2j}(2\xi_j-\delta)\left(\frac{Y_{1j}(R_j)}{\gamma_j^2}\right)\int_0^T\dot{\Psi}_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t$$
$$+2\sum_{j=1}^{n} C_{2j}(2\xi_j-\delta)\left(\frac{Y_{1j}(R_j)}{\gamma_j^2}\right)\int_0^T a_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t.$$

Similarly, we may obtain $\pi(R_1, R_2, \ldots, R_n)$ for $\xi_j = \delta/2$. Now, we have to maximize $\pi(R_1, R_2, \ldots, R_n)$ such that $\sum_{j=1}^{n}\beta_j R_j - CAP \leq 0$. The problem is solved here by *Kuhn–Tucker* method. This method gives us the following conditions:

$$\frac{\partial \pi}{\partial R_j} - \lambda = 0, \quad (j = 1, 2, \ldots, n); \tag{14}$$

$$\lambda\left(\sum_{j=1}^{n}\beta_j R_j - CAP\right) = 0; \tag{15}$$

$$\lambda \geq 0. \tag{16}$$

The Eq. (14) gives us

$$-C_{pj}\frac{(1-\mathrm{e}^{-\delta T})}{\delta} + \frac{(S_{pj}\gamma_j - C_{1j})(2\xi_j-\delta)(\mathrm{e}^{\xi_j T}-1)}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})} - \frac{C_{hj}(\mathrm{e}^{\xi_j T}-1)}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})} + \frac{C_{hj}(1-\mathrm{e}^{-(\xi_j-\delta)T})}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}$$
$$- \frac{2C_{2j}(2\xi_j-\delta)(1-\mathrm{e}^{-(2\xi_j-\delta)T})}{\gamma_j^2(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})^2}\{(R_j-\Psi_j(0))\mathrm{e}^{-(\xi_j-\delta)T}+\Psi_j(T)\}$$
$$- \frac{2C_{2j}(2\xi_j-\delta)\mathrm{e}^{\xi_j T}}{\gamma_j^2(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}\left\{\xi_j\int_0^T\Psi_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t - \int_0^T\dot{\Psi}_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t - \int_0^T a_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t\right\} - \lambda\beta_j = 0,$$
$$\Rightarrow \frac{2C_{2j}(2\xi_j-\delta)(1-\mathrm{e}^{-(2\xi_j-\delta)T})}{\gamma_j^2(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})^2}\{(R_j-\Psi_j(0))\mathrm{e}^{-(\xi_j-\delta)T}+\Psi_j(T)\}$$
$$= -C_{pj}\frac{(1-\mathrm{e}^{-\delta T})}{\delta} + \frac{(S_{pj}\gamma_j - C_{1j})(2\xi_j-\delta)(\mathrm{e}^{\xi_j T}-1)}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})} + \frac{C_{hj}(2-\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}$$
$$- \frac{2C_{2j}(2\xi_j-\delta)\mathrm{e}^{\xi_j T}}{\gamma_j^2(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}\left\{\xi_j\int_0^T\Psi_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t - \int_0^T\dot{\Psi}_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t - \int_0^T a_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t\right\} - \lambda\beta_j.$$

After simplification, it reduces to

$$R_j = \Psi_j(0)\{1-\mathrm{e}^{(3\xi_j-\delta)T}\} + \Psi_j(T)\{\mathrm{e}^{(2\xi_j-\delta)T}-\mathrm{e}^{(\xi_j-\delta)T}\} + \frac{\gamma_j^2(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})^2}{2C_{2j}(2\xi_j-\delta)(1-\mathrm{e}^{-(2\xi_j-\delta)T})\mathrm{e}^{-(\xi_j-\delta)T}}\left[-C_{pj}\frac{(1-\mathrm{e}^{-\delta T})}{\delta}\right.$$
$$\left. + \frac{(S_{pj}\gamma_j - C_{1j})(2\xi_j-\delta)(\mathrm{e}^{\xi_j T}-1)}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})} + \frac{C_{hj}(2-\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}{\xi_j(\mathrm{e}^{\xi_j T}-\mathrm{e}^{-(\xi_j-\delta)T})}\right] + \mathrm{e}^{(3\xi_j-\delta)T}\left\{\int_0^T a_j(t)\mathrm{e}^{-\xi_j t}\mathrm{d}t + \lambda\beta_j\right\}.$$

Case 1. **When the first parts of the demands** D_j $(j = 1, 2, \ldots, n)$ **are** $a_j(t) = d_{1j} + d_{2j}t + d_{3j}t^2$.

This trend in demand applies to some established products (like essential commodities, seasonal goods, etc.) having a demand rate steadily increasing or accelerating over time with the increase of customers. On the other hand, the demand rate of some products (like off-seasonal goods, obsolete items, etc.) steadily decreases or de-accelerates over time with the decrease of customers. We have $d_{1j} > 0,\ d_{2j} > 0,\ d_{3j} > 0$ for increasing demands and $d_{1j} > 0,\ d_{2j} < 0,\ d_{3j} < 0$ for decreasing demands. When $\xi_j \neq \delta/2$, the particular integrals are

$$
\begin{aligned}
\Psi_j(t) &= \int^t G_j(t-s)F_j(s)\mathrm{d}s \\
&= \frac{1}{\delta-2\xi_j}\int^t \{\mathrm{e}^{-(\xi_j-\delta)(t-s)} - \mathrm{e}^{\xi_j(t-s)}\}\{A_j - (\xi_j-\delta)a_j(s) - \dot{a}_j(s)\}\mathrm{d}s \\
&= \frac{1}{\delta-2\xi_j}\Bigg[(A_j - (\xi_j-\delta)d_{1j} - d_{2j})\left\{\mathrm{e}^{-(\xi_j-\delta)t}\int^t \mathrm{e}^{(\xi_j-\delta)s}\mathrm{d}s - \mathrm{e}^{\xi_j t}\int^t \mathrm{e}^{-\xi_j s}\mathrm{d}s\right\} \\
&\quad - ((\xi_j-\delta)d_{2j} + 2d_{3j})\left\{\mathrm{e}^{-(\xi_j-\delta)t}\int^t \mathrm{e}^{(\xi_j-\delta)s}s\mathrm{d}s - \mathrm{e}^{\xi_j t}\int^t \mathrm{e}^{-\xi_j s}s\mathrm{d}s\right\} \\
&\quad - (\xi_j-\delta)d_{3j}\left\{\mathrm{e}^{-(\xi_j-\delta)t}\int^t \mathrm{e}^{(\xi_j-\delta)s}s^2\mathrm{d}s - \mathrm{e}^{\xi_j t}\int^t \mathrm{e}^{-\xi_j s}s^2\mathrm{d}s\right\}\Bigg] \\
&= \frac{1}{\delta-2\xi_j}\Bigg[(A_j - (\xi_j-\delta)d_{1j} - d_{2j})\left(\frac{1}{\xi_j-\delta} + \frac{1}{\xi_j}\right) - ((\xi_j-\delta)d_{2j} + 2d_{3j})\left\{\frac{t}{\xi_j-\delta} - \frac{1}{(\xi_j-\delta)^2} + \frac{t}{\xi_j} + \frac{1}{\xi_j^2}\right\} \\
&\quad - (\xi_j-\delta)d_{3j}\left\{\frac{t^2}{\xi_j-\delta} - \frac{2t}{(\xi_j-\delta)^2} + \frac{2}{(\xi_j-\delta)^3}\right\} + (\xi_j-\delta)d_{3j}\left\{-\frac{t^2}{\xi_j} - \frac{2t}{\xi_j^2} - \frac{2}{\xi_j^3}\right\}\Bigg] \\
&= -\left[A_j - (\xi_j-\delta)d_{1j} - d_{2j} + \frac{\delta\{(\xi_j-\delta)d_{2j} + 2d_{3j}\}}{\xi_j(\xi_j-\delta)} - 2\left\{\frac{\xi_j(\xi_j-\delta)+\delta^2}{\xi_j^2(\xi_j-\delta)}\right\}d_{3j}\right]\frac{1}{\xi_j(\xi_j-\delta)} \\
&\quad - \frac{1}{\xi_j(\xi_j-\delta)}\left[-\{(\xi_j-\delta)d_{2j} + 2d_{3j}\} + \frac{2\delta}{\xi_j}d_{3j}\right]t + \frac{d_{3j}}{\xi_j}t^2.
\end{aligned}
$$

When $\xi_j = \delta/2$, the particular integrals are

$$
\begin{aligned}
\Psi_j(t) &= \int^t G_j(t-s)F_j(s)\mathrm{d}s \\
&= \int^t (t-s)\mathrm{e}^{\delta(t-s)/2}\{A_j - (\xi_j-\delta)d_{1j} - d_{2j} - \{(\xi_j-\delta)d_{2j} + 2d_{3j}\}s - (\xi_j-\delta)d_{3j}s^2\}\mathrm{d}s \\
&= \Bigg[t\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}\{A_j - (\xi_j-\delta)d_{1j} - d_{2j} - \{(\xi_j-\delta)d_{2j} + 2d_{3j}\}s - (\xi_j-\delta)d_{3j}s^2\}\mathrm{d}s \\
&\quad - \mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}\{(A_j - (\xi_j-\delta)d_{1j} - d_{2j})s - ((\xi_j-\delta)d_{2j} + 2d_{3j})s^2 - (\xi_j-\delta)d_{3j}s^3\}\mathrm{d}s\Bigg] \\
&= (A_j - (\xi_j-\delta)d_{1j} - d_{2j})t\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}\mathrm{d}s - ((\xi_j-\delta)d_{2j} + 2d_{3j})t\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}s\mathrm{d}s \\
&\quad - (\xi_j-\delta)d_{3j}t\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}s^2\mathrm{d}s - (A_j - (\xi_j-\delta)d_{1j} - d_{2j})\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}s\mathrm{d}s \\
&\quad + ((\xi_j-\delta)d_{2j} + 2d_{3j})\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}s^2\mathrm{d}s + (\xi_j-\delta)d_{3j}\mathrm{e}^{\delta t/2}\int^t \mathrm{e}^{-\delta s/2}s^3\mathrm{d}s \\
&= (A_j - (\xi_j-\delta)d_{1j} - d_{2j})t\mathrm{e}^{\delta t/2}\left(-\frac{2}{\delta}\mathrm{e}^{-\delta t/2}\right) - ((\xi_j-\delta)d_{2j} + 2d_{3j})t\mathrm{e}^{\delta t/2}\left(-\frac{2}{\delta}t\mathrm{e}^{-\delta t/2} - \frac{4}{\delta^2}\mathrm{e}^{-\delta t/2}\right) \\
&\quad - (\xi_j-\delta)d_{3j}t\mathrm{e}^{\delta t/2}\left\{-\frac{2}{\delta}t^2\mathrm{e}^{-\delta t/2} + \frac{4}{\delta}\left\{-\frac{2}{\delta}t\mathrm{e}^{-\delta t/2} - \frac{4}{\delta^2}\mathrm{e}^{-\delta t/2}\right\}\right\} \\
&\quad - (A_j - (\xi_j-\delta)d_{1j} - d_{2j})\mathrm{e}^{\delta t/2}\left(-\frac{2}{\delta}t\mathrm{e}^{-\delta t/2} - \frac{4}{\delta^2}\mathrm{e}^{-\delta t/2}\right) \\
&\quad + ((\xi_j-\delta)d_{2j} + 2d_{3j})\mathrm{e}^{\delta t/2}\left\{-\frac{2}{\delta}t^2\mathrm{e}^{-\delta t/2} + \frac{4}{\delta}\left\{-\frac{2}{\delta}t\mathrm{e}^{-\delta t/2} - \frac{4}{\delta^2}\mathrm{e}^{-\delta t/2}\right\}\right\} \\
&\quad + (\xi_j-\delta)d_{3j}\mathrm{e}^{\delta t/2}\left\{-\frac{2}{\delta}t^3\mathrm{e}^{-\delta t/2} + \frac{6}{\delta}\left\{-\frac{2}{\delta}t^2\mathrm{e}^{-\delta t/2} + \frac{4}{\delta}\left\{-\frac{2}{\delta}t\mathrm{e}^{-\delta t/2} - \frac{4}{\delta^2}\mathrm{e}^{-\delta t/2}\right\}\right\}\right\} \\
&= -\frac{4}{\delta^2}(\xi_j-\delta)d_{3j}t^2 + \left[-\frac{4}{\delta^2}((\xi_j-\delta)d_{2j} + 2d_{3j}) - \frac{32}{\delta^3}(\xi_j-\delta)d_{3j}\right]t
\end{aligned}
$$

$$+\left[\frac{4}{\delta^2}(A_j-(\xi_j-\delta)d_{1j}-d_{2j})-\frac{16}{\delta^3}((\xi_j-\delta)d_{2j}+2d_{3j})-\frac{96}{\delta^4}(\xi_j-\delta)d_{3j}\right].$$

Case 2. **When the first parts of the demands** D_j $(j = 1, 2, \ldots, n)$ **are varying exponentially with time** t.

Generally, some products (like new computer chips, new spare parts of aeroplanes, new electronic chips, etc.), the demand rate is likely to increase very fast, almost exponentially, with time while in the case of some other products (e.g., spare parts of obsolete machines, off-seasonal goods, etc.), the demand rate is expected to decrease very fast, almost exponentially, with time. In this case, demand rates are $a_j(t) = u_j e^{v_j t}$, $v_j > 0$ or $v_j < 0$. When $\xi_j \neq \delta/2$, $v_j \neq \xi_j$ and $v_j \neq \delta - \xi_j$, the particular integrals are

$$\begin{aligned}
\Psi_j(t) &= \int^t G_j(t-s)F_j(s)\mathrm{d}s \\
&= \frac{1}{\delta-2\xi_j}\int^t \{e^{-(\xi_j-\delta)(t-s)}-e^{\xi_j(t-s)}\}\{A_j-(\xi_j-\delta+v_j)u_j e^{v_j s}\}\mathrm{d}s \\
&= \frac{1}{\delta-2\xi_j}\left[A_j\left\{e^{-(\xi_j-\delta)t}\int^t e^{(\xi_j-\delta)s}\mathrm{d}s-e^{\xi_j t}\int^t e^{-\xi_j s}\mathrm{d}s\right\}\right. \\
&\quad \left.-\ (\xi_j-\delta+v_j)u_j\left\{e^{-(\xi_j-\delta)t}\int^t e^{(\xi_j-\delta+v_j)s}\mathrm{d}s-e^{\xi_j t}\int^t e^{(v_j-\xi_j)s}\mathrm{d}s\right\}\right] \\
&= \frac{1}{\delta-2\xi_j}\left[A_j\left(\frac{1}{\xi_j-\delta}+\frac{1}{\xi_j}\right)-(\xi_j-\delta+v_j)u_j\left(\frac{1}{\xi_j-\delta+v_j}-\frac{1}{v_j-\xi_j}\right)e^{v_j t}\right] \\
&= -\frac{A_j}{\xi_j(\xi_j-\delta)}-\frac{u_j}{v_j-\xi_j}e^{v_j t}.
\end{aligned}$$

When $\xi_j \neq \delta/2$, $v_j = \xi_j$ and $v_j \neq \delta - \xi_j$, the particular integrals are

$$\psi_j(t) = -\frac{A_j}{\xi_j(\xi_j-\delta)}+\frac{u_j}{2v_j-\delta}\{1-(2v_j-\delta)t\}e^{v_j t}.$$

When $\xi_j \neq \delta/2$, $v_j \neq \xi_j$ and $v_j = \delta - \xi_j$, the particular integrals are

$$\psi_j(t) = -\frac{A_j}{\xi_j(\xi_j-\delta)}.$$

When $\xi_j = \delta/2$ and $v_j \neq \delta/2$, the particular integrals are

$$\begin{aligned}
\psi_j(t) &= \int^t \{(t-s)e^{\delta(t-s)/2}\}\{A_j-(\xi_j-\delta+v_j)u_j e^{v_j s}\}\mathrm{d}s \\
&= \frac{4}{\delta^2}A_j+\frac{2u_j}{\delta-2v_j}e^{v_j t}.
\end{aligned}$$

When $\xi_j = \delta/2$ and $v_j = \delta/2$, the particular integrals are

$$\begin{aligned}
\psi_j(t) &= \int^t (t-s)e^{\delta(t-s)/2}A_j\mathrm{d}s \\
&= \frac{4}{\delta^2}A_j.
\end{aligned}$$

4. Numerical example

Example 1.1. Let us consider two items only, i.e., $j = 1, 2$. The first item is ameliorating and demand increases with time with the parameter values in appropriate units such that $\xi_1 = 0.08$, $r = 0.16$, $i = 0.14$, $S_{p1} = 90.0$, $C_{p1} = 50.0$, $C_{h1} = 0.5$, $C_{11} = 0.5$, $C_{21} = 0.2$, $C_{31} = 25.0$, $T = 1$, $d_{11} = 100$, $d_{21} = 10$, $d_{31} = 40$, $\beta_1 = 4.0$, $\gamma_1 = 0.4$, $CAP = 1500$. The second item is deteriorating and demand increases with time with parameter values: $\xi_2 = -0.05$, $S_{p2} = 100.0$, $C_{p2} = 75.0$, $C_{h2} = 0.25$, $C_{12} = 0.6$, $C_{22} = 0.3$, $C_{32} = 30.0$, $d_{12} = 90$, $d_{22} = 20$, $d_{32} = 50$, $\beta_2 = 9.0$, $\gamma_2 = 0.5$. The optimal solutions, by *Kuhn–Tucker* method, are found to be $\{R_1^* = 128.366, R_2^* = 129.218, \lambda = 0\}$ and $\{R_1^* = 122.121, R_2^* = 112.391, \lambda = 4.22324\}$. The first solution does not satisfy the constraint but the second solution satisfies the capacity constraints. So the required optimal solution is $\{R_1^* = 122.121, R_2^* = 112.391, \lambda = 4.22324\}$ with profit $\pi^* = 7538.72$.

Example 1.2. When both the items are ameliorating, and demand of first item decreases with time and the other increases with time, we consider the parameter values: $\xi_1 = 0.08$, $r = 0.16$, $i = 0.14$, $S_{p1} = 90.0$, $C_{p1} = 50.0$, $C_{h1} = 0.5$, $C_{11} = 0.5$, $C_{21} = 0.2$, $C_{31} = 25.0$, $T = 1$, $d_{11} = 100$, $d_{21} = -10$, $d_{31} = -40$, $\beta_1 = 4.0$, $\gamma_1 = 0.4$, $CAP = 1200$, $\xi_2 = 0.05$, $S_{p2} = 100.0$, $C_{p2} = 75.0$, $C_{h2} = 0.25$, $C_{12} = 0.6$, $C_{22} = 0.3$, $C_{32} = 30.0$, $d_{12} = 90$, $d_{22} = 20$, $d_{32} = 50$, $\beta_2 = 9.0$, $\gamma_2 = 0.5$. In this case, the optimal solutions are $\{R_1^* = 94.0161, R_2^* = 123.884, \lambda = 0\}$ and $\{R_1^* = 82.7958, R_2^* = 96.5352, \lambda = 7.58865\}$.

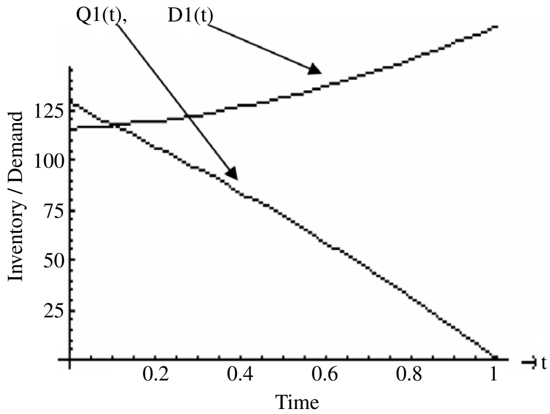

Fig. 1.1a. Inventory and demand of the item 1 versus time.

The first solution does not satisfy the constraint but the second solution satisfies the capacity constraint. So the required optimal solution is $\{R_1^* = 82.7958, R_2^* = 96.5352, \lambda = 7.58865\}$ with profit $\pi^* = 5781.12$.

Example 1.3. When both items are deteriorating, and demand of first item decreases with time and that of the other increases with time, we consider the parameter values: $\xi_1 = -0.09, r = 0.16, i = 0.14, S_{p1} = 90.0, C_{p1} = 50.0, C_{h1} = 0.5, C_{11} = 0.5, C_{21} = 0.2, C_{31} = 25.0, T = 1, d_{11} = 100, d_{21} = -10, d_{31} = -40, \beta_1 = 4.0, \gamma_1 = 0.4, CAP = 1200, \xi_2 = -0.05, S_{p2} = 100.0, C_{p2} = 75.0, C_{h2} = 0.25, C_{12} = 0.6, C_{22} = 0.3, C_{32} = 30.0, d_{12} = 90, d_{22} = 20, d_{32} = 50, \beta_2 = 9.0, \gamma_2 = 0.5$. In this case, the optimal solutions are $\{R_1^* = 100.177, R_2^* = 129.218, \lambda = 0\}$ and $\{R_1^* = 85.1775, R_2^* = 95.4767, \lambda = 8.4685\}$. The first solution does not satisfy the constraint but the second solution satisfies the capacity constraint. So the required optimal solution is $\{R_1^* = 85.1775, R_2^* = 95.4767, \lambda = 8.4685\}$ with profit $\pi^* = 4436.02$.

Example 1.4. When demands of the items vary linearly over time with capacity constraint 1700 square units, then $d_{31} = 0$ and $d_{32} = 0$. In this case, the required solution of Examples 1.1–1.3 are $\{R_1^* = 114.444, R_2^* = 111.913, \pi^* = 6092.87\}$, $\{R_1^* = 106.91, R_2^* = 107.83, \pi^* = 6973.14\}$ and $\{R_1^* = 116.392, R_2^* = 111.913, \pi^* = 6968.31\}$ respectively.

Example 2.1. Let both the items be ameliorating with the following parameter values in appropriate units: $\xi_1 = 0.25, r = 0.16, i = 0.14, S_{p1} = 150, C_{p1} = 90.0, C_{h1} = 0.5, C_{11} = 0.5, C_{21} = 0.2, C_{31} = 25.0, T = 1, u_1 = 50, v_1 = 0.2, \beta_1 = 4.0, \gamma_1 = 0.4, CAP = 800, \xi_2 = 0.30, S_{p2} = 200.0, C_{p2} = 150.0, C_{h2} = 0.9, C_{12} = 0.6, C_{22} = 0.3, C_{32} = 30.0, u_2 = 40, v_2 = -0.35, \beta_2 = 9.0, \gamma_2 = 0.5$. In this case the optimal solution is $\{R_1^* = 73.03, R_2^* = 53.8811\}$ with profit $\pi^* = 6702.89$.

Example 2.2. Let the first item be ameliorating and demand increases over time, and the second item is deteriorating and demand decreases with time with the following parameter values in appropriate units: $\xi_1 = 0.4, r = 0.16, i = 0.14, S_{p1} = 5000, C_{p1} = 4000, C_{h1} = 0.5, C_{11} = 0.5, C_{21} = 0.2, C_{31} = 25.0, T = 1, u_1 = 50, v_1 = 0.2, \beta_1 = 4.0, \gamma_1 = 0.4, CAP = 4000, \xi_2 = -0.05, S_{p2} = 2000, C_{p2} = 1500, C_{h2} = 0.9, C_{12} = 0.6, C_{22} = 0.3, C_{32} = 30.0, u_2 = 40, v_2 = -0.35, \beta_2 = 9.0, \gamma_2 = 0.5$. In this case, the optimal solutions are $\{R_1^* = 593.311, R_2^* = 230.915, \lambda = 0\}$ and $\{R_1^* = 580.865, R_2^* = 186.282, \lambda = 11.2023\}$. In this case, the first solution does not satisfy the capacity constraint and so the required optimal solution is $\{R_1^* = 580.865, R_2^* = 186.282\}$ with profit $\pi^* = 589\,581.00$.

Example 2.3. If both the items be deteriorating with the same parameter values as in Example 2.2, with $\xi_1 = -0.04, \xi_2 = -0.05$ and $CAP = 4000$ square units, then the required optimal solution is $\{R_1^* = 430.415, R_2^* = 230.915\}$ with profit $\pi^* = 56148.10$.

4.1. Discussion of the figures

Form Figs. 1.1a–1.1d, we see that the salesmen's initiatives (E_1) for first item increases with time. This happens to clear stock due to amelioration of the item. In 2nd item, the inventory decreases due to deterioration and increasing demand over time that causes less investment in E_2, i.e., E_2 decreases over time.

We observe in Figs. 1.2a–1.2d that the demand is decreasing over time and E_1 and E_2 are increasing over time. It is rational to think that the investment in salesmen' initiatives is higher to clear the stock of obsolete items whose demands decrease over time. In the case of second item, salesmen' initiatives increase over time to clear additional stock due to amelioration. This case is also feasible in real situation.

From Figs. 1.3a–1.3d, it is seen that the salesmen' initiatives E_1 and E_2 decrease with time for deterioration of the items. Quite often, the investment in salesmen' initiatives decreases when inventory decreases with time due to demand and deterioration.

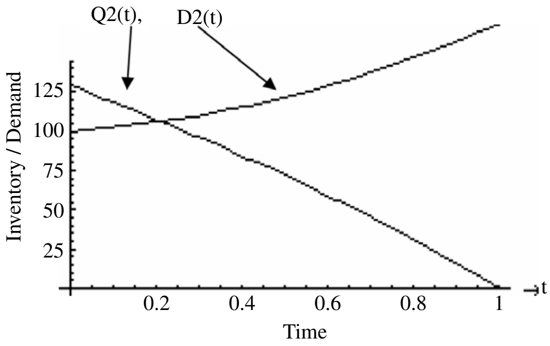

Fig. 1.1b. Inventory and demand of the item 2 versus time.

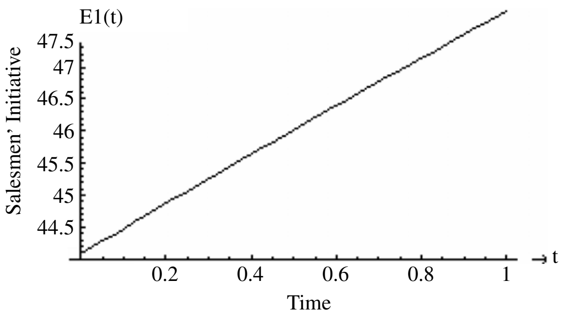

Fig. 1.1c. Salesmen' initiative versus time for the item 1.

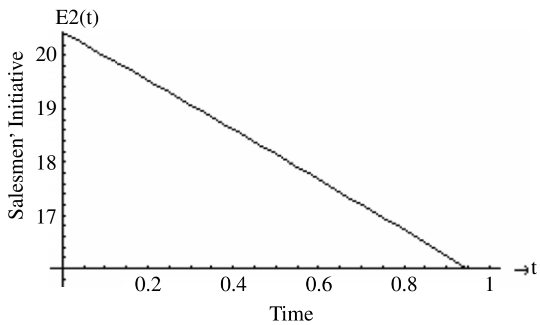

Fig. 1.1d. Salesmen' initiative versus time for the item 2.

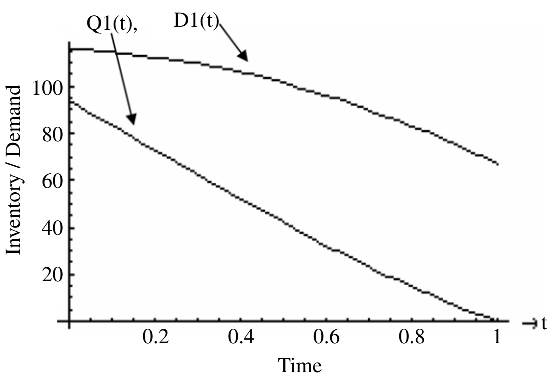

Fig. 1.2a. Inventory and demand of the item 1 versus time.

From Figs. 2.1a–2.1d, we find that E_1 decreases with time and E_2 increases with time. In this case, both the items are ameliorating, the first part of the demand function D_1 is an exponentially increasing function of time t and the first part

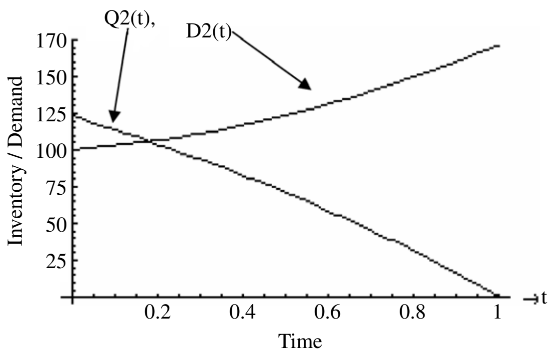

Fig. 1.2b. Inventory and demand of the item 2 versus time.

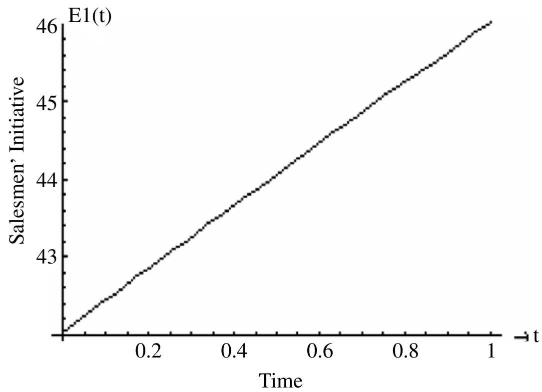

Fig. 1.2c. Salesmen' initiative versus time for the item 1.

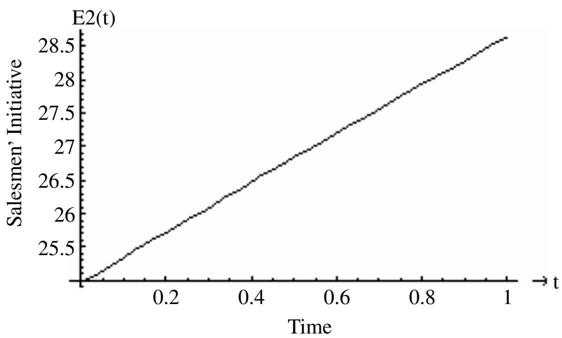

Fig. 1.2d. Salesmen' initiative versus time for the item 2.

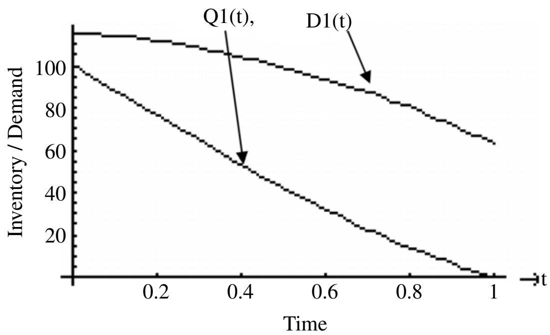

Fig. 1.3a. Inventory and demand of the item 1 versus time.

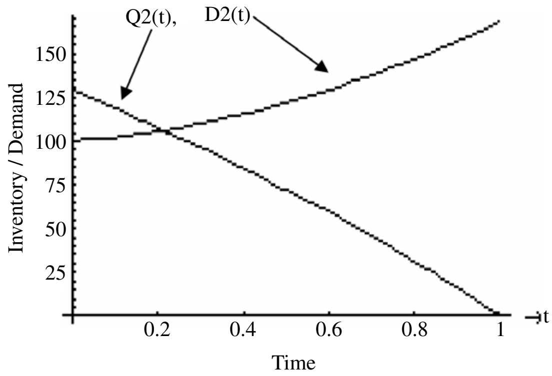

Fig. 1.3b. Inventory and demand of the item 2 versus time.

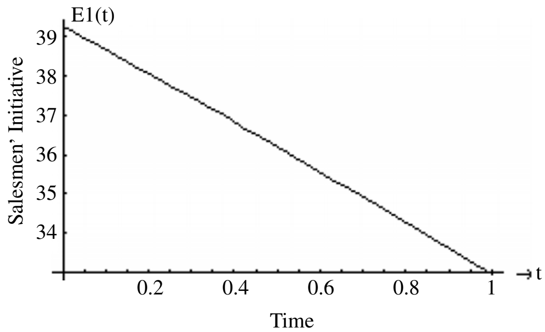

Fig. 1.3c. Salesmen' initiative versus time for the item 1.

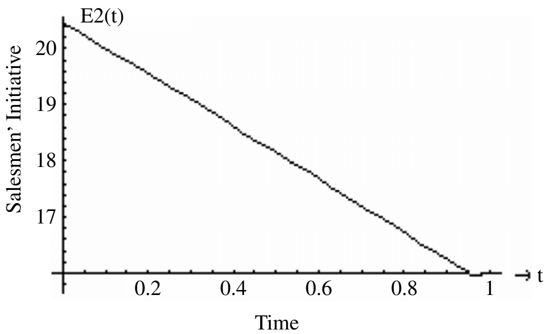

Fig. 1.3d. Salesmen' initiative versus time for the item 2.

of the demand function D_2 is an exponentially decreasing function of time t. It may happen that some managements do not invest more for salesmen' initiatives as the demand is exponentially increasing over time whereas, the management is bound to invest more in salesmen' initiatives (i.e., E_2 increases over time) to clear the stock of obsolete items (i.e., the demand decreases exponentially over time).

In the Figs. 2.2a–2.2d, it is observed that E_1 increases with time that increases the demand of the ameliorating item and E_2 decreases over time because of deterioration and decreasing demand of the second item. Generally, a good management decreases the investment in salesmen' initiatives for a smaller stock size due to deterioration. On the other hand, investment in salesmen's initiatives for ameliorating items should be increased because of excessive sales.

In Figs. 2.3a–2.3d, E_1 and E_2 decrease over time. In this case, both the items are deteriorating that results in less stock of the items. Consequently, E_1 and E_2 decrease with time. As both the items are deteriorating so the inventory level reaches downwards to adjust the demand of the customers and deterioration. As a result, the salesmen' initiatives decrease over time that causes the decreasing demand in the both cases.

In Example 1.1, the profit per unit item of the first type is higher than second type that results in more investment in E_1 and R_1 compared to E_2 and R_2. Whereas, in the Examples 1.2 and 1.3, the investment in R_2 is higher than in R_1 in spite of more profit per unit of the first item compared to second item. It happens for increasing demand of second item. Same fact

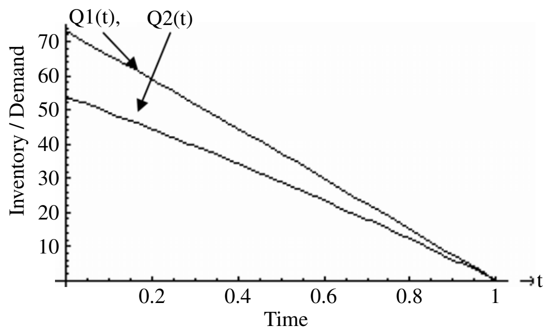

Fig. 2.1a. Inventories of items 1 and 2 versus time.

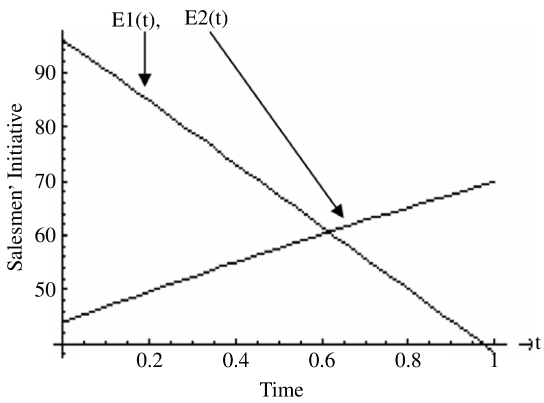

Fig. 2.1b. Salesmen' initiatives of items 1 and 2 versus time.

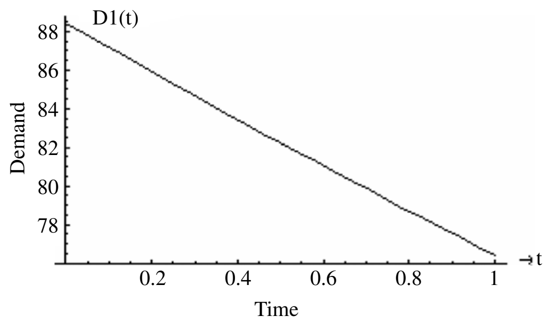

Fig. 2.1c. Demand of the item 1 versus time.

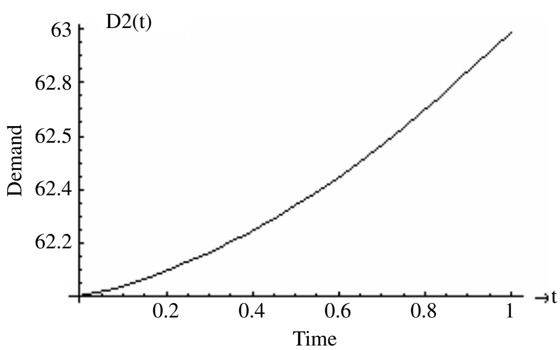

Fig. 2.1d. Demand of the item 2 versus time.

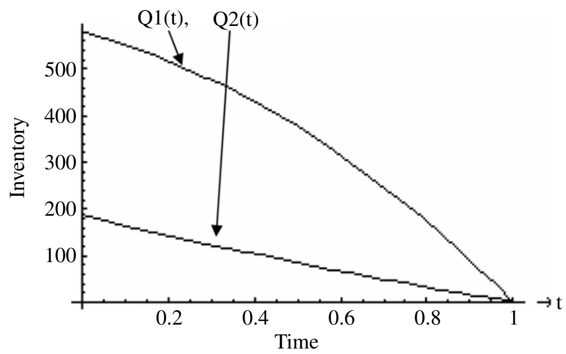

Fig. 2.2a. Inventories of items 1 and 2 versus time.

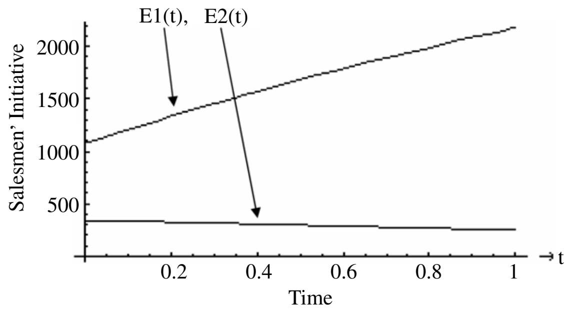

Fig. 2.2b. Salesmen' initiatives of items 1 and 2 versus time.

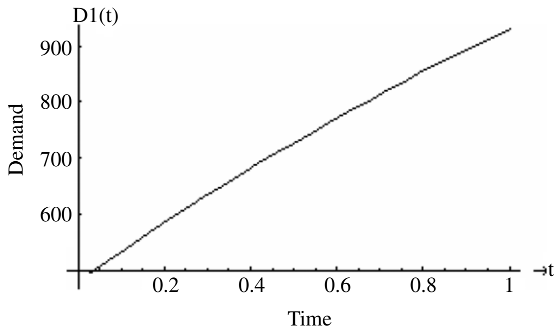

Fig. 2.2c. Demand of the item 1 versus time.

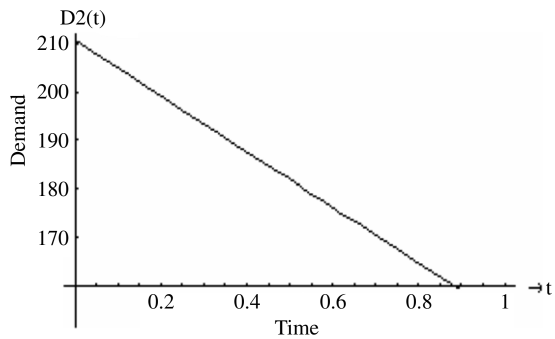

Fig. 2.2d. Demand of the item 2 versus time.

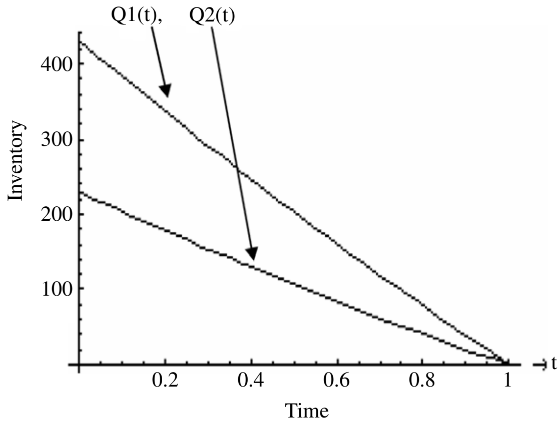

Fig. 2.3a. Inventories of items 1 and 2 versus time.

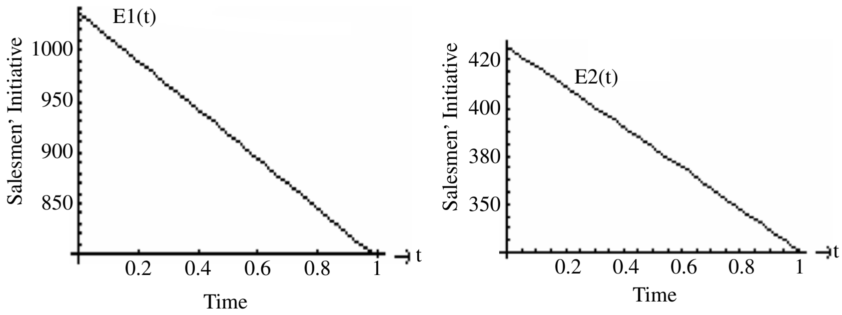

Fig. 2.3b. Salesmen' initiatives of items 1 and 2 versus time.

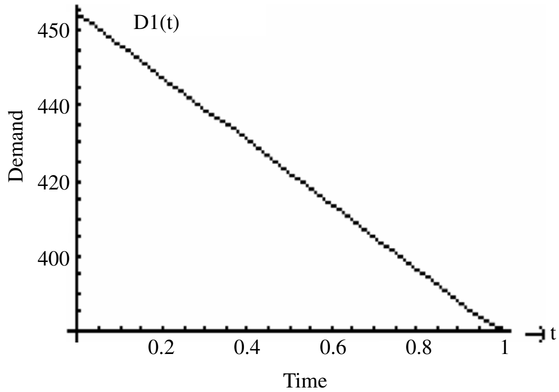

Fig. 2.3c. Demand of the item 1 versus time.

occurs in Example 1.4 also. In Examples 2.1–2.3, profit per unit of first item is higher than that of the second item. That is why initial lot-size R_1 is higher than R_2.

5. Conclusion

In an oligopolistic marketing system, quite often, the advertising and salesmen' effort play a crucial role to boost sale of the items that results in earning more money from a business sector in a given economy. Determination of demands and costs due to advertising and salesmen' effort is quite difficult. We have formulated the demands which are influenced by the above media. The approach in this paper is to concentrate on investment for the purpose of advertising and salesmen' effort which maximize the profit. It is also demonstrated that the optimum order sizes of multi-items are affected by inflation and time value of money, with the effects becoming more significant at higher values of inflation rates. This study illustrates the importance of taking into account inflation and time discounting, especially when time varying demands are influenced

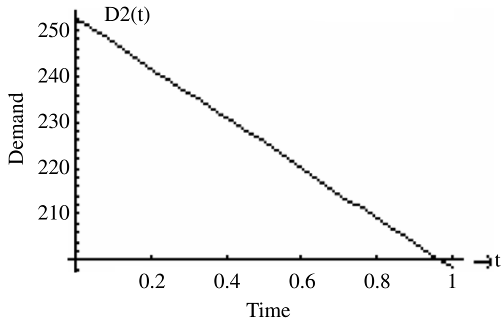

Fig. 2.3d. Demand of the item 2 versus time.

by advertising media and salesmen's initiatives. The model provides the major new contribution — the effect of advertising and salesmen's initiatives on demand — to operations in management practice.

Acknowledgements

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Appendix A. Determination of the parameters C_{1j}, C_{2j} and C_{3j}

Let, at an arbitrary time t, a market researcher observe the costs with various data of $E_j(t)$ of jth item. Suppose the costs of various values of $E_j(t)$ are as follows:

$E_j(t)$	$H_j(t)$
E_{1j}	H_{1j}
E_{2j}	H_{2j}
E_{3j}	H_{3j}
.	.
.	.
.	.
$E_{n-1,j}$	$H_{n-1,j}$
E_{nj}	H_{nj}

Let us consider a parabolic curve given by

$$H_j = C_{1j}E_j^2 + C_{2j}E_j + C_{3j}. \tag{17}$$

Now multiplying Eq. (16) by E_j and E_j^2, we have

$$E_jH_j = C_{1j}E_j^3 + C_{2j}E_j^2 + C_{3j}E_j \tag{18}$$

and

$$E_j^2H_j = C_{1j}E_j^4 + C_{2j}E_j^3 + C_{3j}E_j^2 \tag{19}$$

respectively. Putting the values of (E_{1j}, H_{1j}), (E_{2j}, H_{2j}), (E_{3j}, H_{3j}), ..., (E_{nj}, H_{nj}) in the Eqs. (17)–(19) and summing these, we have

$$\sum_{i=1}^{n} H_{ij} = C_{1j}\sum_{i=1}^{n} E_{ij}^2 + C_{2j}\sum_{i=1}^{n} E_{ij} + nC_{3j}, \tag{20}$$

$$\sum_{i=1}^{n} E_{ij}H_{ij} = C_{1j}\sum_{i=1}^{n} E_{ij}^3 + C_{2j}\sum_{i=1}^{n} E_{ij}^2 + C_{3j}\sum_{i=1}^{n} E_{ij} \tag{21}$$

and

$$\sum_{i=1}^{n} E_{ij}^2H_{ij} = C_{1j}\sum_{i=1}^{n} E_{ij}^4 + C_{2j}\sum_{i=1}^{n} E_{ij}^3 + C_{3j}\sum_{i=1}^{n} E_{ij}^2. \tag{22}$$

From the Eqs. (20)–(22), we have

$$C_{3j}=\left[\left\{\left(\sum_{i=1}^{n}E_{ij}^{4}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\right\}\left\{\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}H_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}H_{ij}\right)\right\}\right.$$
$$\left.-\left\{\left(\sum_{i=1}^{n}E_{ij}^{4}\right)\left(\sum_{i=1}^{n}H_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}^{2}H_{ij}\right)\right\}\left\{\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\right]$$
$$\Big/\left[\left\{\left(\sum_{i=1}^{n}E_{ij}^{4}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\right\}\left\{n\sum_{i=1}^{n}E_{ij}^{3}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}\right)\right\}\right.$$
$$\left.-\left\{n\sum_{i=1}^{n}E_{ij}^{4}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\left\{\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\right]$$

$$C_{2j}=\left[\left\{\left(\sum_{i=1}^{n}E_{ij}^{4}\right)\left(\sum_{i=1}^{n}H_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}^{2}H_{ij}\right)\right\}\left\{n\sum_{i=1}^{n}E_{ij}^{3}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}\right)\right\}\right.$$
$$\left.-\left\{n\sum_{i=1}^{n}E_{ij}^{4}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\left\{\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}H_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}H_{ij}\right)\right\}\right]$$
$$\Big/\left[\left\{\left(\sum_{i=1}^{n}E_{ij}^{4}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\right\}\left\{n\sum_{i=1}^{n}E_{ij}^{3}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)\left(\sum_{i=1}^{n}E_{ij}\right)\right\}\right.$$
$$\left.-\left\{n\sum_{i=1}^{n}E_{ij}^{4}-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\left\{\left(\sum_{i=1}^{n}E_{ij}^{3}\right)\left(\sum_{i=1}^{n}E_{ij}\right)-\left(\sum_{i=1}^{n}E_{ij}^{2}\right)^{2}\right\}\right]$$

and

$$C_{1j}=\left(\sum_{i=1}^{n}H_{ij}-C_{2j}\sum_{i=1}^{n}E_{ij}-nC_{3j}\right)\Big/\left(\sum_{i=1}^{n}E_{ij}^{2}\right).$$

In this way, the parameters C_{1j}, C_{2j} and C_{3j} can be estimated from marketing research.

Appendix B

The value of π depends on the path $Q_j = Q_j(t)$ $(j = 1, 2, \ldots, n)$ taken between the end points $(t = 0)$ and $(t = T)$. Suppose π has an extreme value when the path taken C_j^o is given by $Q_j = Q_j^o(t), 0 \le t \le T$. Let us consider the class of neighboring curves C_j^ϵ given by $Q_j = Q_j^\epsilon(t) = Q_j^o + \epsilon\eta_j(t)$ where ϵ is a small quantity and $\eta_j(t)$ is an arbitrary differentiable function of t. Therefore, the value of π for the path C_j^ϵ is given by $\pi(\epsilon) = \sum_{j=1}^{n}\int_0^T \phi_j^\epsilon\,\mathrm{d}t$, where $\phi_j^\epsilon = \phi_j(Q_j^o(t) + \epsilon\eta_j(t), \dot{Q}_j^o(t) + \epsilon\dot{\eta}_j(t), t)$. For maximum value of $\pi(\epsilon)$, we must have $\frac{\mathrm{d}\pi(\epsilon)}{\mathrm{d}\epsilon}\,|_{\epsilon=0} = 0$ and $\frac{\mathrm{d}^2\pi(\epsilon)}{\mathrm{d}\epsilon^2}\,|_{\epsilon=0} < 0$. Now

$$\begin{aligned}\frac{\mathrm{d}\pi(\epsilon)}{\mathrm{d}\epsilon} &= \sum_{j=1}^{n}\int_0^T\left\{\eta_j(t)\frac{\partial\phi_j^\epsilon}{\partial Q_j}+\dot{\eta}_j(t)\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right\}\mathrm{d}t\\
&= \sum_{j=1}^{n}\left\{\int_0^T\eta_j(t)\frac{\partial\phi_j^\epsilon}{\partial Q_j}\mathrm{d}t+\left[\eta_j(t)\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right]_0^T\right\}-\sum_{j=1}^{n}\int_0^T\eta_j(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right)\mathrm{d}t\\
&= \sum_{j=1}^{n}\left[\eta_j(t)\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right]_0^T+\sum_{j=1}^{n}\int_0^T\eta_j(t)\left\{\frac{\partial\phi_j^\epsilon}{\partial Q_j}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right)\right\}\mathrm{d}t\\
&= \sum_{j=1}^{n}\int_0^T\eta_j(t)\left\{\frac{\partial\phi_j^\epsilon}{\partial Q_j}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\phi_j^\epsilon}{\partial\dot{Q}_j}\right)\right\}\mathrm{d}t\end{aligned}$$

because $\eta_j(0) = \eta_j(T) = 0$ as Q is fixed at the end points $t = 0$ and $t = T$. Therefore, $\frac{d\pi(\epsilon)}{d\epsilon}|_{\epsilon=0} = 0$ gives us $\frac{\partial \phi_j}{\partial Q_j} - \frac{d}{dt}(\frac{\partial \phi_j}{\partial \dot{Q}_j}) = 0$, which is the necessary condition for an extreme value of π. Again,

$$\frac{d^2\pi(\epsilon)}{d\epsilon^2} = \sum_{j=1}^{n} \int_0^T \left\{ \eta_j^2 \frac{\partial^2 \phi_j^\epsilon}{\partial Q_j^2} + 2\eta_j\dot{\eta}_j \frac{\partial^2 \phi_j^\epsilon}{\partial Q_j \partial \dot{Q}_j} + \dot{\eta}_j^2 \frac{\partial^2 \phi_j^\epsilon}{\partial \dot{Q}_j^2} \right\} dt$$

so that

$$\left.\frac{d^2\pi}{d\epsilon^2}\right|_{\epsilon=0} = \sum_{j=1}^{n} \int_0^T \left\{ \eta_j^2 \frac{\partial^2 \phi_j}{\partial Q_j^2} + 2\eta_j\dot{\eta}_j \frac{\partial^2 \phi_j}{\partial Q_j \partial \dot{Q}_j} + \dot{\eta}_j^2 \frac{\partial^2 \phi_j}{\partial \dot{Q}_j^2} \right\} dt. \quad (23)$$

Now, we have

$$\frac{\partial \phi_j}{\partial Q_j} = e^{-\delta t} \left\{ S_{pj}\xi_j - C_{hj} - \frac{C_{1j}\xi_j}{\gamma_j} + \frac{2C_{2j}\xi_j}{\gamma_j^2}(\dot{Q}_j - \xi_j Q_j + a_j) \right\},$$

$$\frac{\partial^2 \phi_j}{\partial Q_j^2} = -\frac{2C_{2j}\xi_j^2}{\gamma_j^2} e^{-\delta t},$$

$$\frac{\partial^2 \phi_j}{\partial Q_j \partial \dot{Q}_j} = \frac{2C_{2j}\xi_j}{\gamma_j^2} e^{-\delta t},$$

$$\frac{\partial \phi_j}{\partial \dot{Q}_j} = e^{-\delta t} \left[-S_{pj} + \frac{C_{1j}}{\gamma_j} - \frac{C_{2j}}{\gamma_j^2}(\dot{Q}_j - \xi_j Q_j + a_j) \right]$$

and

$$\frac{\partial^2 \phi_j}{\partial \dot{Q}_j^2} = -\frac{2C_{2j}}{\gamma_j^2} e^{-\delta t}.$$

Using the above partial derivatives in Eq. (23), we have

$$\left.\frac{d^2\pi}{d\epsilon^2}\right|_{\epsilon=0} = -\sum_{j=1}^{n} \frac{2C_{2j}}{\gamma_j^2} \int_0^T e^{-\delta t}[\xi_j^2\eta_j^2(t) - 2\xi_j\eta_j(t)\dot{\eta}_j(t) + \dot{\eta}_j^2(t)]dt$$

$$= -\sum_{j=1}^{n} \frac{2C_{2j}}{\gamma_j^2} \int_0^T e^{-\delta t}\{\xi_j\eta_j(t) - \dot{\eta}_j(t)\}^2 dt < 0,$$

because $e^{-\delta t}\{\xi_j\eta_j(t) - \dot{\eta}_j(t)\}^2 > 0$. Hence, the above sufficient condition, $\frac{d^2\pi}{d\epsilon^2}|_{\epsilon=0} < 0$, implies that the functional π has a maximum value only.

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