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Charged Scalar Fields in an External Magnetic Field: Renormalisation and Universal Diamagnetism

Debnarayan Jana *
Raman Research Institute, Bangalore 560 080, India
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The physical and mathematical mechanism behind diamagnetism of N (finite) spinless bosons (relativistic or non-relativistic) is well known. The mathematical signature of this diamagnetism follows from Kato's inequality while its physical way of understanding goes back to Van Leewen. One can guess that it might be true in the field theoretic case also. While the work on systems with a finite number of degrees of freedom suggests that the same result is true in a field theory, it does not by any means prove it. In the field theoretic context one has to develop a suitable regularisation scheme to renormalise the free energy. We show that charged scalar fields in (2+1) and (3+1) dimensions are always diamagnetic, even in the presence of interactions and at finite temperatures. This generalises earlier work on the diamagnetism of charged spinless bosons to the case of infinite degrees of freedom. We also discuss possible applications of the theory.

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I. INTRODUCTION

A classical gas of charged point particles is non-magnetic, by Van Leeuwen's Theorem [1]. But quantum mechanically, free spinless Bose particles in a uniform magnetic field show diamagnetism [2]. One finds that the energy of the system in a magnetic field is higher than in the absence of the field. Simon [3] proved quite generally that the ground state of non-relativistic spinless bosons interacting through an arbitrary potential *always increases* in a magnetic field. He went on to extend this result by showing that the free energy in the presence of a magnetic field is always greater than the free energy in the absence of a magnetic field at all temperatures [3]. An alternative proof of this result is given in [4]. All this work which deals with systems with a finite number of degrees of freedom suggests that diamagnetism is a universal property of spinless bosons. In field theory (which describes systems with an infinite number of degrees of freedom) charged spinless bosons are described by complex scalar fields. One might therefore expect that charged scalar fields would also show diamagnetic behaviour. In fact Brydges et. al. [5] have shown this diamagnetic inequality for the interacting scalar fields on a lattice. This result insisted on an ultraviolet cutoff. Here we want to extend to the continuum case with proper account of the renormalization. With this motivation we study the magnetic behavior of scalar field theories in both (2+1) and (3+1) dimensions [1].

The paper is organised as follows. The first part deals with finite temperature free scalar field theory in the presence of an external homogeneous magnetic field. Here, we explicitly calculate the partition function and the free energy as a function of the applied magnetic field in (2+1) dimension. This expression is formally divergent. Using a suitable regularization scheme we compute the *difference* in the free energy (with and without the magnetic field) and obtain a finite answer. This difference is also shown to be positive, thus establishing the diamagnetic behaviour of free charged scalar fields. We establish the renormalisation of the free energy through numerical and analytical calculations. Next we discuss the renormalization of the free energy in (3+1) dimension. In this connection we also show the quantitative difference between the responses in odd and even spatial dimension. More specifically, we point out the drastic change of the behaviour of the free energy at strong magnetic field in the zero temperature limit.

We then move on to interacting scalar field theory in the second part. Here, we cannot evaluate the partition function explicitly. Nevertheless we prove the universal diamagnetism of scalar fields by assuming a finite momentum cutoff in the theory. If the theory is renormalizable, then one can take this cutoff to infinity while maintaining finiteness of all physical quantities. In both cases the results obtained are exact. In fact we present a *non-perturbative* approach to prove this diamagnetic inequality for this interacting case in an arbitrary spatial dimension. Before the conclusion, we illustrate few examples of spinless bose systems and point out the variation of the susceptibility with temperature above the critical temperature in a physical way.

Although this work is rederivation of a known result, we hope that it has a value beyond the purely pedagogical. Here in most of the paper we will talk about the response of the vacuum. The vacuum as a medium has

[1] At this point we would like to emphasize that it is extremely naive and dangerous to conclude the response of the field theory from the corresponding N particle systems. For example, N fermions at $T = 0$ show Pauli paramagnetism while its counter field theory QED vacuum is diamagnetic.

been a fruitful picture in trying to develop intuition about abelian and non-abelian gauge theories. In fact we point out several interesting points regarding the nature of the vacuum say for example QED and QCD in presence of an external magnetic field.

II. FREE CASE

A. (2+1) dimension

In this subsection we calculate the free energy of free scalar fields in the presence of an external uniform magnetic field. For ease of presentation we work here first in two spatial dimensions. The interesting physics takes place in the plane normal to the applied field. In the next subsection we will discuss the renormalization of free energy in (3+1) dimension.

Let Φ be a complex scalar field which describes charged spinless Bosons. The Lagrangian density of a free charged scalar field in the presence of a constant homogeneous external magnetic field is given by

$$\mathcal{L} = (D_\mu \Phi)^* (D^\mu \Phi) - m^2 (\Phi^* \Phi) \tag{1}$$

where $\mu = 0, 1, 2,$

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu \tag{2}$$

and m and e are the mass and charge respectively. (We set $\hbar = 1$ and $c = 1$). Now, we write the complex field in term of two real fields Φ_1 and Φ_2.

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}, \Phi^* = \frac{\Phi_1 - i\Phi_2}{\sqrt{2}} \tag{3}$$

This theory has a global U(1) symmetry and therefore a conserved Noether charge Q, given by

$$Q = \int d^2x \, (\Pi_1 \Phi_2 - \Phi_1 \Pi_2) \tag{4}$$

where

$$\Pi_i = \partial_0 \Phi_i \tag{5}$$

The Hamiltonian density of the system is given by

$$\mathcal{H} = \frac{1}{2}(\Pi_1^2 + \Pi_2^2) + \frac{1}{2}(\nabla\Phi_1)^2 + \frac{1}{2}(\nabla\Phi_2)^2 + \frac{1}{2}(m^2 + e^2 A^2)(\Phi_1^2 + \Phi_2^2) - \vec{j} \cdot \vec{A}, \tag{6}$$

where the current density j is given by

$$\vec{j} = -\mathrm{e} \, (\Phi_1 \vec{\nabla} \Phi_2 - \Phi_2 \vec{\nabla} \Phi_1). \tag{7}$$

We now suppose that the external magnetic field is uniform in the $x - y$ plane.

We choose the temporal gauge ($A_0 = 0$). The constant magnetic field B is

$$B = \partial_x \, A_y - \partial_y \, A_x \tag{8}$$

where A_x and A_y are independent of t.

The action of this theory is

$$S = \int_0^\beta \int d^2x \; d\tau \; \mathcal{L}(\Phi, \Phi^*, A), \tag{9}$$

where τ is the imaginary time variable which runs from 0 to β (=$1/(k_B T)$), the inverse temperature. The action defined above is quadratic and so the partition function can be evaluated exactly. As is usual in finite temperature field theory [6], we impose periodic boundary conditions for Bosonic fields

$$\Phi(\mathbf{x}, 0) = \Phi(\mathbf{x}, \beta). \tag{10}$$

Now, the partition function of this theory can be written as

$$Z(B) = \int \mathcal{D}[\Pi_1]\mathcal{D}[\Pi_2] \int \mathcal{D}[\Phi_1]\mathcal{D}[\Phi_2] \exp\Bigg[\int d\tau \; d^2x \Bigg(\mathrm{i}\Pi_1 \frac{\partial \Phi_1}{\partial \tau} + \mathrm{i}\Pi_2 \frac{\partial \Phi_2}{\partial \tau} - H(\Phi_1, \Phi_2, \Pi_1, \Pi_2) + \mu(\Phi_2 \Pi_1 - \Phi_1 \Pi_2)\Bigg)\Bigg] \tag{11}$$

Here μ is the chemical potential associated with the conserved charge Q. The charge density (Q/A) has to be contrasted with the usual number density. The charge density refers here to the difference between the particle density and the antiparticle density and hence can take any sign while the number density by definition is always positive. In that sense μ is not the usual chemical potential used in Grand Canonical Ensemble. We pick the gauge in which the vector potential $\mathbf{A}$ is (-By, 0), and expand the complex scalar field in terms of modes adapted to the present situation. These modes solve the Klein-Gordon equation in an external magnetic field. The eigenfunctions are labelled by one discrete (l) and one continuous p_x quantum number and the spectrum depends on l only. In the gauge we choose, the modes are plane waves in the x direction and harmonic oscillator (i.e. gaussian) wavefunctions in the y direction.

The spectrum is given by

$$\omega_l^2 = m^2 + (2l + 1)eB, \quad l = 0, 1, 2.......\infty \tag{12}$$

The degeneracy of these states for fixed l is $eAB/2\pi$, where A is the area of the system. So, these modes can be thought of as quantized harmonic oscillators. Expanding the fields Φ_1 and Φ_2 in these modes the system reduces to a collection of harmonic oscillators with frequency ω_l.

By standard manipulations [6], we get the free energy as

$$F(B) = 2\pi \mathrm{e} A B \sum_{l=0}^{\infty} \left[\omega_l + \frac{1}{\beta} \ln(1 - \exp(-\beta(\omega_l - \mu)) + \frac{1}{\beta} \ln(1 - \exp(-\beta(\omega_l + \mu)) \right] \quad (13)$$

The first term in the square brackets corresponds to the zero point fluctuation of the vacuum and the other two terms are finite temperature contributions of the particles and antiparticles respectively. Without the above formalism one can also calculate the vacuum energy by summing up the zero point energies for the modes of all the fields. Of course when we sum over all modes the energy actually diverges. One has to introduce a regularisation scheme or a cutoff so that the vacuum energy becomes finite.

It is easy to notice that this zero point energy is divergent due to the summation of infinity number of modes (Landau levels). In conventional field theory this infinite zero point energy is always discarded; since it can be reabsorbed in a suitable redefinition of the zero of energy. This is justified in the sense that the infinite zero point energy is unobservable. However, the change in zero point energy caused by the external constraint is finite and observable. So, according to Casimir's [7] idea, the physical vacuum energy can be defined as the difference between the zero point energy corresponding to the vacuum configurations with constraints and the one corresponding to the free vacuum configurations. This definitions must be supplemented in general with a regularisation prescription in order to obtain a finite final convergent expression. We will first compare the free energy of the system with and without the magnetic field at zero temperature.

1. Zero Temperature Case

The free energy of the system in presence of the magnetic field at zero temperature is given by

$$F_0(B) = 2\pi A \ eB \sum_{l=0}^{\infty} \ \omega_l, \quad (14)$$

where $\omega_l^2 = m^2+(2l+1)eB$. Obviously, this sum diverges. In order to obtain a finite answer we need to impose a cutoff L in the sum (14). Then the free energy becomes

$$F_0(B, L) = 2\pi A \ eB \sum_{l=0}^{L} \ \omega_l. \quad (15)$$

The free energy in the absence of the magnetic field at zero temperature is given by the divergent expression

$$F_0(0) = 2\pi A \int_0^{\infty} \ pdp \ \sqrt{(p_x^2 + p_y^2) + m^2} \quad (16)$$

We regularize this expression by imposing a cutoff Λ. Then the free energy (16) becomes

$$F_0(0, \Lambda) = 2\pi A \int_0^{\Lambda} \ pdp \ \sqrt{(p_x^2 + p_y^2) + m^2} \quad (17)$$

In order to compare the free energies in equations (15) and (17), we choose the cutoffs L and Λ in such a way that both systems have the same number of modes. Such a procedure can be justified on physical grounds if one imagines that the magnetic field is turned on adiabatically [8]. In the Appendix A and B, we show that this mode matching scheme is the correct regularisation procedure. More specifically, in Appendix A we compare this scheme with another scheme which looks apparently correct and in Appendix B we justify this mode matching scheme from the field theoretic point of view.

Counting the modes upto the L-th Landau level we find

$$2\pi A \ eB \sum_{l=0}^{L} \ 1 = 2\pi A \ eB(L+1) \quad (18)$$

Similarly, for the momentum cutoff upto Λ we get the modes without the magnetic field as

$$2\pi A \int_0^{\Lambda} pdp = \pi \ A\Lambda^2 \quad (19)$$

Equating these gives us

$$\Lambda^2 = 2 \ eB \ (L+1) \quad (20)$$

Now, the free energy in absence of the magnetic field depends on magnetic field through the momentum cutoff and is given by

$$F_0(0, L) = 2\pi A \int_0^{\Lambda(B)} \ pdp \ \sqrt{p^2 + m^2} \quad (21)$$

The difference between the two free energies is given by

$$\Delta F(B, L) = F_0(B, L) - F_0(0, L) \quad (22)$$

We define $f(B) = F_0(B, L)/2\pi A$, $\tilde{f}(B) = F_0(0, L)/2\pi A$ and $\Delta f(B) = f(B) - \tilde{f}(B)$. Numerically evaluating these sums one can show that for finite L, $\Delta f(B)$ the difference between two large quantities is positive. As the cutoff L goes to infinity, $\Delta f(B)$ becomes the difference between two infinities. In this limit we find that $\Delta f(B)$ tends to a finite value. Thus, the susceptibility at zero temperature in the relativistic case is non-zero. This vacuum susceptibility can be interpreted as due to virtual currents.

We now show analytically that $\Delta F(B)$ is positive i.e. the vacuum is diamagnetic. Note that

$$\Delta f(B) \ = \ f(B) - \tilde{f}(B) = \sum_{l=0}^{\infty} \ a_l(B, m) \quad (23)$$

where $a_l(B,m)$ is given by

$$a_l(B,m) = eB\left[\sqrt{m^2+(2l+1)eB} - \int_0^1 d\alpha\ \sqrt{m^2+2(l+\alpha)eB}\right] \tag{24}$$

Introducing a dimensionless quantity $\rho = \frac{eB}{m^2}$ the above equation becomes

$$a_l(\rho) = \rho\left[\sqrt{1+(2l+1)\rho} - \int_0^1 d\alpha\ \sqrt{1+2(l+\alpha)\rho}\right] \tag{25}$$

The positivity of $a_l(\rho)$ for each l can be proved geometrically. Defining $z_l = (1+2l\rho)/2\rho$ and $f(\alpha) = \sqrt{z_l+\alpha}$, the coefficient $a_l(\rho)$ can be rewritten in terms of $c_l(\rho)$ as

$$c_l(\rho) = \frac{a_l(\rho)}{\sqrt{2}\ \rho^{3/2}} = f(1/2) - \int_0^1 d\alpha\ f(\alpha). \tag{26}$$

Since, the function $f(\alpha)$ is convex, the area under the tangent drawn at $\alpha = 1/2$ is greater than the area under the curve (see the figure below).

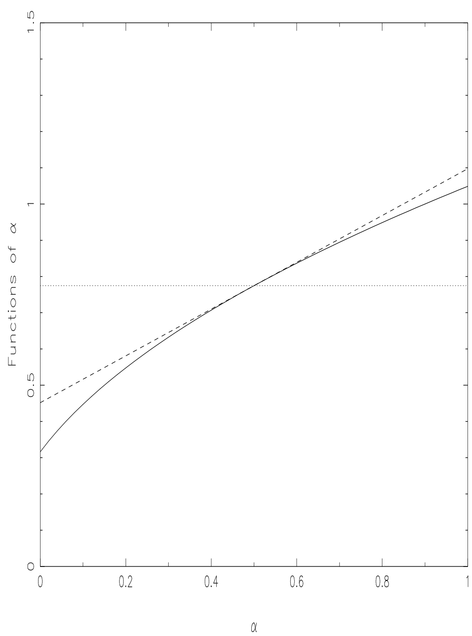

Figure 1: The full line is the curve $f(\alpha) = \sqrt{.1+\alpha}$. The dashed line is the tangent to the above curve at $\alpha = .5$. It is straightforward to see that the area under the curve is less than the area under the tangent. Also the area under the tangent is the same as the area of the rectangle. The area of this rectangle is given by $f(1/2) \times 1 = f(1/2)$. Hence, the positivity of $a(.1)$ is proved. This can be generalised to any positive value of z_l.

This shows that $c_l(\rho)$ is positive. To show the convergence of the sum (23) we note that

$$\int_0^1 d\alpha\ f(\alpha)\ \leq\ \left[f(1/2)\ -\ \frac{f(0)+f(1)}{2}\right] \tag{27}$$

Now, applying mean value theorem twice one can easily show that

$$\int_0^1 d\alpha\ f(\alpha) \leq \frac{1}{16(z_l+\alpha)}3/2 \tag{28}$$

Thus the coefficient $c_l(\rho)$ is positive for each l and the sum converges, hence the diamagnetic inequality is established.

2. Massless limit

In this section we want to discuss the behaviour of the leading term of the free energy in the massless limit and its consequences. It is easy to see from the zero temperature free energy that the magnetisation in the zero mass limit is given by

$$\mathrm{M(B)} \sim\ -\sqrt{B} \tag{29}$$

So, the susceptibility in this zero mass limit is given by

$$\chi(B)\ \sim\ -\ \frac{1}{\sqrt{B}} \tag{30}$$

which diverges [9] as B goes to zero. This divergence of the susceptibility is reminiscent of the fact that the free energy $F(B) \sim B^{3/2}$. This variation of the free energy in the massless limit can also be understood from dimensional arguments as follows. Since we are working in natural units $\hbar = 1$ and $c = 1$, then $[m] \sim [L]^{-1}$. Then the free energy density (i.e. per unit area) varies as $[L]^{-3}$. However, the dimension of B is $[L]^{-2}$. So, the massless limit restricts the free energy density variation with magnetic field B to $B^{3/2}$ only. This feature of the susceptibility has already been noticed in the magnetised pair Bose gas [10]. This divergence of the susceptibility can be compared with that of ideal charged bose gas at the condensation point [11]. The magnetic susceptibility is defined by

$$M = \chi B \tag{31}$$

where B is the applied field; what is actually computed, however, is a quantity χ' defined by

$$M = \chi' B' \tag{32}$$

where B' is the "acting" field. The relation between B and B' requires special attention; in fact if the "acting" field is identified with the average microscopic field then we have for this case

$$B' = B + 2\pi M \tag{33}$$

This equation immediately gives a relation between χ and χ' as

$$\chi = \frac{\chi'}{1 - 2\pi\chi'} \tag{34}$$

This shows that on approaching $B \to 0$ limit, where $\chi' \to -\infty$, $\chi \to -\frac{1}{2\pi}$ so that the permeability tends to zero. So, in this case the external field will be totally expelled. This happens because of large number of virtual particle and antiparticle produced in the ground state so that the overall diamagnetism of the system is high enough to totally expel the external field. That the magnetisation is non-analytic in the limit $B \to 0$ is the actual signature of a phase transition as a function of external field strength. Note that this behaviour is equivalent to the behaviour of the free energy in strong magnetic field limit. This can be understood the way m and B appears in $\omega = \frac{eB}{m}$ which is the only energy scale in the problem. Hence, one should expect the same behaviour as $m \to 0$ or $B \to \infty$ because in both situations higher Landau levels become less and less occupied. In fact in this situation one can construct the theory in the lowest Landau level.

Another interesting point is that there exists a critical magnetic field below which magnetic field will be totally expelled. This critical field can be estimated as follows. The effective magnetic field can be defined as $B_{eff} = B + 2\pi M$. Here, the magnetisation M varies as $- const\sqrt{B}$. Therefore, there exists a critical magnetic field B_c where the effective field B_{eff} vanishes.

3. Finite Temperature Case

Now, for the finite temperature case one can regulate the free energy through the same mode matching regularisation method. Finally, one can write down the free energy difference in dimensionless form as before

$$\Delta F(B) = F(B) - F(0) = \sum_{l=0}^{\infty} b_l(\rho, \delta, \zeta) \tag{35}$$

where,

$$b_l(\rho, \delta, \zeta) = \frac{\rho}{\delta}\left[g(\rho, l, 1/2) - \int_0^1 d\alpha\ g(\rho, l, \alpha)\right]. \tag{36}$$

The dimensionless variables are defined as $\delta = \beta m$ and $\zeta = \beta\mu$.

The coefficient $g(\rho, l, \alpha)$ is given by

$$g(\rho, l, \alpha) = \log\left(1 - \exp(-\delta(\sqrt{1 + 2(l+\alpha)\rho} - \zeta))\right) + \log\left(1 - \exp(-\delta(\sqrt{1 + 2(l+\alpha)\rho} + \zeta))\right). \tag{37}$$

Now, defining $z_l = \frac{\delta^2(1+2l\rho)}{2\rho}$ we can write the equation (37)

$$g(\rho, l, \alpha) = \log\left(1 - \exp(-(\sqrt{z_l + \alpha} - \zeta))\right) + \log\left(1 - \exp(-(\sqrt{z_l + \alpha} + \zeta))\right). \tag{38}$$

The function $g(\rho, l, \alpha)$ is convex, so the zero temperature argument applies unchanged. It follows that the free energy satisfies the following inequality

$$F(B) \geq F(0) \tag{39}$$

Thus the response of the system to the magnetic field will be diamagnetic.

B. (3+1) dimension

In this subsection we would like to renormalize the free energy in presence of an external uniform magnetic field in (3+1) dimension. For the sake of repetition we do not present here the derivation of the free energy in (3+1) dimension which can be easily be done in the framework presented earlier for (2+1) dimension case. In fact the final result can be understood easily from the dispersion relation (energy spectrum) for (3+1) case which is given by

$$\omega_l = \sqrt{p_z^2 + m^2 + (2l+1)eB}, \quad l = 0, ...\infty \tag{40}$$

Thus, we get the free energy of charged scalar fields in 3d in presence of a constant homogeneous external magnetic field as

$$F(B) = 2\pi \mathrm{e} \mathrm{V} B \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} dp_z \left[\omega_l + \frac{1}{\beta}\ln(1 - e^{-\beta(\omega_l - \mu)}) + \frac{1}{\beta}\ln(1 - e^{-\beta(\omega_l + \mu)})\right] \tag{41}$$

From the above equation (41) we notice that the zero point free energy is given by

$$F(B, m) = 2\pi eBV \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} dp_z \left[\sqrt{m^2 + p_z^2 + (2l+1)eB}\right] \tag{42}$$

We regularise this free energy by the prescription given in [12] which was used in case of QED in an external magnetic field. In principle, one can use the prescription through a novel mode matching already developed for (2+1) dimension to regularise this free energy . But we use this regularisation scheme to compute quantitative evaluation of the regularised free energy in two interesting limits in the zero temperature case. As it is clear from (2+1) dimension case that we cannot evalute the response function quantatively that's why we choose this scheme for that reason.

In field theory there are various regularisation schemes for example Schwinger's proper time regularisation scheme [13] or invariant regularisation scheme due to Pauli and Villars [14]. Through this simple regularisation scheme we demonstrate the renormalisation of field strength and charge as shown in the context of vacuum polarisation [13].

1. Zero Temperature Case

The divergence of the above integral (Eq. (42)) is eliminated by subtracting the value of the sum when $B = 0$. Hence, to carry out this renormalisation procedure it is appropriate to calculate first the convergent expression. Defining the free energy density $f = F/V$ we can write

$$\frac{\partial f}{\partial (m^2)} = 2\pi eB \int_0^\infty \sum_{l=0}^{\infty} \left[\frac{dp_z}{\sqrt{m^2 + p_z^2 + (2l+1)eB}} \right] \quad (43)$$

Differentiating once again we get $\Theta = \frac{\partial^2 f}{\partial (m^2)^2}$

$$\Theta = -\pi eB \int_0^\infty \left[\sum_{l=0}^{\infty} (m^2 + p_z^2 + (2l+1)eB)^{-3/2} \right] dp_z \quad (44)$$

Notice that the integral is now absolutely convergent. Performing the integral and the sum, we obtain

$$\begin{aligned}\Theta &= -\pi eB \left[\sum_{l=0}^{\infty} \frac{1}{m^2 + (2l+1)eB} \right] \\ &= \pi eB \int_0^\infty e^{-(m^2+eB)\eta} \left[\frac{1}{1 - e^{-2eB\eta}} \right] d\eta, \end{aligned} \quad (45)$$

which reduces to a simple integral form given by

$$\begin{aligned}\Theta &= \frac{\pi eB}{2} \int_0^\infty \frac{d\eta \;\; e^{-m^2\eta}}{\sinh(eB\eta)} \\ &= \frac{\pi eB}{2} \int_0^\infty d\eta \; e^{-m^2\eta} \;\; g(eB\eta), \end{aligned} \quad (46)$$

where $g(eB\eta) = 1/\sinh(eB\eta)$. Now to get back the free energy we have to integrate Θ twice. Integrating once we get with a undetermined constant

$$\begin{aligned}\Theta_1 &= \int \Theta \; d(m^2) \\ &= \frac{\pi eB}{2} \int_0^\infty (-1/\eta) \; e^{-m^2\eta} \; g(eB\eta) \; d\eta + C_2. \end{aligned} \quad (47)$$

Similarly integrating once more one gets back

$$\begin{aligned}\Theta_2 &= \int \Theta_1 \; d(m^2) \\ &= \frac{\pi eB}{2} \int_0^\infty \frac{1}{\eta^2} e^{-m^2\eta} g(eB\eta) \;\; d\eta \\ &\quad + C_2 \; m^2 + C_1 \end{aligned} \quad (48)$$

These two undetermined constants depend only on B but not on m^2. Now it is easy to take $B \to 0$ limit in the above formula. This gives

$$\Theta_2(B=0) = \frac{\pi}{2} \int_0^\infty \frac{1}{\eta^3} \; e^{-m^2\eta} d\eta \quad (49)$$

Here we have assumed that two constants C_1 and C_2 vanish as $B \to 0$. Therefore the difference between the two free energies can be written in an integral form as

$$\begin{aligned}\Delta f = \frac{\pi}{2} &\int_0^\infty \frac{1}{\eta^3} \;\; e^{-m^2\eta} \; d\eta \left[\frac{eB\eta}{\sinh(eB\eta)} - 1 \right] \\ &+ C_2 \; m^2 + C_1 \end{aligned} \quad (50)$$

Till now the constants C_1 and C_2 are undetermined. These are determined from the following considerations. Parity and dimensional analysis give us a particular form [15] of Δf given by

$$\Delta f = m^4 \; g\left(\frac{B^2}{m^4} \right) \quad (51)$$

From equation (51) it is clear immediately that there will be no odd terms of m^2 in the expression for Δf. Hence, C_2 is zero. Now to find out the coefficient C_1 we follow the following prescription. Notice that the expansion of Δf in B^2 begins with a term linear in B^4. Therefore, a term in B^2 would alter the coefficient in the Lagrangian $L_0 = \frac{B^2}{8\pi}$. This essentially modifies the definition of charge. Hence, the elimination of B^2 term thus corresponds to a renormalization of charge.

Expanding $\sinh(eB\eta)$ in leading order upto B^2 we get

$$C_1 = \frac{\pi}{2} \int_0^\infty \frac{e^2 B^2 \eta^2 \; e^{-m^2\eta} \; d\eta}{6\eta^3} \quad (52)$$

Rescaling $m^2\eta \to \eta$ and defining $b = \frac{eB}{m^2}$ finally we obtain

$$\Delta f = \frac{\pi m^4}{2} \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left[\frac{b\eta}{\sinh(b\eta)} - 1 + \frac{1}{6} \; b^2\eta^2 \right] \quad (53)$$

This renormalized energy difference Δf is one of the main results of the paper. Notice that the difference of free energy Δf goes to zero as $B \to 0$ as it should be since we have subtracted the zero field free energy. One can easily show through numerical integration that this free energy is positive definite and it converges. Before we end this subsection we want to make a remark on this regularisation scheme. A careful reader will immediately notice that this scheme will not work in (2+1) dimension. This shows the important role played by the p_z integral. In the next two sections we consider two interesting limits of the free energy difference.

2. Small Magnetic Field Limit

At small magnetic field ($b\eta$ being very small) we can approximate the free energy difference upto the leading factor in b^4 as

$$\Delta f = \frac{7\pi m^4 b^4}{720} \int_0^\infty e^{-\eta}\ \eta\ d\eta = \frac{7\pi e^4 B^4}{720 m^4} \tag{54}$$

Thus the free energy difference at small magnetic field limit goes as B^4. This is obvious from the fact that we have subtracted the leading divergence of B^2 term from the free energy. So, the next order term in the free energy difference is B^4 according to symmetry. Notice that the free energy difference is positive definite and thus establishes the diamagnetism of the charged scalar fields at zero temperature.

3. Strong Magnetic Field Limit

In this section we would like to address various responses of the system in the strong magnetic field limit.

In a strong magnetic field ($b \gg 1$) rescaling $b\eta \to \eta$, we get the free energy difference from equation (53)

$$\Delta f = \frac{\pi\ m^4\ b^2}{2} \int_0^\infty e^{-\eta/b}\ \frac{d\eta}{\eta^3} \left[\frac{\eta}{\sinh(\eta)} - 1 + \frac{1}{6}\eta^2 \right] \tag{55}$$

When $b \gg 1$, the important range in this integral is $1 \ll \eta \ll b$, in which $e^{-\eta/b} \approx 1$, then terminating the range of integration (with logarithmic accuracy) at $\eta \approx 1$ and $\eta \approx b$, we get

$$\Delta f \approx \frac{\pi m^4 b^2}{12} \ln b \tag{56}$$

It is important to note that in the massless limit the free energy in 3d turns out [16] as

$$\Delta f \approx \frac{\pi m^4 b^2}{12} \ln \left(\frac{\overline{\Lambda}^2}{eB} \right) \tag{57}$$

where $\overline{\Lambda}$ is some UV cutoff in the theory. The form immediately suggests the equivalence of massless limit and large magnetic field limit. This is quite obvious from the way B and m occur in $\omega = \frac{eB}{m}$ which is the only energy scale in this problem. If one compares with L_0 then one finds

$$\frac{\Delta f}{L_0} = \frac{2\pi^2 e^2}{3} \ln \left(\frac{eB}{m^2} \right) \tag{58}$$

which implies that the radiative correction to the field equations may become of order of unity in strong magnetic fields given by

$$B_0 \sim \frac{m^2}{|e|} \exp \left(\frac{3}{8\pi^3 \alpha} \right) \tag{59}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. Before we go to discuss the response of the system in an external magnetic field we would like to make some comments on the response of electric field alone. Since an external electric field always gives a non-vanishing probability for pair creation, it is more convenient to discuss the response to an external magnetic field and calculate the magnetic permeability μ rather than ϵ. Then we can deduce ϵ from μ if we assume that the vacuum is Lorentz-invariant. If that is the case then we should have

$$\epsilon\mu = 1 \tag{60}$$

Depending on the value of $\mu < 1$ or $\mu > 1$ one can classify the nature of the vacuum.

From equation (56) it is easy to calculate the magnetisation and the susceptibility in this limit. The magnetisation is given by

$$M(B) = -\frac{\pi e B\ m^2}{12} - \frac{\pi e^2 B}{6} \ln \left(\frac{eB}{m^2} \right) \tag{61}$$

Another interesting point before we go to estimate the susceptibility is that there exists a critical magnetic field below which magnetic field will be totally expelled. This critical field can be estimated as follows. The effective magnetic field can be defined as $B_{eff} = B + 4\pi M$. Here, the magnetisation M varies according to equation (61). Therefore, there exists a critical magnetic field B_c where the effective field B_{eff} vanishes. This critical magnetic field B_c is given by the following equation

$$\ln \left(\frac{eB_c}{m^2} \right) = \frac{6}{\pi e^2} \left(1 - \frac{\pi e m^2}{12} \right) = \tilde{g}(e, m^2) \tag{62}$$

and in a more simple form

$$B_c = \frac{m^2}{|e|} \exp(\tilde{g}). \tag{63}$$

Therefore, the susceptibility is

$$\chi(B) = -\frac{\pi e m^2}{4} - \frac{\pi e^2}{6}\ \ln \left(\frac{eB}{m^2} \right) \tag{64}$$

Notice that in 3d, the zero field susceptibility has a peculiar logarithmic form in the free energy. Even for QED [17] one also gets same logarithmic behaviour of the free energy (and hence in the susceptibility) in this strong magnetic field limit. For the sake of comparison we notice that

$$\chi_{2d}(B) \sim -\frac{1}{\sqrt{B}}$$
$$\chi_{3d}(B) \sim -\ln(B) \tag{65}$$

This interesting behaviour of the susceptibility in the massless limit is reminiscent of the fact that the free energy variation in two cases as

$$f_{2d}(B) \sim B^{3/2}$$
$$f_{3d}(B) \sim B^2 \ \ln(B) \tag{66}$$

Also it is interesting to note that in both cases the free energy is a non-analytic function of B.

One can also calculate the critical magnetic field for which the effective permeability becomes zero. The permeability is given by

$$\mu(B) = 1 + 4\pi\chi(B)$$
$$= 1 - \pi^2 em^2 - \frac{2\pi^2 e^2}{3} \ln\left(\frac{eB}{m^2}\right) \tag{67}$$

Thus, the critical magnetic field B_1 is simply

$$B_1 = \frac{m^2}{|e|} \exp(d(e, m^2)), \tag{68}$$

where the function $d(e, m^2)$ is determined from the equation

$$d(e, m^2) = \frac{3}{2\pi^2 e^2}\left(1 - \pi e m^2\right) \tag{69}$$

Thus we have three critical values of magnetic field for which (i) B_{eff} vanishes, (ii) $\mu(B)$ vanishes and (iii) logarithmic term in the susceptibility goes away and $\chi(B)$ becomes $-\frac{\pi e m^2}{4}$. One thing to notice here that all the critical magnetic fields are of order $\frac{m^2}{e}$ and these critical fields should be compared with the field on which the classical theory of electrodynamics will break down. This classical magnetic field can be estimated from the comparision of two length scale

$$\frac{e^2}{m} \sim \frac{m}{eB} \tag{70}$$

This gives the critical field

$$B_{class} \sim \frac{m^2}{e^3} \tag{71}$$

On comparison we notice that

$$\frac{B_{class}}{B_c} \sim \frac{1}{e^2} \tag{72}$$

This is true for all the other critical fields also. Thus, the ratio between the classical field and the critical field is of the order of fine structure constant.

Next we would like to compute the dielectric constant $\epsilon(r)$ from the effective permeability. Since we are working in natural units $\hbar = 1$ and $c = 1$ then the dielectric constant ϵ and the permeability μ is related by $\epsilon\mu = 1$. However, one can show from the effective action [18] formalism that under certain circumstances one can still use the relation in presence of an external magnetic field though the physical situation is not Lorentz-invariant. Hence, the effective permeability $\mu(B)$ should be related to the dielectric constant $\epsilon(r)$ by the relation given by

$$\epsilon(r) = \frac{1}{\mu(B)} \ |_{eB \to 1/r^2} \tag{73}$$

This immediately suggests that

$$\epsilon(r) = \frac{1}{1 - \pi^2 em^2 - \frac{4\pi e^2}{3} \ln\left(\frac{1}{mr}\right)} \tag{74}$$

From equation (67) we notice that $\mu(B) < 1$ which means that $\epsilon(r) > 1$. This is easily seen from the above equation. Hence, the charged scalar field vacuum (diamagnetic) is screening one. Before we go to finite temperature case scheme we would like to make a comment on the vacuum energy of QCD in an external magnetic field. In fact through this regularisation scheme one can explicitly show the paramagnetic nature of this spin-1 field [19].

4. Finite Temperature Case

Now, for the finite temperature case one can regulate the free energy through the same mode matching regularisation method used in (2+1) dimension. Finally, one can write down the free energy difference in dimensionless form as before

$$\Delta F(B) = F(B) - F(0) = \sum_{l=0}^{\infty} d_l(b, \delta, \zeta, \rho) \tag{75}$$

where,

$$d_l(\rho, \delta, \zeta) = \frac{\rho}{\delta}\int_{-\infty}^{\infty} dp_z \left[g(\rho, l, 1/2) - \int_0^1 d\alpha \ g(\rho, l, \alpha)\right]. \tag{76}$$

The dimensionless variables are defined as $\delta = \beta m$ and $\rho = \beta\mu$.

The coefficient $g(\rho, l, \alpha)$ is given by

$$g(\rho, l, \alpha) = \log\left(1 - \exp(-\delta(\sqrt{1 + 2(l+\alpha)\rho} - \zeta))\right) + \log\left(1 - \exp(-\delta(\sqrt{1 + 2(l+\alpha)\rho} + \zeta))\right). \tag{77}$$

Now, defining $z_l = \frac{(1+2l\rho)}{2\rho}$ we can write the equation (77)

$$g(\rho, l, \alpha) = \log\left(1 - \exp(-(\sqrt{z_l + \alpha} - \zeta))\right) + \log\left(1 - \exp(-(\sqrt{z_l + \alpha} + \zeta))\right). \tag{78}$$

The function $g(\rho, l, \alpha)$ is convex, so the zero temperature argument applies unchanged. It follows that the free energy satisfies the following inequality

$$F(B) \geq F(0) \tag{79}$$

This proves the diamagnetism for charged scalar fields at finite temperature. It would be interesting to look into the various limits of this free energy difference. But here we do not address these limits at all.

III. INTERACTING CASE

In this section we want to extend the diamagnetic inequality to the self-interacting field theory case including the dynamical interaction between scalar fields. We present here the proof of diamagnetic inequality for a generalised d dimensional case. The partition function of this charged self-interacting field theory in the presence of the magnetic field can be written as

$$\mathrm{Z(B)} = \int\int \mathcal{D}[\Phi]\mathcal{D}[\Phi^*]\mathrm{exp}(-S(\Phi,\Phi^*,A)), \tag{80}$$

where the action S is defined as

$$\mathrm{S} = \int\int d^{\,d}x \; d\tau \left[(D_\mu\Phi)(D^\mu\Phi)^* + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\right]. \tag{81}$$

The action is not quadratic and $Z(B)$ cannot be evaluated in closed form. Nevertheless, we can show that the response of the system to an external magnetic field is diamagnetic. Since the formal expression for the partition function may not exist (the integrals may not exist) we impose a cut off in momentum space. The functional integral in (80) signifies that one only integrates over those field configurations whose Fourier transforms have support within a sphere of radius Λ_0 in momentum space. The partition function then explicitly depends on Λ_0. We do not explicitly indicate the Λ_0 and μ dependence of $Z(B,\Lambda_0,\mu)$ below.

We divide the action into two parts S_0 and $\mathrm{S_{int}}$, where S_0 is the action in the absence of the external field.

$$\mathrm{S} = \mathrm{S_0} + \mathrm{S_{int}}, \tag{82}$$

where

$$\mathrm{S_0} = \int\int d^dx \; d\tau \left[(\partial_\mu\Phi)(\partial^\mu\Phi)^*) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\right], \tag{83}$$

$$\mathrm{S_{int}} = \int\int d^dx \; d\tau \left[-\mathrm{i}e(\partial_\mu\Phi)(A^\mu\Phi^*) + \mathrm{i}e(A_\mu\Phi)(\partial^\mu\Phi^*) + e^2(\mathbf{A}\cdot\mathbf{A})(\Phi\Phi^*)\right]. \tag{84}$$

Notice that $\exp(-S_0)$ is a positive measure on the space of field configurations. The ratio $Z(B)/Z(0)$ can therefore be regarded as the expectation value of $\exp(-\mathrm{S_{int}})$. Since $\exp(-\mathrm{S_{int}})$ is an oscillatory function whose modulus is less than or equal to 1, we conclude that

$$\frac{\mathrm{Z(B)}}{\mathrm{Z(0)}} = \ll \mathrm{exp}{-S_{int}} \gg \leq 1 \tag{85}$$

This implies that

$$F(B) \geq F(0) \tag{86}$$

This result is an *exact* and *non-perturbative* one. Hence, it is more general and strong compared to perturbative one.

In this derivation, we have not assumed any form for the vector potential. So, the result derived above is true for *both homogeneous or inhomogeneous* magnetic fields of any strength. Since β is arbitrary, the result holds at *all temperatures*. The argument presented here works for any arbitrary interaction $V(\Phi^*\Phi)$ (Generally, it is assumed that $V(\Phi^*\Phi)$ is a smooth function, for instance, a polynomial).

Upto now we have considered the cases of charged scalar fields interacting through a potential. It is also possible to consider interaction mediated by a dynamical electromagnetic field A_μ. The fields in the system are now Φ (charged scalar fields) and A_μ. If one applies an external magnetic field A_{ext} then the full Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu\Phi)^*(D^\mu\Phi) - m^2(\Phi^*\Phi) - V(\Phi^*,\Phi) \tag{87}$$

where $D_\mu = \partial_\mu - \mathrm{i}eA_\mu^{ext} - \mathrm{i}eA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$.

The argument given above can be modified as follows. The definition of S_0 changes slightly while $\mathrm{S_{int}}$ remains the same.

$$\mathrm{S_0} = \int\int d^dx \; d\tau \left[-\frac{1}{4}F^2 + (\partial_\mu - ieA_\mu\Phi)^*(\partial^\mu + ieA^\mu\Phi) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\right], \tag{88}$$

and

$$\mathrm{S_{int}} = \int\int d^dx \; d\tau \left[-\mathrm{i}e(\partial_\mu\Phi)(A^\mu\Phi^*) + \mathrm{i}e(A_\mu\Phi)(\partial^\mu\Phi^*) + e^2(\mathbf{A}\cdot\mathbf{A})(\Phi\Phi^*)\right]. \tag{89}$$

Again one can repeat the same argument to establish the diamagnetic inequality by noting that $\exp(-S_0)$ is a positive measure and the ratio $Z(B)/Z(0)$ as an expectation value of $\exp(-\mathrm{S_{int}})$. This universal inequality follows from basic principles and does not depend on the details of the interaction.

A. Comment on the regularisation scheme in interacting case

The effective potential $V(\Phi^*\Phi)$ is gauge invariant. What we have done is that we have suitably redefined the measure of the integral through the inclusion of self-interaction. Thus now, the bare measure has been changed to so called "dressed" measure. This has the advantage that the effects of self-interaction and mutual interaction have been separated. Also In field theory one would also require that the interaction $V(\Phi^*\Phi)$ be renormalizable. In $(d+1)$ dimensions this would restrict the interaction $(\Phi^*\Phi)^m$ only. From simple power counting, one can notice easily that the required value of m is $\frac{d+1}{d-1}$. In condensed matter physics, where a natural cutoff exists, higher order powers of $(\Phi^*\Phi)$ may also be present.

Under normal conditions $k_BT \ll m$ we expect the self interaction to merely affect the particle's mass. This is the same concept one uses in quasiparticle picture of Landau Fermi Liquid theory [20]. The effect of all the interactions can be put in the definition of the effective mass of the quasi particles. Therefore, the error committed would be small if we were to ignore the self-interactions while keeping each mass at a fixed observalble value. Hence, we drop the superscript "dress" and work with the ordinary Wiener measure.

Notice that the bare mass enters the Wiener measure. However, in the expression of the $S_{\rm int}$ neither refers to nor uses the bare mass; it never enters the interaction of the scalar particle with the electromagnetic field. On the otherhand, it is important to realize that the bare mass m differs from the physical or "effective" mass m_{eff}. The differnece is induced by the interaction. Adjusting the bare mass so as to gurantee that m_{eff} agrees with the observed mass of the particle constitutes the central issue of the renormalization program. It is generally believed that this issue ought to be resolved outside the perturbation theory. In this paper we do not attempt to answer these questions.

In the interacting case it turns out that the partition function and hence the free energy cannot be evaluated in closed form but still one can show the diamagnetism of charged scalar fields. This shows that diamgnetism of charged scalar fields is *universal* i.e. robust in the sense that it is true for all dimensions and independent of the interaction between the scalar fields. One more comment regarding the meaning of universal behaviour of this system towards an external magnetic field. The proof of the diamagnetic inequality in the interacting case does not assume the form of the vector potential. Thus, in that sense the theory is true whether the applied magnetic field is homogeneous or inhomogeneous. The word *universality* also reflects this fact that the behaviour of the system towards the magnetic field is independent of the nature of an applied magnetic field. We would like to emphasize one point regarding our theory which is as follows. If a phase transition takes place changing the drastic nature of the system, then of course our theory fails. As long as a phase transition does not take place that alters the bosonic behaviour, this theory can predict the response of the system in an external magnetic field as diamagnetic.

IV. APPLICATIONS

It is evident from the previous sections that though the results obtained are quite general and important from the theoretical point of view but fail to connect directly with the experiment. What we have shown is basically the nature of the response of a system under an external magnetic field. Thus, though the results are of interesting one but fails to compute explicitly the susceptibility as a function of magnetic field, mass and temperature. Only in zero temperature case we have been able to compute the susceptibility in two limits. However, one can resort to perturbation method to compute approximately the response of the system in case of finite temperature. In this section we would like to discuss the possible situations which could be well-connected with the theory presented. From an experimental point of view one is more interested in the quantitative variation of the susceptibility with the temperature. Therefore, in this section we will mainly address the temperature variation of the diamagnetic susceptibility assuming an effective Landau-Ginzburg free energy functional apart from giving some illustration of spinless bose systems. Notice that the order parameter in Landau-Ginzburg model is a scalar one which has a characteristic $U(1)$ symmetry. Only the mass parameter in Landau-Ginzburg model has a different meaning in contrast to the theory presented here.

One obvious example spinless bose systems in the laboratory is the Cooper pair formation in superconductors which shows perfect diamagnetism (known as the Meissner [21] effect) below the critical temperature. Cooper pairs also exist in Neutron Stars [22] where the magnetic field is very high compared to any laboratory field. Of course, the operators which create and destroy Cooper pairs are not strictly Bose operators, so this is only an analogy. In fact one can also calculate the diamagnetic susceptibility of the Cooper pairs above the critical temperature, more specifically the temperature variation of the susceptibility. The diamagnetic susceptibility above the critical temperature has been observed recently in High-T_c compounds [23]. In literature there exists a model [24] which does predict that above the critical temperature all or at any rate some of the carriers are bosons. Here, of course, as has been pointed out [25] that the definition of the "critical temperature" bears a different meaning in contrast to usual one. The "critical temperature" in that context refers to the temperature at which the gas of bosons become wholely *non-degenerate*. However, in the following we do not address all these issues regarding the mechanism of formation of bosons.

With the usual Landau-Ginzburg effective field theory [26] it turns out that the diamagnetic susceptibility in d dimension $\chi_d \sim -(T-T_c)^{-\frac{4-d}{2}}$. Without going much details into this calculation [27], the above peculiar variation of the susceptibility with temperature can be explained as follows. Following the argument given for 3d [28], we notice that above T_c, droplet of Cooper pairs will grow and decay as a result of thermodynamic fluctuations. Their mean radius is approximately equal to ξ which is the coherence length of the Cooper pairs and phenomenologically the simplest variation of this coherence length is taken as $\xi^2 \sim (T-T_c)^{-1}$. The "mass parameter" in this theory is inversely proportional to this coherence length. Therefore, the amount of energy required to produce a droplet is given by

$$\delta E \sim \frac{1}{\xi} \times \xi^d \times |\phi|^2 \tag{90}$$

Here $|\phi|^2$ is the density of the Cooper pairs in the framework of phenomenological Landau-Ginzburg free energy functional. This energy must be equal to the thermal energy $k_B T_c$ and hence

$$|\phi|^2 \sim \frac{k_B T_c}{\xi^{d-1}} \tag{91}$$

Now consider the expression for the diamagnetic susceptibility of atoms in scaled form as

$$\chi_d \sim -\frac{|\phi|^2 e^2 \xi^2}{1/\xi} \sim -(T-T_c)^{-\frac{4-d}{2}} \tag{92}$$

In otherwords for a general variation of the coherence length $\xi^2 \sim (T-T_c)^{-\nu}$ the susceptibility variation turns out as $\chi_d \sim -(T-T_c)^{-\nu(4-d)/2}$. However, as it is evident for dimension $d \geq 4$ the above formula does not make any sense; but this formula of course correctly reproduces the expected temperature variations for 2d and 3d. In fact, in 3d the exponent of the susceptibility ($1/2$) has been observed in the experiment [23]. Notice that inspite of the strong correlation among the Cooper pairs in the system the pairs behave as if they are non-interacting. Thus, within this simple Landau-Ginzburg approach one can give a strong hint towards the role of dimensionality of the system in High-T_c material.

It has been shown explicitly by J. Daicic et. al. [10] that the magnetised pair Bose systems are relativistic superconductors. These systems are not covered by previous analysis [1,29,3,4] which apply to non-relativistic quantum mechanical systems. Pions would be suitable candidates for application of this theory with π^+ and π^- regarded as the particles and antiparticles. The choice is also motivated by the fact that these pions are massive ($mc^2 = 139.5673$ Mev), obey Bose-Einstein statistics and that they possess no spin. It is also well known that hard core bosons [30] in any dimension on any lattice show a preference for zero flux.

V. CONCLUSIONS AND PERSPECTIVES

The response of a system to an electric field is completely different from its response to a magnetic field. The basic difference between the responses of a system on application of an electric field or a magnetic field lies in the Hamiltonian of the system.

The Lagrangian of a system in the presence of an electric field can be written as

$$\mathcal{L} = (D_0\Phi)^*(D_0\Phi) - (\nabla\Phi)^*(\nabla\Phi) - m^2(\Phi^*\Phi) - V(\Phi^*\Phi) \tag{93}$$

where

$$D_0 = \partial_0 - \mathrm{i}eA_0 \tag{94}$$

For statistical mechanics to make sense, the Hamiltonian H must be independent of time. Then it follows that

$$\begin{aligned}\mathcal{H} = (\Pi^*)(\Pi) + (\nabla\Phi)^*(\nabla\Phi) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\\ -\mathrm{i}e\left[(\Pi^*)(A_0\Phi) - (\Pi)(A_0\Phi^*)\right]\end{aligned} \tag{95}$$

The electric field appears in the Hamiltonian through the linear vector potential A_0 term. Now, from finite temperature second order perturbation [31,32] theory, one can show easily that the free energy of the system always decreases with the electric field. Hence, the dielectric susceptibility is always positive in thermal equilibrium.

But in the case of a magnetic field the Hamiltonian contains both linear and quadratic terms in A. The net effect of an applied magnetic field is not *a priori* clear. However, as our analysis makes clear, for charged scalar field theories the net effect is always diamagnetic.

In case of spinless bosons, there is no Zeeman term coupled with a magnetic field and hence the system consisting of spinless bosons always has higher energy in a magnetic field than without the magnetic field. It has been already pointed [3,4,30,5] out that there is no corresponding theorem for fermions. So, in case of fermions having spin the general tendency is to show paramagnetism [33].

Before we end we would like to comment on a recent work in the literature [34] on ultrarelativistic hot scalar plasma. The results of their perturbative treatment are consistent with our results. In contrast, as we have already pointed out that our treatment is a non-perturbative one. Since any non-perturbative treatment is always welcome in field theory we thought it would be interesting to present these results and arguments. Besides we have been able to show the renormalisation of the free energy. These regularisation schemes help one to understand various interesting renormalization of the quantity involved (in this case charge) and a proper justification of the scaling functions discussed in the paper.

In summary, we have shown exactly, that charged scalar fields in (2+1) and (3+1) dimension at all temperatures are diamagnetic.

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APPENDIX A: COMMENT ON AN ANOTHER REGULARISATION SCHEME

In this appendix we would like to discuss about another regularisation scheme which at first sight one might think of as a correct regularisation scheme. However as we show below this method is not a correct scheme. This scheme can be thought of as relating L and Λ by the criterion that the maximum energy of the two systems should be matched. The matching of the energy cutoff gives the relation between the momentum cut off Λ and the Landau level L as

$$E_L = \sqrt{m^2 + (2L+1)eB} = \sqrt{\Lambda^2 + m^2} \tag{A1}$$

This immediately suggests that

$$\Lambda^2 = (2L+1)eB \tag{A2}$$

Note that this relation is different from the mode matching relation(20) we used previously for the charged scalar field theory. However, it turns out that this renormalisation procedure gives a divergent answer for the free energy difference Δf. We support this statement through both numerical as well as analytical calculations.

Below we present numerically a comparison between the two schemes. The values of Δf^A (scheme due to anonymous) and Δf^J (used in the paper) are tabulated for $B = 2$, $m = .1$ and for various values of the cutoff L.

Different values of cutoffs(L)	Δf^A	Δf^J
L=10	6.774	0.2157
L=100	20.308	0.2325
L=1000	63.507	0.2382
L=10000	200.248	0.2400
L=50000	447.457	0.2404
L=80000	565.929	0.2405
L=100000	632.698	0.241

From this table it is easy to see that our method gives a finite convergent, cutoff independent answers while the other method gives cutoff dependent, divergent answers for the free energy. In fact one can show analytically in this case that Δf diverges with the cutoff as $\sqrt{L}$. This unphysical and meaningless answer is due to the comparision of systems with different number of degrees of freedom. Note that in our analysis Δf converges as $\frac{1}{\sqrt{L}}$. Hence, the scheme due to matching of highest energy is therefore clearly unphysical. However, as already discussed our scheme can be justified physically from the adiabtically turning on the magnetic field. Then the energy of each mode is affected by the magnetic field. Comparing systems with the same number of modes is the logically and physically correct procedure. What the other method does is that it compares the systems with different degrees of freedom. Hence, regularisation scheme based on energy cutoff does not give a renormalisable answer. Also notice that phase space is invariant in any Lorentz frame while the energy is not. We present below analytical work to show that in our case δf varies with $\frac{1}{\sqrt{L}}$ and as $L \to \infty$, it does converge while the other case it diverges as $\sqrt{L}$ in the same limit as indicated above. Since the ultimate aim is to take the cutoff to infinity let us calculate the behaviour of the leading term in free enrgy difference as a function of cutoff. The difference between the free energy (using the two regularisation schemes) is given by

$$\delta f_L(B,m) = \frac{1}{3}\Big[(m^2 + (2L+1)eB)^{3/2} - (m^2 + 2(L+1)eB)^{3/2}\Big] \tag{A3}$$

Now defining a dimensionless small parameter $\eta = \frac{m^2}{(2L+1)eB}$ (which is reasonable in large cutoff L and in strong magnetic field) it is easy to write the above difference as

$$\delta f_L(\eta) \sim \sqrt{L} + O(\frac{1}{\sqrt{L}}) \tag{A4}$$

This is consistent with the numerical results presented in the above table. Another way of qualitatively understanding the above divergence is the following . In a strong magnetic field limit all the higher Landau levels are far away and one is interested only in the lowest Landau level. Now in this situation if one countes the energy states and matches with the free case, then there will be leading order diveregnce with the cutoff due to mismatch of number of degrees of freedom. Thus we notice that though the regularisation scheme provides a finite positive free energy differenece, it fails to give a renormalised difference. Hence this scheme should not be considered as a correct and physical regularisation prescription.

APPENDIX B: UNIQUENESS OF THE REGULARISATION SCHEME USED

In this appendix we want to show the uniqueness of the regularisation method used in this paper. In particular, we would like to argue that the mode matching relation (20) can be obtained from the condition that the difference between the two free energies is independent of the cutoff and tends to some finite value. Here too we assume that the difference is finite for all values of $B > 0$ and $m > 0$. As we have noted before for charged scalar fields the difference between the free energies can be written as

$$\Delta f = eB \sum_{l=0}^{L} \sqrt{m^2 + (2l+1)eB} - \int_0^{\Lambda} pdp\ \sqrt{p^2 + m^2} \tag{B1}$$

We want to find the relation between Λ and L for which the difference tends to a finite value independent of the cutoffs. In otherwords as $L \to \infty$ or $\Lambda \to \infty$ the difference between the free energy becomes unique and finite. Though in principle Δf depends on both Λ and L separately, we will notice that because of a unique relation between them we will get a finite value independent of both of them. Hence the free energy difference can be written as

$$\Delta f = \Delta f_0 + O(\frac{1}{g(L)}), \tag{B2}$$

where the first term is the universal component which is independent of the cutoff (its value of course depends on eB and m) and the second term is the correction to it. The function $g(L)$ is assumed to be a smooth function of L and in the present analysis we are not interested in the actual form of it. We assume the cutoff to be high enough so that we can replace the sum by an integral. Now if we demand that Δf_0 should be independent of the cutoff then we get an equation of the form

$$\Lambda \frac{\partial \Lambda}{\partial L} \sqrt{m^2 + \Lambda^2} \to eB\sqrt{m^2 + 2LeB} \tag{B3}$$

We have used the symbol $\to$ instead of an equal sign to signify "asymptotically approaching" rather than equality between the two relations. In fact they are strictly equal only if the cutoff is taken to infinity. But, because of finite value of L or Λ however large, we do not expect strict equality in the above relation. From this equation in the first order approximation it is quite evident that there should be a relation between Λ and L such that

$$\Lambda \frac{\partial \Lambda}{\partial L} = eB \tag{B4}$$

This immediately gives a relation between the two cutoff within an undetermined constant C as

$$\Lambda^2 = 2eBL + 2C \tag{B5}$$

It is important to notice that this relation has been obtained in the limit of large enough cutoff. In fact as we have seen in the previous subsection the relation is valid for small as well as large value L. However, the physical energy difference approaches to finite value as $\frac{1}{\sqrt{L}}$ as we vary L. In the next we will try to fix this constant from the fact that the difference goes to a finite value. We substitute this value in the free energy difference equation (B1) and plot the difference as a function of various values of L for fixed values of B and m. Then it turns out that the only allowed value of C for the free energy to be finite and independent of the cutoff is eB. Thus we find that the exact relation between the two cutoff is

$$\Lambda^2 = 2eBL + 2eB = 2eB(L + 1) \tag{B6}$$

It is interesting to note that no other values of C are allowed for the free energy difference to be finite and independent of cutoff. Thus we have justified the mode matching relation (20). In fact, this is the only method by which one can get a unique finite cutoff-independent free energy difference.