

Comparative study of spanning cluster distributions in different dimensions

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April 10, 2017

Abstract

The probability distributions of the masses of the clusters spanning from top to bottom of a percolating lattice at the percolation threshold are obtained in all dimensions from two to five. The first two cumulants and the exponents for the universal scaling functions are shown to have simple power law variations with the dimensionality. The cases where multiple spanning clusters occur are discussed separately and compared.

Percolation is a subject which has been studied extensively for the last few decades. The relevance of percolation in various areas of physics is also well established. Although many of the properties of percolating systems are well understood and studied, there still remains a lot of details to be explored and intricate questions to be addressed[1, 2].

At the critical point (percolation threshold) there appears for the first time a cluster spanning the whole lattice. The spanning cluster is a fractal in the sense its mass M scales with the length as L^D where $D < d$, d being the spatial dimension and D the fractal dimension.

Distribution of cluster masses at and away from criticality has been studied in detail [1]. Conditional probability distributions for spanning cluster (SC) masses, their moments, and other variables like the shortest path etc. also appear in the literature [3, 4, 5, 6, 7, 8]. While the distribution for the

cluster masses show a power law behaviour at criticality [1], the probabilities of spanning clusters masses have an entirely different variation [4, 7].

In this article we report a study of the probability distribution functions of the masses of the spanning clusters which span the lattice from top to bottom and a comparative analysis for different dimensions. Here the condition that the cluster spans along *one* particular direction of the lattice is necessary and sufficient and hence the condition of the spanning along all directions is relaxed.

We examine the distribution functions separately for the two cases (a) when there exists only one SC (b) when there are more than one coexisting spanning clusters. Although case (a) occurs predominantly, case (b) has recently been established [9] to have a finite non-zero probability of occurrence even in two dimensions. Little is known about the distribution functions of masses in case (b) and we attempt to extract as much information for this as possible.

We have simulated L^d hypercubic lattices in d dimensions with helical boundary conditions where each site is occupied with a probability p . The clusters are identified using the Hoshen Kopelman algorithm. The largest lattices considered have sizes $L = 800$ in $d = 2$, $L = 60$ in $d = 3$, $L = 30$ in $d = 4$ and $L = 15$ in $d = 5$. A maximum of 10^6 initial configurations (for the smallest lattices) were generated at the percolation threshold p_c where the values of p_c given in ref [1] have been used.

As it is known that M scales as L^D , where D is the fractal dimension of the spanning cluster, we have directly measured the probability distribution of M/L^D , i.e., the bin sizes are chosen to be proportional to $1/L^D$. We normalise the probabilities so that the total probability is unity. We find that the *normalised* probabilities plotted against $m = M/L^D$ all collapse on a single curve for different system sizes. This happens in all dimensions from two to five. As an example, the collapses in two and three dimensions are shown in Fig. 1. Finite sizes effects are stronger in higher dimensions.

The probability distribution is of the form:

$$P(M/L^D) \propto f(M/L^D) \tag{1}$$

where

$$f(x) = Ax^\alpha \exp(-\gamma x^\beta) \tag{2}$$

Fitting the universal scaling function in the above form is best in two dimensions. However, for the tail of the distribution, the above form gives a

very good fit even in higher dimensions. It maybe added here that the tail of the distribution becomes important in many problems, e.g., in problems related to stock market fluctuations [10].

The form of the probability distribution obtained here is very close to that studied in [4]. We do not get any prefactor for the scaling functions here as the bin sizes are proportional to $1/L^D$. (This factor, as $1/M$ appears when the normalised probabilities are also divided by the bin sizes as in [4].) However, the exponents obtained in the present study are totally different. For example, in two dimensions, $\beta = 6.7 \pm 0.1$ and $\gamma \sim 10$ while in [4] the corresponding values are ~ 19 and $\sim 10^{-8}$ respectively. The possible reasons for this discrepancy are discussed later.

Quantitative comparison of the distributions for different dimensions is done by calculating the first and second cumulants of the distributions and studying their behaviour with the dimensionality. In each dimension, we extrapolate these results for $1/L \rightarrow 0$ as there are some finite size effects. In general we fit the cumulants as linear functions of $1/L$ to extrapolate. The extrapolated values vary as simple power laws as given below (see Fig. 2)

$$\langle m \rangle \sim d^{-a} \quad (3)$$

$$\sigma^2 = \langle m^2 \rangle - \langle m \rangle^2 \sim d^{-2b} \quad (4)$$

with $a = 1.65 \pm 0.1$ and $b = -0.25 \pm 0.01$.

We find the scaling form of the distribution by fitting with appropriate values of A, α, γ and β . Again α, A and β show simple power law variations with dimensionality: the powers are close to -2 for α and β (see Fig. 2), γ apparently has no dependence on the dimensionality.

The case when there exists more than one spanning clusters has also been explored. We rank the spanning clusters by their sizes and obtain separately the distributions for the r th largest cluster when the total number of spanning clusters is n . For two dimensions, when there exists two SC's, the distribution function for the larger SC is clearly different from that of the unique SC (see Fig. 3). In particular, the distribution is more symmetric in comparison to that of the unique SC and more sharply peaked. One concludes that there is a different universal function for the SC's when $n > 1$. However, an attempt to find the exact form of this distribution is difficult because of the fluctuations in the data. This fluctuation is unavoidable as the probability of cases with $n > 1$ is very small.

However, certain features of the distribution of the masses are available from the present study. The mean value of M/L^D varies appreciably, for example, in the $n = 2$ case in two dimensions: for the largest SC $\langle m \rangle_{n=2,r=1} \sim 0.42$ compared to $\langle m \rangle_{n=1,r=1} \sim 0.58$, where $m_{n,r}$ is the mass of the r th largest SC when the number of SC is n . Whereas, in higher dimensions, the largest spanning clusters in the multiple SC cases become comparable in size whatever be the number of SC's. Such a result indicates that in the higher dimensions, the largest SC is unaffected by the presence of others - consistent with the fact that it is easier to conceive independent coexisting spanning clusters along one direction in large dimensions.

However, the width becomes smaller and σ shows a power law behaviour with n :

$$\sigma = (\langle m^2 \rangle - \langle m \rangle^2)_{n,r=1} \sim n^{-2c} \quad (5)$$

with $c \sim 0.3 \pm 0.02$. This is shown for the case for five dimensions where one can obtain an appreciable number of SC's numerically (Fig. 4).

We also get conclusive results for the ratios of the SC masses when e.g., $n = 2$. This ratio is around 1.4 in case of $d = 2$ (also obtained in [11]) and 2.2 for $d = 5$. This indicates that the larger SC becomes dominant in higher dimensions.

In summary, we have obtained several quantities related to the distribution of the spanning cluster masses systematically varying with the dimensionality. A universal scaling function is obtained in each dimension, having similar form with dimension dependent exponents. These dependences appear as simple power law variations of the dimensionality. The dependence of the exponents on the dimensionality is not surprising; in fact, the variation of the cumulants with the dimension is related to the dimension-dependent exponents. One can, in principle, also fit these exponents as polynomial functions of $(6 - d)$ (as 6 is the upper critical dimension in percolation). We keep the question of exact variation of the exponents as function of dimension open as it is difficult to obtain a concrete form from the numerical simulations only.

As mentioned before, these exponents do not match with some earlier results [4, 7]. However, it must be noted that in these studies, the conditional probabilities were obtained with different boundary conditions in the sense that the clusters were required to span in *all* directions. There are differences in the values of $\langle m \rangle$ and σ as well, $\langle m \rangle$ being smaller when the SC spans along

one direction only. Hence the universal function seems to be highly dependent on the boundary conditions and in that sense only weakly universal. We also do not attempt to fit the scaling function by an alternative form as that admits yet another parameter and the fitting becomes difficult to handle. Another less important point is that in [4], the distributions are obtained for any number of SC present. Although the distributions in the one SC case and two SC case are quite different, it should however not matter as the latter has a very small probability of occurrence.

We have also shown qualitatively that the distributions for the SC's in multiple SC case are different from that of the one SC case. The second cumulant of the largest SC, in particular, apparently varies as a power law with n . The mean $\langle m \rangle$ for the largest SC varies appreciably in low dimensions but becomes a constant in higher dimensions.

The author is grateful to B. K. Chakrabarti for discussions and D. Stauffer for critical comments on the manuscript. She also thanks A. Aharony for bringing ref [4] to notice.

References

- [1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1994).
- [2] D. Stauffer, *Physica A* **242**, 1 (1997).
- [3] S. Havlin, B. L. Trus, G. H. Weiss and D. Ben-Avraham, *J. Phys A* **18** L247 (1985).
- [4] A. U. Neumann and S. Havlin, *J. Stat. Phys.* **52** 203 (1988)
- [5] H. Saleur and B. Derrida, *J. Physique* **46** 1043 (1985).
- [6] P. Sen and P. Ray, *J. Stat. Phys.* **59** 1573 (1990).
- [7] J.-P. Hovi and A. Aharony, *Phys. Rev* **56** 172 (1997).
- [8] N. Jan, D. Stauffer and A. Aharony, *J. Stat Phys.* **92** 325 (1998)
- [9] L. de Arcangelis, *J. Phys. A* **20**, 3057 (1987); M. Aizenman, *The IMA Volumes in Mathematics and its Applications* (Springer-Verlag, 1996);

Nucl. Phys. B **485** 551 (1997); C.-K. Hu and C.-Y. Lin, Phys. Rev. Lett. **77**, 8 (1996); P. Sen, Int. J. Mod. Phys. C **7**, 603 (1996); L. Shchur and S. S. Kosyakov, Int. J. Mod. Phys. C **8** 443 (1997); P. Sen, Int. J. Mod. Phys. C **8** 229 (1997); J. Cardy J. Phys. A **31** L105 (1998)

- [10] R.N. Mantegna und H. E. Stanley, *Econophysics: An introduction*, Cambridge University Press, Cambridge 1999, in press; J.P. Bouchaud et al., Physics World **12**, 25 (January 1999).
- [11] P. Sen and A. Aharony, Int. J. Mod. Phys. C, **8** 1169 (1997).

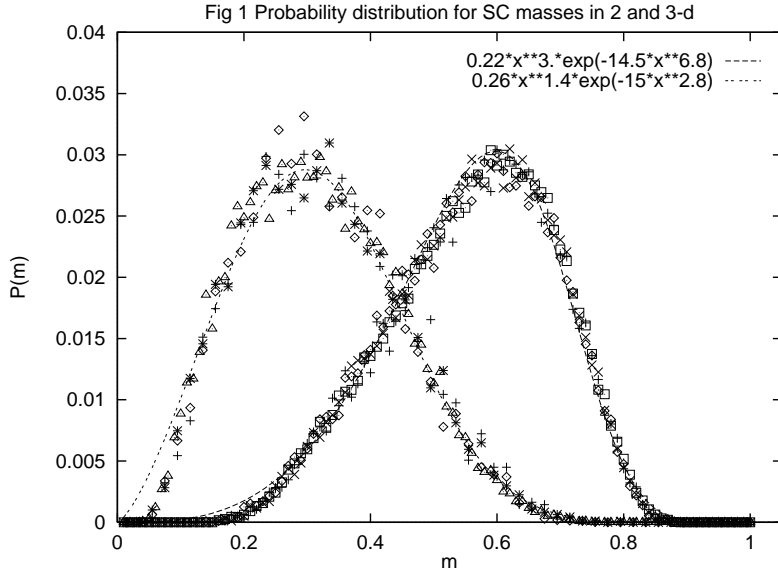


Figure 1: The probability distribution for SC masses for different lattice sizes are shown for $d = 3$ (left) and $d = 2$ (right). The dashed curves are possible fittings for the universal functions.

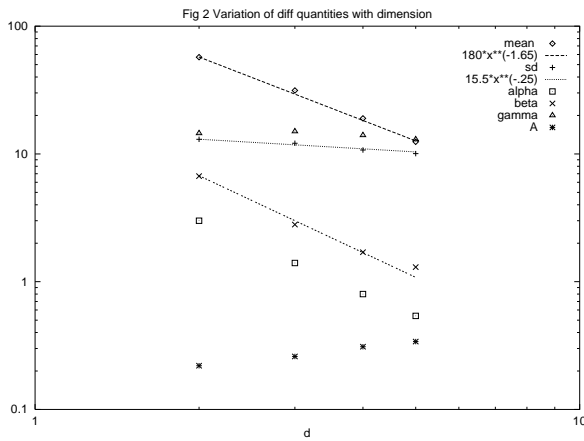


Figure 2: Variations of different quantities with dimension. Power law fittings for the first two cumulants are shown. The dashed line has a slope equal to -2.

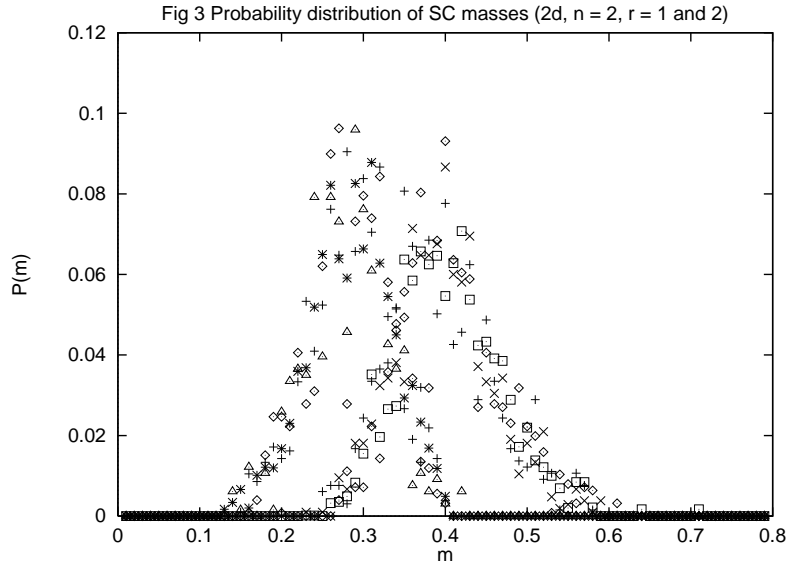


Figure 3: The probability distribution for the two coexisting SC's are shown for different lattice sizes in two dimensions.

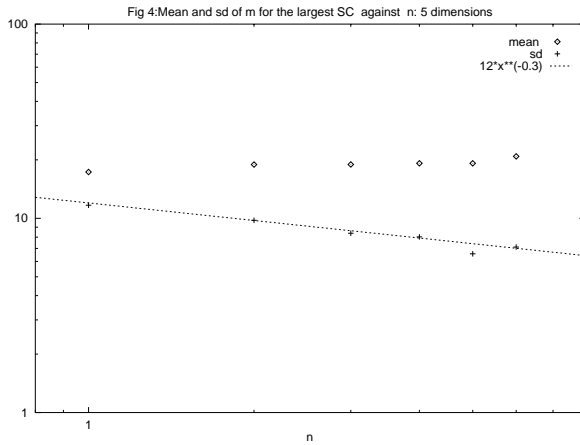


Figure 4: The variation of the first two cumulants of the largest SC is shown against the number of SC in five dimensions. The lattice size is 13^5 .