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An optimal shipment strategy for imperfect items in a stock-out situation

Monami Das Roy [a], Shib Sankar Sana [b,*], Kripasindhu Chaudhuri [c]

[a] Department of Mathematics, Bethune College, Calcutta University, Kolkata-700006, India

[b] Department of Mathematics, Bhangar Mahavidyalaya, University of Calcutta, Bhangar, 24 Pgs(South), W.B., India

[c] Department of Mathematics, Jadavpur University, Kolkata-700032, India

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ABSTRACT

The objective of this paper is to minimize the expected total cost by consolidating a number of batches of imperfect quality products of different cycles for a single shipment and maximize the expected average profit function. Shortages are allowed at the end of each ordering cycle and are partially backlogged. The stock-out period is assumed to be exponentially distributed. The ordered lot size and cycle length of all the ordering cycles of a shipping cycle are considered to be equal. The model is analyzed for both infinite and finite planning horizons. The percentage of defective items in a lot is assumed to follow a uniform distribution. All the fractions of imperfect quality items in the ordering cycles of each shipping phase are assumed to be independent and identically distributed. After the completion of a screening process, one portion of the perfect quality products is used to serve partial backlogging at a cost per unit and the rest of the stock is used to adjust the demand. Calculus method is used to obtain the optimal number of order cycles for shipment. Optimal shortage period, optimal lot size and expected average profit for the model are developed on both of the finite and infinite time horizons. A comparison between the our models and Maddah and Jaber's (2008) [1] model is done through numerical studies which are also used to illustrate the models graphically.

1. Introduction

In a manufacturing process, it is impossible to produce 100% perfect quality products. Production of imperfect items is a very natural phenomenon. These imperfect items are either reworked or used in another inventory system. Again, now-a-days, '*Stock-out*' and '*partial backlogging*' are well known events in any inventory control system. Generally, shortages occur due to the following reasons:

(i) Presence of defective items.
(ii) Damage, spoilage, evaporation, etc., reduces the stock.
(iii) To avoid such damages (specially, in case of food, medicines, etc.) and the corresponding monetary losses, sellers do not like to overstock.
(iv) Insufficient space for storage is also a reason of shortage.
(v) Sometimes shortage is a created event of the seller himself to increase the market demand and price of a product.

* Corresponding author.
E-mail addresses: mdasroy2009@yahoo.in, monamidasroy@gmail.com (M. Das Roy), shib_sankar@yahoo.com (S.S. Sana), k.s.chaudhuri@gmail.com, chaudhuriks@gmail.com (K.S. Chaudhuri).

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The demand which remains unsatisfied during the shortage period are backlogged either completely or partially. Most of the time, it is impossible to meet the demand completely. Only a part of the demand is possible to satisfy which is called '*partial back ordering*'. This happens because, some customers lose their patience and go elsewhere to buy the product. Many inventory models are developed including the partial backlogging of the shortages which are discussed later.

Another key factor of any production control and management system is '*Shipment*'. '*Shipping cost*' means the cost which the seller or buyer must have to pay for '*Shipment*' of the product. Two scenarios may arise relating to the '*shipping cost*'. Either the seller has to pay the '*shipping cost*' which increases the total cost function or the buyer has to pay it, which increases the total revenue cost of the seller. So, we can easily recognize how the '*shipping cost*' reflects the objective function of the problem. Therefore, it is a very important part of an inventory management system. The '*Shipping cost*' varies with the distance between the supplier and the receiver. In case the customer is located in a very remote area or abroad, the '*Shipping cost*' becomes very high. Free shipping service facility is also offered by many companies, specially international companies. Usually free shipping service facility is available when the order is very large. Companies provide such offers for avoiding frequently placed small orders and to provoke customers to place large order. This is one of the business tricks along with the others.

In this paper, we have emphasized on both, partial backlogging and shipment. Though frequent shipping decreases the inventory holding cost, but it increases the shipping cost. On the other hand, infrequent shipping decreases shipping cost, but increases inventory carrying cost. Therefore we have focused on determining the optimal number of ordering cycles between two consecutive shipments of imperfect quality items, optimal shortage period, optimal lot size and expected average profit for the model in both of the time horizons, infinite and finite. Modification of the expected average profit by introducing purchase cost and screening cost in Maddah and Jaber's model [1] is shown in Section 5 of the present paper. Finally, our models in both finite and infinite time horizons are compared with Maddah and Jaber's model [1] through numerical examples.

The rest of the paper is organized as follows. The literature review is discussed in Section 2. Section 3 provides the notations and assumptions of the models. Mathematical formulation and solution are derived in Section 4. In Section 5, modification in Maddah and Jaber's [1] paper is performed. Section 6 presents numerical results together with graphical illustrations. Comparison are made in Section 7. At the end, summary and conclusion are drawn along with future research proposal in Section 8.

2. Literature review

A lot of work has been done on partial backlogging and also shipment of products. Many models are developed including any one of them or both of them. Regarding this, a brief literature survey is performed below:

Montgomery et al. [2] studied an EOQ (*Economic Ordered Quantity*) model involving partial backlogging and lost sale. Rosenberg [3], Park [4] also developed inventory models by considering partial backlogging. Abad [5] framed an inventory model by allowing partial backlogging. Wee [6] considered quantity discount, pricing and partial back ordering where inventory deterioration takes place with time. Papachristos and Skouri [7] determined optimal replenishment policy for deteriorating items. They used exponential partial back ordering. Teng et al. [8] also allowed partial backlogging in their model and obtained the optimal replenishment policy for deteriorating items. Teng and Yang [9] developed an EOQ model including partial back ordering where demand and cost fluctuate with time. Pal et al. [10] analyzed a deterministic inventory model involving partial backlogging and deteriorating items. They considered that the demand rate is affected by the selling price and advertizement. San Jose et al. [11] also considered an EOQ model where exponential function was used to do partial backlogging. Dye et al. [12] developed a model with deteriorating items and negative exponential partial backlogging where price-dependent demand is considered. Chern et al. [13] extended the traditional EOQ model by introducing not only deteriorating items but also partial backlogging and inflation. Das Roy et al. [14] analyzed an EOQ model where the items are categorized into two types such as perfect quality and imperfect quality. Imperfect quality items are sold at a discount price in a single batch. Exponential partial backlogging and lost sale were considered in their model. Again, a comparative study between the two models of shortages occurring at the beginning and at the end of the ordering cycle with partial back ordering is done by Das Roy et al. [15]. Several authors such as Cardenas-Barron [16,17], Abad [18,19], Ouyang and Chang [20], Zhou and Yang [21], Wee et al. [22,23], Wu et al. [24], Aksen [25], Lodree [26], Leung [27,28], Thangam and Uthayakumar [29], Vijayan and Kumaran [30], Panda et al. [31], San Jose et al. [32], Roy et al. [33], etc., analyzed different models with partial backlogging and obtained various results.

There are several authors who framed different types of inventory models including shipment. Goyal [34] introduced lot-for-lot shipments policy from the vendor to the buyer by assuming infinite rate of production. Banerjee [35], Goyal [36] developed models involving shipment. Gallego and Simchi-Levi [37] discussed the effects of direct shipping strategy. Blumenfeld et al. [38], Barnes-Schuster and Bassak [39] and Jones and Qian [40] developed models by considering direct shipping between supplier and customer. Russell and Krajewski [41] proposed a model with over-declaring shipments and quantity discounts. They determined the optimal purchase cost and transportation cost. Chien [42] discussed an inventory model where stochastic demand and one-to-one direct shipping policies are allowed. Again, Goyal [43] considered a model where size of the shipment increased by a factor which is equal to the ratio of the production rate and the demand rate. Hill [44] studied Goyal's [43] model and generalized it by considering the geometric growth factor as a decision variable. Geunes and Zeng [45] worked on one-to-one base-stock distribution system and proposed the impacts of inventory shortage policies. Abad and Aggarwal [46] structured an inventory model where the demand is price-sensitive. The over-declaring option, less-than-truckload or truckload size shipments are also considered. An integrated inventory model discussed by Hoque and Goyal [47] where the batches which are to be shipped are equal or unequal in sizes. A multiple shipment policy is determined by Siajadi et al. [48]. Ayanso et al. [49] framed model by using drop-shipping mechanism. Ertogral et al. [50] studied a vendor–buyer supply chain inventory model including shipment. Darwish [51] constructed a model where the issues of purchasing and transportation are emphasized. He considered the quantity and freight discounts and obtained the lot size and reorder point for continuous review inventory model. Maddah and Jaber [1] revisited Salameh and Jaber's [52] work and extended Salameh and Jaber's [52] paper by introducing shipment. They studied the consolidation of imperfect quality items and their shipment. Sajadieh et al. [53] framed an inventory model where shortages are allowed and the lead time between the vendor and the buyer is stochastic. They minimized the expected total cost for both vendor and the buyer and obtained the optimal production and shipment policy where all the shipments are considered to be equal sized batches. Zhou et al. [54] considered an inventory model using the concept of free shipping option. They used stochastic demand and offered free shipping policy if the ordered quantity is less than a certain quantity which is called free shipping quantity.

3. Notations and assumptions of the models

The notations used in this paper are as follows:

3.1. Notations

H: Length of the finite horizon
D: Demand rate in units per unit time
y: Order size for each cycle
B: Maximum back order level allowed
K: Fixed cost of placing an order
c: Purchasing cost per unit
p: Percentage of defective items in y
$f(p)$: Probability density function of p
s: Unit selling price of perfect quality items
v: Unit selling price of imperfect quality items, $v \leq c$
h: Holding cost per unit item per unit time
c_b: Back order cost per unit per unit time
c_l: Cost of lost sale per unit item
x: Screening rate in units per unit time
d: Unit screening cost
K_s: Shipping cost
$E(.)$: Expected value operator
$Q_{I_i}(t)$: On hand inventory at time t during ith ordering cycle, where $0 \leq t \leq t_1$ and $i = 1, 2, 3, \ldots, n$.
$Q_{S_i}(t)$: The level of negative inventory at time t during ith ordering cycle, where $t_1 \leq t \leq t_1 + t_2$ and $i = 1, 2, 3, \ldots, n$.
$Q_{l_i}(t)$: The lost sale quantity at time t during ith ordering cycle, where $t_1 \leq t \leq t_1 + t_2$ and $i = 1, 2, 3, \ldots, n$.

3.2. Assumptions

1. The inventory model is developed only for a single type of product.
2. Imperfect quality items are present.
3. Shortages are allowed at the end of the ordering cycle.
4. Partial backlogging is permitted.
5. Replenishment is instantaneous.
6. Lead time is zero.
7. The percentage of defective items in a lot follows a uniform distribution. All the fractions of imperfect quality items p_i in the ordering cycle i $(i = 1, 2, \ldots, n)$ of a shipping phase are independent and identically distributed.
8. T is the length of a shipping cycle and T_i $(i = 1, 2, 3, \ldots, n)$ is the length of each ordering cycle. All the ordering cycles are identical. $T_i = t_1 + t_2$, where t_1 is the time during which demand is met and t_2 is the shortage period. Also $T = \sum T_i$ where $i = 1, 2, 3, \ldots, n$.
9. Stock-out period is exponentially distributed.
10. $(1 - p)y - B$ items fulfill the demand during $(0, t_1)$ and remaining B items meet the demand during the stock-out period $(t_1, t_1 + t_2)$.
11. After the completion of the screening process, one portion of the perfect quality products is used to serve partial backlogging at a cost of c_b per unit and the rest of the stock is used to adjust the demand.
12. Shipment of the batches of imperfect quality products of n ordering cycles is considered.
13. Both the time horizons, infinite and finite, are considered.

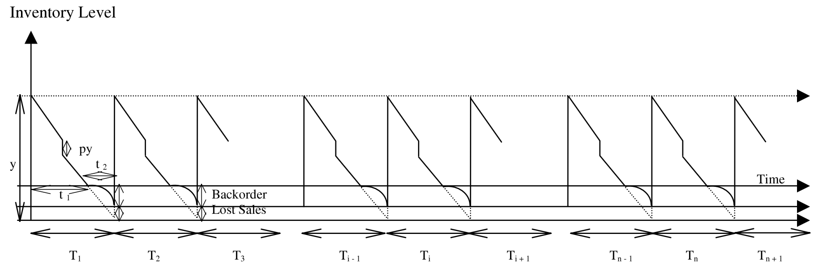

Fig. 1. Inventory model in infinite planning horizon.

4. Mathematical formulation and solution of the model

We consider that the ordering lot size of each cycle T_i $(i = 1, 2, 3, \ldots, n)$ which are equal in length is y. Among the lot of size y, the number of imperfect quality products is py whereas $(1 - p)y$ is the number of perfect quality products where $0 \leq p \leq 1$. The fraction of defective items in each cycle is independent and identically distributed (say p) that follows a uniform distribution function. Shortages occurred at the end of each ordering cycle during the time span $[t_1, t_1 + t_2]$. After 100% screening at a rate of x units per unit time, $\left(y - \frac{B}{1-p}\right)$ units of y are used to serve the demand Dt_1 during $[0, t_1]$ and the rest $\frac{B}{1-p}$ units are used for serving partial backlogging during the time period $[t_1, t_1 + t_2]$ at a cost of c_b per unit. In stock-out period $[t_1, t_1 + t_2]$, the demand $De^{-\delta(t_1+t_2-t)}$ at any time t is satisfied whereas the remaining part of the demand, $D(1 - e^{-\delta(t_1+t_2-t)})$ remains unsatisfied, where δ is a positive constant and $(t_1 + t_2 - t)$ is the waiting time until the next replenishment begins at time $(t_1 + t_2)$. During the shortage period, the demand of some customers who have no patience to wait is considered as the case of lost sale. This causes financial losses of the seller and he also loses his goodwill with the unsatisfied customers. Consequently, lost sale cost c_l per unit for such losses is considered here.

The demand D is always less than the screening rate x per unit time. To avoid shortages in $[0, t_1]$, i.e., $(1 - p)x \geq D$ must hold. The total shortages B at time $(t_1 + t_2)$ will be satisfied if $(1 - p)x \geq B + D$ because shortage can be avoided within the screening time if $E(1 - p)x \geq \text{Max}(B + D, D)$, i.e., $E(1 - p)x \geq B + D$.

The imperfect quality products of each ordering cycle are sold in a single batch with a discount price of v per unit and we consolidate a number of such batches of imperfect quality items to release those for shipment with a shipping cost K_s per shipment.

4.1. Infinite time horizon

Suppose that Q_{S_i} is the level of negative inventory at any time t during the ith ordering cycle where $t_1 \leq t \leq t_1 + t_2$ and $i = 1, 2, 3, \ldots, n$. (See Fig. 1). Then the governing differential equation of the negative inventory level at any time t during the ith ordering cycle is

$$\frac{dQ_{S_i}(t)}{dt} = -De^{-\delta(t_1+t_2-t)}, \quad t_1 \leq t \leq t_1 + t_2 \text{ with } Q_{S_i}(t_1) = 0. \tag{1}$$

Eq. (1) gives

$$Q_{S_i}(t) = -\frac{D}{\delta}(e^{-\delta(t_1+t_2-t)} - e^{-\delta t_2}).$$

The level of maximum back order per cycle is

$$B = -Q_{S_i}(t_1 + t_2) = \frac{D}{\delta}(1 - e^{-\delta t_2}).$$

The back ordering cost per shipping cycle is (see Appendix A)

$$\text{BC} = \frac{nc_b D}{\delta^2}(1 - \delta t_2 e^{-\delta t_2} - e^{-\delta t_2}).$$

The lost sale quantity at any time t is

$$Q_{l_i}(t) = D(1 - e^{-\delta(t_1+t_2-t)}), \quad t_1 \leq t \leq t_1 + t_2.$$

The lost sale cost per shipping cycle is (see Appendix B)

$$\text{LSC} = \frac{nc_l D}{\delta}(\delta t_2 - 1 + e^{-\delta t_2}).$$

Let $Q_{I_i}(t)$ be the inventory level at any time t during the ith ordering cycle where $0 \le t \le t_1$ and $i = 1, 2, 3, \ldots, n$. Then the governing differential equation of the inventory level at any time t during the ith ordering cycle is

$$\frac{dQ_{I_i}(t)}{dt} = -D, \quad 0 \le t \le t_1 \text{ with } Q_{I_i}(0) = (1-p_i)y - B.$$

The solution is

$$Q_{I_i}(t) = [(1-p_i)y - B] - Dt, \quad \text{where } i = 1, 2, 3, \ldots, n \text{ and } 0 \le t \le t_1.$$

Let y_1 be the initial order and y_2 is the next order. B is good quality items.

Then $(1-p_i)y - B = x_1$ and $y_1(1-p_i) = x_1$ so that $y_1 = \frac{x_1}{1-p_i} = \left(y - \frac{B}{1-p_i}\right)$.

Again, $(1-p_i)y_2 = B$ so that $y_2 = \frac{B}{1-p_i}$.

Now, $(1-p_i)y - B = Dt_1$.

Therefore, the total cycle length of an ordering cycle is

$$\begin{aligned} T_i &= t_1 + t_2 \\ &= \frac{(1-p_i)y - B}{D} + t_2 \\ &= \frac{(1-p_i)y + (Dt_2 - B)}{D}. \end{aligned} \tag{2}$$

The length of the shipping cycle is

$$\begin{aligned} T &= \sum_{i=1}^{n} T_i \\ &= \sum_{i=1}^{n} \frac{(1-p_i)y + (Dt_2 - B)}{D}. \end{aligned}$$

Let p_i's be independent and identically distributed. Therefore, $p_1 = p_2 = \cdots = p_n = p$ (say) so that

$$T = \frac{n[(1-p)y + (Dt_2 - B)]}{D}.$$

The inventory holding cost of perfect quality items per shipping cycle HC_p is (see Appendix C)

$$\begin{aligned} \text{HC}_p &= \sum_{i=1}^{n} \text{HC}_{p_i} \\ &= \frac{nh}{2D}[(1-p)y - B]^2. \end{aligned}$$

The expected holding cost of perfect quality items per shipping cycle is

$$\begin{aligned} E\text{HC}_p &= \frac{nh}{2D} E[(1-p)y - B]^2 \\ &= \frac{nh}{2D}[E[(1-p)^2]y^2 - 2By(1-E[p]) + B^2]. \end{aligned}$$

The inventory holding cost of imperfect quality items per shipping cycle HC_{imp} is (see Appendix D)

$$\begin{aligned} \text{HC}_{\text{imp}} &= h\left[\sum_{i=1}^{n} p_i y T_i + \sum_{i=1}^{n-1} p_i y \sum_{j=i+1}^{n} T_j + \sum_{i=1}^{n} \frac{p_i y_1^2}{x} + \sum_{i=1}^{n} \frac{p_i y_2^2}{x}\right] \\ &= h\left[\sum_{i=1}^{n} p_i y \left(\frac{(1-p_i)y + (Dt_2 - B)}{D}\right) + \sum_{i=1}^{n-1} p_i y \sum_{j=i+1}^{n} \left(\frac{(1-p_j)y + (Dt_2 - B)}{D}\right)\right. \\ &\quad \left. + \sum_{i=1}^{n} \frac{p_i}{x}\left(y - \frac{B}{1-p_i}\right)^2 + \sum_{i=1}^{n} \frac{p_i}{x}\left(\frac{B}{1-p_i}\right)^2\right] \\ &= nh\left[\frac{(n+1)p(1-p)y^2}{2D} + \frac{(n+1)(Dt_2 - B)py}{2D} + \frac{py^2}{x} - \frac{2B}{x}\left(\frac{p}{1-p}\right)y + \frac{2B^2}{x}\left(\frac{p}{(1-p)^2}\right)\right]. \end{aligned}$$

The expected holding cost of imperfect quality items per shipping cycle is

$$\mathrm{EHC_{imp}} = \frac{nh}{2D}\Bigg[(n+1)E[p(1-p)]y^2 + (n+1)(Dt_2 - B)E[p]y + \frac{2DE[p]y^2}{x} - \frac{4BD}{x}E\left[\frac{p}{1-p}\right]y + \frac{4DB^2}{x}E\left[\frac{p}{(1-p)^2}\right]\Bigg].$$

The total expected holding cost per shipping cycle is

$$\begin{aligned}\mathrm{EHC} &= \mathrm{EHC}_p + \mathrm{EHC_{imp}}\\ &= \frac{nh}{2D}\Bigg[E[(1-p)^2]y^2 - 2By(1-E[p]) + B^2 + (n+1)E[p(1-p)]y^2\\ &\quad + (n+1)(Dt_2 - B)E[p]y + \frac{2DE[p]y^2}{x} - \frac{4BD}{x}E\left[\frac{p}{1-p}\right]y + \frac{4DB^2}{x}E\left[\frac{p}{(1-p)^2}\right]\Bigg].\end{aligned}$$

The ordering cost per shipping cycle $\mathrm{OC} = K$.
The purchasing cost per shipping cycle $\mathrm{PC} = cy$.
The screening cost per shipping cycle $\mathrm{SC} = dy$.
The shipping cost of n cycles $\mathrm{SHC} = K_s$.

The total cost per shipping cycle is

$$\begin{aligned}TC(y, n, t_2) &= \mathrm{OC} + \mathrm{PC} + \mathrm{SC} + \mathrm{SHC} + \mathrm{HC} + \mathrm{BC} + \mathrm{LSC}\\ &= nK + ncy + ndy + K_s + \mathrm{HC} + \frac{nc_bD}{\delta^2}(1 - \delta t_2 \mathrm{e}^{-\delta t_2} - \mathrm{e}^{-\delta t_2}) + \frac{nc_lD}{\delta}(\delta t_2 - 1 + \mathrm{e}^{-\delta t_2}).\end{aligned}$$

The expected total cost per shipping cycle is

$$\begin{aligned}\mathrm{ETC}(y, n, t_2) &= n\Bigg[K + cy + dy + \frac{K_s}{n} + \frac{h}{2D}\bigg(E[(1-p)^2]y^2 - 2By(1-E[p]) + B^2\\ &\quad + (n+1)E[p(1-p)]y^2 + (n+1)(Dt_2 - B)E[p]y + \frac{2DE[p]y^2}{x} - \frac{4BD}{x}E\left[\frac{p}{1-p}\right]y\\ &\quad + \frac{4DB^2}{x}E\left[\frac{p}{(1-p)^2}\right]\bigg) + \frac{c_bD}{\delta^2}(1 - \delta t_2\mathrm{e}^{-\delta t_2} - \mathrm{e}^{-\delta t_2}) + \frac{c_lD}{\delta}(\delta t_2 - 1 + \mathrm{e}^{-\delta t_2})\Bigg].\end{aligned}$$

The expected revenue per shipping cycle is

$$\mathrm{ETR}(y, n, t_2) = n(s(1 - E[p]) + vE[p])y.$$

The expected length of the shipping cycle is

$$E[T] = \frac{n[(1 - E[p])y + (Dt_2 - B)]}{D}. \tag{3}$$

The expected average profit per shipping cycle is [55]

$$\begin{aligned}\mathrm{ETP}(y, n, t_2) &= \frac{\mathrm{ETR}(y, n, t_2) - ETC(y, n, t_2)}{E(T)}\\ &= \frac{D}{[(1 - E(p))y + (Dt_2 - B)]}\Bigg[s(1 - E(p))y + vE(p)y - K - cy - \mathrm{d}y\\ &\quad - \frac{K_s}{n} - \frac{h}{2D}\bigg(E[(1-p)^2]y^2 - 2By(1-E[p]) + B^2 + (n+1)E[p(1-p)]y^2\\ &\quad + (n+1)(Dt_2 - B)E[p]y + \frac{2DE[p]y^2}{x} - \frac{4BD}{x}E\left[\frac{p}{1-p}\right]y\\ &\quad + \frac{4DB^2}{x}E\left[\frac{p}{(1-p)^2}\right]\bigg) - \frac{c_bD}{\delta^2}(1 - \delta t_2\mathrm{e}^{-\delta t_2} - \mathrm{e}^{-\delta t_2}) - \frac{c_lD}{\delta}(\delta t_2 - 1 + \mathrm{e}^{-\delta t_2})\Bigg]\end{aligned}$$

$$= \frac{1}{[e_1 y + (Dt_2 - B)]}\left[e_2 y - D\left(K + \frac{K_s}{n}\right) - e_3 y^2 - (n+1)e_4 y^2 + e_5 By - (n+1)e_6(Dt_2 - B)y - e_7 B^2 - \frac{D^2}{\delta^2}\{(c_b - c_l\delta)(1 - \mathrm{e}^{-\delta t_2}) - \delta t_2(c_b \mathrm{e}^{-\delta t_2} - c_l\delta)\}\right] \quad (4)$$

where

$$e_1 = 1 - E(p) > 0, \qquad e_2 = D[(s - v)e_1 + (v - c - d)],$$
$$e_3 = \frac{h}{2}\left[E[(1-p)^2] + \frac{2E(p)D}{x}\right] = \frac{h}{2}\left[e_1^2 + \mathrm{Var}(p) + \frac{2E(p)D}{x}\right] > 0, \quad \text{(see Appendix E)}$$
$$e_4 = \frac{h}{2}E[p(1-p)], \qquad e_5 = \frac{h}{2}\left[2e_1 + \frac{4D}{x}E\left[\frac{p}{1-p}\right]\right] > 0,$$
$$e_6 = \frac{h}{2}E[p] > 0, \qquad e_7 = \frac{h}{2}\left[1 + \frac{4D}{x}E\left[\frac{p}{(1-p)^2}\right]\right] > 0.$$

Our aim is to determine the optimum number of ordering cycles per shipment, optimal lot size y and optimal shortage period t_2 which maximize the expected average profit function.

Setting $\frac{\partial \mathrm{ETP}}{\partial n} = 0$, $\frac{\partial \mathrm{ETP}}{\partial y} = 0$, and $\frac{\partial \mathrm{ETP}}{\partial t_2} = 0$ to obtain the optimum values of n, y and t_2, we have as follows: $\frac{\partial \mathrm{ETP}}{\partial n} = 0$ gives

$$n^*(y, t_2) = \sqrt{\frac{DK_s}{y[e_4 y + e_6(Dt_2 - B)]}}. \quad (5)$$

$\frac{\partial \mathrm{ETP}}{\partial y} = 0$ gives

$$e_1(e_3 + (n+1)e_4)y^2 + 2(e_3 + (n+1)e_4)\alpha(t_2)y - \left[De_1\left(K + \frac{K_s}{n}\right) + \alpha(t_2)\beta(t_2) + e_1\gamma(t_2) - (n+1)e_6(\alpha(t_2))^2\right] = 0$$

where

$$\alpha(t_2) = Dt_2 - B, \qquad \beta(t_2) = e_2 + e_5 B > 0,$$
$$\gamma(t_2) = \frac{D^2}{\delta^2}[e_7(1 - \mathrm{e}^{-\delta t_2})^2 + (c_b - c_l\delta)(1 - \mathrm{e}^{-\delta t_2}) - \delta t_2(c_b \mathrm{e}^{-\delta t_2} - c_l\delta)].$$

The above equation can be written as $Ly^2 + My + N = 0$ where

$$L = e_1(e_3 + (n+1)e_4) > 0, \qquad M = 2(e_3 + (n+1)e_4)\alpha(t_2) > 0,$$
$$N = De_1\left(K + \frac{K_s}{n}\right) + \alpha(t_2)\beta(t_2) + e_1\gamma(t_2) - (n+1)e_6(\alpha(t_2))^2.$$

The solution of the above equation is

$$y^*(n, t_2) = \frac{1}{2L}[-M + \sqrt{M^2 + 4LN}] \quad \text{as } y^* > 0, \quad (6)$$

where $M^2 + 4LN$ must be positive for real values of y^*.

Also, $\frac{\partial \mathrm{ETP}}{\partial t_2} = 0$ gives

$$\left[e_1\frac{\partial\beta(t_2)}{\partial t_2} + \frac{\partial\alpha(t_2)}{\partial t_2}(e_3 + (n+1)(e_4 - e_1e_6))\right]y^2 - \left[e_1\frac{\partial\gamma(t_2)}{\partial t_2} + \frac{\partial\alpha(t_2)}{\partial t_2}\beta(t_2) - \alpha(t_2)\frac{\partial\beta(t_2)}{\partial t_2}\right]y - \left[\alpha(t_2)\frac{\partial\gamma(t_2)}{\partial t_2} - \frac{\partial\alpha(t_2)}{\partial t_2}\gamma(t_2) - D\left(K + \frac{K_s}{n}\right)\frac{\partial\alpha(t_2)}{\partial t_2}\right] = 0.$$

The above equation can be written as $Uy^2 - Vy - W = 0$ where

$$U = e_1\frac{\partial\beta(t_2)}{\partial t_2} + \frac{\partial\alpha(t_2)}{\partial t_2}(e_3 + (n+1)(e_4 - e_1e_6)),$$
$$V = e_1\frac{\partial\gamma(t_2)}{\partial t_2} + \frac{\partial\alpha(t_2)}{\partial t_2}\beta(t_2) - \alpha(t_2)\frac{\partial\beta(t_2)}{\partial t_2},$$
$$W = \alpha(t_2)\frac{\partial\gamma(t_2)}{\partial t_2} - \frac{\partial\alpha(t_2)}{\partial t_2}\gamma(t_2) - D\left(K + \frac{K_s}{n}\right)\frac{\partial\alpha(t_2)}{\partial t_2}.$$

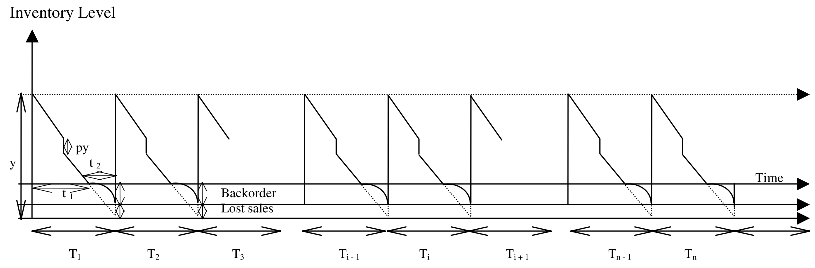

Fig. 2. Inventory model in finite planning horizon.

The solution of the above equation is

$$y^{**}(n, t_2) = \frac{1}{2U}[V + \sqrt{V^2 + 4UW}] \quad \text{as } y^{**} > 0, \tag{7}$$

where $V^2 + 4UW$ must be positive for real values of y^{**}.

From Eqs. (6) and (7), we get

$$\frac{1}{2L}[-M + \sqrt{M^2 + 4LN}] = \frac{1}{2U}[V + \sqrt{V^2 + 4UW}]. \tag{8}$$

Solving Eq. (8), we get the value of t_2 in terms of n. Using this value of t_2 in any one of Eqs. (6) or (7), we obtain the value of the lot size y. Substituting these two values of y and t_2 in (5), we get the optimal value of the number of ordering cycle n i.e. n^*. Using these values of n^*, we finally get the optimal lot size y i.e. y^* and shortage period t_2 i.e. t_2^*. (See Fig. 6)

4.2. Finite time horizon

Let us consider the length of the time horizon to be H which is finite (see Fig. 2).

Now, we have from Eq. (3),

$$E[T] = \frac{n[(1 - E[p])y + (Dt_2 - B)]}{D} = H(say).$$

Therefore,

$$H = \frac{n[(1 - E[p])y + (Dt_2 - B)]}{D}$$

$$\text{or,} \quad y = \frac{1}{w_1}\left[\frac{DH}{n} - (Dt_2 - B)\right] \tag{9}$$

where $w_1 = 1 - E[p]$.

Now, substituting the value of y from Eq. (9) in Eq. (4), we have

$$\begin{aligned}
\text{ETP}(n, t_2) = {} & \frac{n}{DH}\Bigg[\frac{1}{w_1}e_2\left\{\frac{DH}{n} - (Dt_2 - B)\right\} - D\left(K + \frac{K_s}{n}\right) - \frac{1}{w_1}e_3\left\{\frac{DH}{n} - (Dt_2 - B)\right\}^2 \\
& - \frac{1}{w_1}(n+1)e_4\left\{\frac{DH}{n} - (Dt_2 - B)\right\}^2 + \frac{1}{w_1}e_5B\left\{\frac{DH}{n} - (Dt_2 - B)\right\} - \frac{1}{w_1}(n+1)e_6(Dt_2 - B) \\
& \times \left\{\frac{DH}{n} - (Dt_2 - B)\right\} - e_7B^2 - \frac{D^2}{\delta^2}\{(c_b - c_l\delta)(1 - \mathrm{e}^{-\delta t_2}) - \delta t_2(c_b\mathrm{e}^{-\delta t_2} - c_l\delta)\}\Bigg] \\
= {} & \frac{1}{DH}\Bigg[\{w_2 + w_5B - (n+1)w_6(Dt_2 - B)\}\{DH - n(Dt_2 - B)\} \\
& - D(nK + K_s) - \left\{\frac{w_3}{n} + \left(1 + \frac{1}{n}\right)w_4\right\}\{DH - n(Dt_2 - B)\}^2 \\
& - n\left\{w_7B^2 + \frac{D^2}{\delta^2}((c_b - c_l\delta)(1 - \mathrm{e}^{-\delta t_2}) - \delta t_2(c_b\mathrm{e}^{-\delta t_2} - c_l\delta))\right\}\Bigg]
\end{aligned} \tag{10}$$

where

$$w_1 = e_1 > 0, \quad w_2 = \frac{e_2}{w_1}, \quad w_3 = \frac{e_3}{w_1^2} > 0, \quad w_4 = \frac{e_4}{w_1^2}, \quad w_5 = \frac{e_5}{w_1} > 0,$$
$$w_6 = \frac{e_6}{w_1} > 0, \quad w_7 = e_7 > 0.$$

Setting $\frac{\partial \text{ETP}}{\partial t_2} = 0 = \frac{\partial \text{ETP}}{\partial n}$ to obtain the optimum values of t_2 and n, we have as follows: $\frac{\partial \text{ETP}}{\partial t_2} = 0$ gives

$$\begin{aligned}&2(w_6 - w_4)(Dt_2 - B)(1 - e^{-\delta t_2})n^2 - [2(w_3 + w_4 - w_6)(Dt_2 - B)(1 - e^{-\delta t_2})\\ &\quad + (w_2 + w_5 B + Dc_l)(1 - e^{-\delta t_2}) + e^{-\delta t_2}(w_5(Dt_2 - B) + 2Bw_7 + Dc_b t_2)\\ &\quad - DH(2w_4 - w_6)(1 - e^{-\delta t_2})]n - DH[(w_6 - 2w_3 - 2w_4)(1 - e^{-\delta t_2}) - w_5 e^{-\delta t_2}] = 0. \qquad (11)\end{aligned}$$

The above equation can be written as $\theta(t_2)n^2 - \phi(t_2)n - \psi(t_2) = 0$ where

$$\begin{aligned}\theta(t_2) &= 2(w_6 - w_4)(Dt_2 - B)(1 - e^{-\delta t_2}),\\ \phi(t_2) &= 2(w_3 + w_4 - w_6)(Dt_2 - B)(1 - e^{-\delta t_2}) + (w_2 + w_5 B + Dc_l)(1 - e^{-\delta t_2})\\ &\quad + e^{-\delta t_2}(w_5(Dt_2 - B) + 2Bw_7 + Dc_b t_2) - DH(2w_4 - w_6)(1 - e^{-\delta t_2}),\\ \psi(t_2) &= DH[(w_6 - 2w_3 - 2w_4)(1 - e^{-\delta t_2}) - w_5 e^{-\delta t_2}].\end{aligned}$$

The solution of the above equation is

$$n^*(t_2) = \frac{1}{2\theta(t_2)}[\phi(t_2) - \sqrt{(\phi(t_2))^2 + 4\theta(t_2)\psi(t_2)}] \quad \text{as } n^* > 0, \qquad (12)$$

where $\phi(t_2)^2 + 4\theta(t_2)\psi(t_2)$ must be positive for real values of n^*.

Again $\frac{\partial \text{ETP}}{\partial n} = 0$ gives

$$\begin{aligned}&2(w_6 - w_4)(Dt_2 - B)^2 n^3 + \Bigg[(w_6 - w_3 - w_4)(Dt_2 - B)^2 + (2DHw_4 - DHw_6 - w_2 - w_5 B)(Dt_2 - B)\\ &\quad - \left\{DK + w_7 B^2 + \frac{D^2}{\delta^2}((c_b - c_l\delta)(1 - e^{-\delta t_2}) - \delta t_2(c_b e^{-\delta t_2} - c_l\delta))\right\}\Bigg]n^2 + D^2H^2(w_3 + w_4) = 0. \qquad (13)\end{aligned}$$

Applying Eq. (11) $\times\, n(Dt_2 - B) - (13) \times (1 - e^{-\delta t_2})$, we get

$$\begin{aligned}&\Bigg[(w_3 + w_4 - w_6)(Dt_2 - B)^2(1 - e^{-\delta t_2}) + Dc_l(Dt_2 - B)(1 - e^{-\delta t_2}) + e^{-\delta t_2}(Dt_2 - B)\{w_5(Dt_2 - B)\\ &\quad + 2Bw_7 + Dc_b t_2\} - \left\{DK + w_7 B^2 + \frac{D^2}{\delta^2}((c_b - c_l\delta)(1 - e^{-\delta t_2}) - \delta t_2(c_b e^{-\delta t_2} - c_l\delta))\right\}(1 - e^{-\delta t_2})\Bigg]n^2\\ &\quad - DH(Dt_2 - B)[(2w_3 + 2w_4 - w_6)(1 - e^{-\delta t_2}) + w_5 e^{-\delta t_2}]n + D^2H^2(w_3 + w_4)(1 - e^{-\delta t_2}) = 0.\end{aligned}$$

The above equation can be written as $\xi(t_2)n^2 - \eta(t_2)n + \zeta(t_2) = 0$ where

$$\begin{aligned}\xi(t_2) &= (w_3 + w_4 - w_6)(Dt_2 - B)^2(1 - e^{-\delta t_2}) + Dc_l(Dt_2 - B)(1 - e^{-\delta t_2}) + e^{-\delta t_2}(Dt_2 - B)\{w_5(Dt_2 - B)\\ &\quad + 2Bw_7 + Dc_b t_2\} - \left\{DK + w_7 B^2 + \frac{D^2}{\delta^2}((c_b - c_l\delta)(1 - e^{-\delta t_2}) - \delta t_2(c_b e^{-\delta t_2} - c_l\delta))\right\}(1 - e^{-\delta t_2}),\\ \eta(t_2) &= DH(Dt_2 - B)[(2w_3 + 2w_4 - w_6)(1 - e^{-\delta t_2}) + w_5 e^{-\delta t_2}],\\ \zeta(t_2) &= D^2H^2(w_3 + w_4)(1 - e^{-\delta t_2}).\end{aligned}$$

The solution is

$$n^{**}(t_2) = \frac{1}{2\xi(t_2)}[\eta(t_2) - \sqrt{(\eta(t_2))^2 - 4\xi(t_2)\zeta(t_2)}] \quad \text{as } n^{**} > 0 \qquad (14)$$

where $(\eta(t_2))^2 - 4\xi(t_2)\zeta(t_2)$ must be positive for real values of n^{**}.

From Eqs. (12) and (14), we get

$$\frac{1}{2\theta(t_2)}[\phi(t_2) - \sqrt{(\phi(t_2))^2 + 4\theta(t_2)\psi(t_2)}] = \frac{1}{2\xi(t_2)}[\eta(t_2) - \sqrt{(\eta(t_2))^2 - 4\xi(t_2)\zeta(t_2)}]. \qquad (15)$$

Solving Eq. (15), we get the optimal shortage period t_2 i.e. t_2^*. Substituting this value of t_2^* in any Eqs. (12) or (14), we get the optimal number of ordering cycles for each shipment i.e. n^*. With the help of these two values of t_2^* and n^*, we obtain the optimal lot size y^* (see Fig. 7) from Eq. (9).

5. Modification in Maddah and Jaber's [1] paper

Maddah and Jaber [1] developed an inventory model where they used shipment of imperfect batches of different ordering cycles. They have neglected the purchase cost and the screening cost. The total expected cost per unit time given by them is

$$\mathrm{ECT}(y,n) = \frac{1}{1-E[p]}\left[\left(K+\frac{K_s}{n}\right)\frac{D}{y}+\frac{hy}{2}\times\left\{E[(1-p)^2]-\frac{2(n-1)}{n}\mathrm{Var}[p]+(n-1)E[p](1-E[p])+2E[p]\frac{D}{x}\right\}\right].$$

If we introduce the purchasing cost and the screening cost, then the total expected cost per unit time becomes

$$\mathrm{ECT}(y,n) = \frac{1}{1-E[p]}\left[D(c+d)+\left(K+\frac{K_s}{n}\right)\frac{D}{y}+\frac{hy}{2}\times\left\{E[(1-p)^2]-\frac{2(n-1)}{n}\mathrm{Var}[p]+(n-1)E[p](1-E[p])+2E[p]\frac{D}{x}\right\}\right].$$

Also the expected total revenue $\mathrm{ETR} = \{s(1-E[p])+vE[p]\}y$

and the expected length of the ordering cycle is $E[T] = \frac{(1-E[P])y}{D}$.

Hence, the expected average profit is

$$\begin{aligned}\mathrm{ETP}(y,n) &= \frac{1}{1-E[p]}\left[D\{s(1-E[p])+vE[p]-c-d\}-\left(K+\frac{K_s}{n}\right)\frac{D}{y}-\frac{hy}{2}\left\{E[(1-p)^2]\right.\right.\\ &\quad\left.\left.-\frac{2(n-1)}{n}\mathrm{Var}[p]+(n-1)E[p](1-E[p])+2E[p]\frac{D}{x}\right\}\right]\\ &= \frac{1}{e_1}\left[D(se_1+vE[p]-c-d)-\left(K+\frac{K_s}{n}\right)\frac{D}{y}-\frac{hy}{2}\left\{e_1^2+\mathrm{Var}(p)+2E[p]\frac{D}{x}\right.\right.\\ &\quad\left.\left.-\frac{2(n-1)}{n}\mathrm{Var}[p]+(n-1)E[p](1-E[p])\right\}\right] \qquad (16)\end{aligned}$$

where $e_1 = 1-E[p]$.

6. Numerical examples

Example 1. We consider an inventory situation where the imperfect quality items are present and batches of such imperfect items are taken together from several ordering cycles and are shipped with a constant cost. Shortages occur at the end of the ordering cycle and are partially backlogged. In an ordered lot, imperfect quality items follow a uniform distribution with the following probability density function

$$f(p) = 25, \quad \text{when } 0 \le p \le 0.04$$
$$= 0, \quad \text{otherwise.}$$

Let the values of the other parameters of the inventory system where the time horizon is considered to be infinite are as follows:

$K = \$100$, $K_s = \$50$, $D = 50\,000$ units, $h = \$5$, $c_b = \$4$, $x = 175\,200$ units, $d = \$0.5$, $c = \$25$, $s = \$50$, $c_l = \$26$, $v = \$20$, $\delta = 0.2$ in appropriate units.

With the help of the above values, we get $e_1 = 0.98$, $\mathrm{Var}(p) = 0.0004/3$, $e_2 = 1.195\times 10^6$, $e_3 = 2.42987$, $e_4 = 0.0486667$, $e_5 = 4.95865$, $e_6 = 0.05$, $e_7 = 2.56026$ and the optimal lot size y, the shortage period t_2, the number of ordering cycles for shipment n and the expected average profit ETP are as follows: $n^* = 4.34529 \cong 4$, $y^* = 1663.41$ units, $t_2^* = 0.00860252 \cong 0.01$ unit and $\mathrm{ETP}^* = \$1\,212\,490$.

We have seen that for $n = 4$, $y^* = 1663.41$ units, $t_2^* = 0.00860252 \cong 0.01$ unit and $\mathrm{ETP}^* = \$1\,212\,490$ and for $n = 5$, $y^* = 1625.48$ units, $t_2^* = 0.0084063 \cong 0.01$ unit and $\mathrm{ETP}^* = \$1\,212\,480$. As $\mathrm{ETP}(4) > \mathrm{ETP}(5)$ (see Fig. 3), the optimal number of ordering cycle for shipment will be $n^* = 4$ and the corresponding $\mathrm{ETP}^* = \$1\,212\,490$.

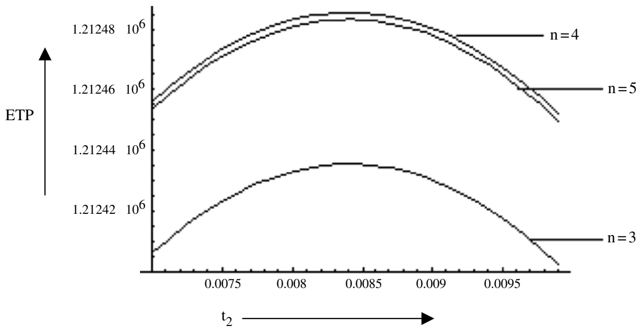

Fig. 3. ETP versus t_2 for various n on infinite time horizon.

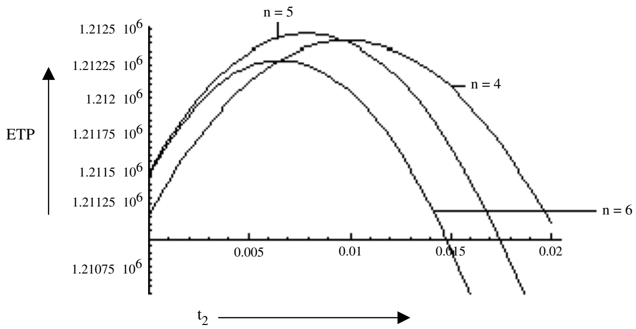

Fig. 4. ETP versus t_2 for various n on finite time horizon.

The Hessian matrix $H = \begin{pmatrix} \frac{\partial^2 \text{ETP}}{\partial n^2} & \frac{\partial^2 \text{ETP}}{\partial n \partial y} & \frac{\partial^2 \text{ETP}}{\partial n \partial t_2} \\ \frac{\partial^2 \text{ETP}}{\partial y \partial n} & \frac{\partial^2 \text{ETP}}{\partial y^2} & \frac{\partial^2 \text{ETP}}{\partial y \partial t_2} \\ \frac{\partial^2 \text{ETP}}{\partial t_2 \partial n} & \frac{\partial^2 \text{ETP}}{\partial t_2 \partial y} & \frac{\partial^2 \text{ETP}}{\partial t_2^2} \end{pmatrix}$ at the optimal solution $n^* = 4, y^* = 1663.41, t_2^* = 0.00860252$ is $\begin{pmatrix} -47.9144 & -0.107257 & -5.08137 \\ -0.107257 & -0.00327898 & 151.783 \\ -5.08137 & 151.783 & -2.93584 \times 10^7 \end{pmatrix}$ which is negative definite, because all the eigenvalues of the matrix are negative ($x = -2.93584 \times 10^7, -47.9146, -0.00225404$).

Hence, the required optimal solution is $n^* = 4, y^* = 1663.41$ units, $t_2^* = 0.00860252$ unit and $\text{ETP}^* = \$1\,212\,490$.

Example 2. Now we consider an inventory situation where all the parameters are the same as mentioned in Example 1, but the time horizon is finite with the parameter value $H = 0.15$ unit. Therefore, the optimal number of ordering cycles for shipment n, optimal shortage period t_2, optimal lot size y and the expected average profit ETP are as follows:

$$n^* = 4.63929 \cong 5, \qquad y^* = 1649.25 \text{ units}, \qquad t_2^* = 0.0085293 \cong 0.01 \text{ unit} \quad \text{and} \quad \text{ETP}^* = \$1\,212\,490.$$

We have seen that for $n = 4, y^* = 1912.77$ units, $t_2^* = 0.00989377 \cong 0.01$ unit and $\text{ETP}^* = \$1\,212\,420$ and for $n = 5, y^* = 1530.29$ units, $t_2^* = 0.0079135 \cong 0.01$ unit and $\text{ETP}^* = \$1\,212\,470$. As ETP(5) $>$ ETP(4) (see Fig. 4), the optimal number of ordering cycles for shipment will be $n^* = 5$ and the corresponding $\text{ETP}^* = \$1\,212\,470$.

Since at the optimal solution $n^* = 5, t_2^* = 0.0079135$, we have

$$\frac{\partial^2 \text{ETP}}{\partial n^2} = -309.688 < 0, \qquad \frac{\partial^2 \text{ETP}}{\partial t_2^2} = -3.19457 \times 10^7 < 0 \quad \text{and}$$

$$H = \frac{\partial^2 \text{ETP}}{\partial n^2} \times \frac{\partial^2 \text{ETP}}{\partial t_2^2} - \left(\frac{\partial^2 \text{ETP}}{\partial n \partial t_2}\right)^2 = 9.8932 \times 10^9 > 0.$$

Therefore, the Hessian matrix $H = \begin{pmatrix} \frac{\partial^2 \text{ETP}}{\partial n^2} & \frac{\partial^2 \text{ETP}}{\partial n \partial t_2} \\ \frac{\partial^2 \text{ETP}}{\partial t_2 \partial n} & \frac{\partial^2 \text{ETP}}{\partial t_2^2} \end{pmatrix}$ is negative definite.

Hence, the required optimal solution is $n^* = 5, y^* = 1530.29$ units, $t_2^* = 0.0079135 \cong 0.01$ unit and $\text{ETP}^* = \$1\,212\,470$.

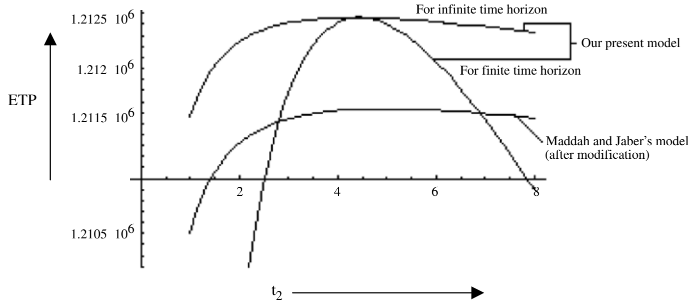

Fig. 5. Comparison of ETP of our models with Maddah and Jaber's modified model.

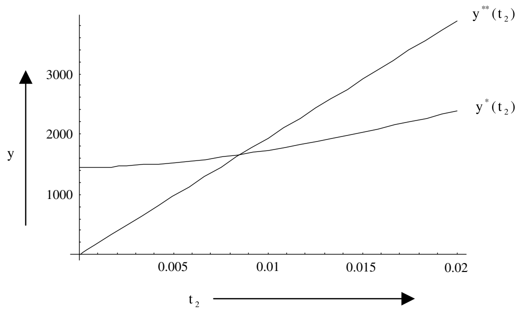

Fig. 6. The value of $y^*(t_2)$ and $y^{**}(t_2)$ versus t_2 for Example 1.

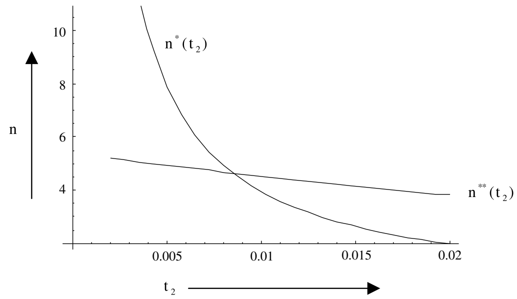

Fig. 7. The value of $n^*(t_2)$ and $n^{**}(t_2)$ versus t_2 for Example 2.

7. Comparison

Maddah and Jaber [1] discussed the issue of consolidation of imperfect quality items. They did not allow the occurrence of shortages in their model. We have also analyzed inventory models including the issue of consolidation of imperfect quality products, adding the occurrence of shortages at the end of an ordering cycle, partial back ordering with a back ordering cost c_b, lost sale cost c_l, etc.

With the help of the same parameter values, as mentioned in Maddah and Jaber's [1] paper, we have got the expected average profit by using modified Maddah and Jaber's [1] model (from Eq. (16)) to be $\text{ETP}^* = \$1\,211\,630$.

Adding some extra parameter values such as $c_b = \$4$, $c_l = \$26$ and $H = 0.15$ unit, we get the expected average profit from Eqs. (4) and (10) to be $\text{ETP}^* = \$1\,212\,490$ and $\text{ETP}^* = \$1\,212\,470$ on the infinite and finite planning horizons respectively.

We have noticed that, in both the time horizons, infinite and finite, the expected average profit ETP^* of our models is much more than that in Maddah and Jaber's [1] model (see Fig. 5).

8. Summary and conclusion

Here, we have framed a deterministic inventory model where we have studied the exponential partial backlogging of the unsatisfied demand and shipment of the batches of imperfect quality items. If the number of ordering cycles is increased in a single shipping cycle, then shipping cost is saved but inventory carrying cost is increased and vice versa. Keeping these in mind, we have developed EOQ ('*Economic Order Quantity*') models to balance these two costs so that the expected cost is reduced and the expected average profit is maximized. We have discussed our models on both the time horizons. In any real life situation, occurrence of shortages, partial backlogging, shipment—all these are very natural phenomena. Through numerical studies, we have shown that our models, where all these events are included, are more profitable than the model given by Maddah and Jaber [1] where shortages are not allowed. In a realistic point of view, we would like to acknowledge the fact that our model is more acceptable and applicable than Maddah and Jaber's [1] model.

Future research efforts can be undertaken by considering that the ordered lot size and the cycle length of the ordering cycles of a single shipping cycle are not all equal. Another interesting issue would be to develop the model for multi-items.

Appendix A

We have from Eq. (1),

$$Q_{S_i}(t) = -\frac{D}{\delta}(\mathrm{e}^{-\delta(t_1+t_2-t)} - \mathrm{e}^{-\delta t_2}).$$

The total back ordering cost per shipping cycle is

$$\begin{aligned}
\mathrm{BC} &= nc_b \int_{t_1}^{t_1+t_2} \{-Q_{S_i}(t)\}\mathrm{d}t \\
&= \frac{nc_bD}{\delta} \int_{t_1}^{t_1+t_2} \{\mathrm{e}^{-\delta(t_1+t_2-t)} - \mathrm{e}^{-\delta t_2}\}\mathrm{d}t \\
&= \frac{nc_bD}{\delta} \left[\frac{\mathrm{e}^{-\delta(t_1+t_2-t)}}{\delta} - \mathrm{e}^{-\delta t_2} t\right]_{t_1}^{t_1+t_2} \\
&= \frac{nc_bD}{\delta^2} [\mathrm{e}^{-\delta(t_1+t_2-t)} - \delta \mathrm{e}^{-\delta t_2} t]_{t_1}^{t_1+t_2} \\
&= \frac{nc_bD}{\delta^2} (1 - \delta t_2 \mathrm{e}^{-\delta t_2} - \mathrm{e}^{-\delta t_2}).
\end{aligned}$$

Appendix B

The total lost sale cost per shipping cycle is

$$\begin{aligned}
\mathrm{LSC} &= nc_l \int_{t_1}^{t_1+t_2} Q_{l_i}(t)\mathrm{d}t \\
&= nc_lD \int_{t_1}^{t_1+t_2} \{1 - \mathrm{e}^{-\delta(t_1+t_2-t)}\}\mathrm{d}t \\
&= nc_lD \left[t - \frac{\mathrm{e}^{-\delta(t_1+t_2-t)}}{\delta}\right]_{t_1}^{t_1+t_2} \\
&= nc_lD \left[t_2 - \frac{1}{\delta}(1 - \mathrm{e}^{-\delta t_2})\right] \\
&= \frac{nc_lD}{\delta} (\delta t_2 - 1 + \mathrm{e}^{-\delta t_2}).
\end{aligned}$$

Appendix C

The inventory holding cost of perfect quality items in the ith ordering cycle HC_{p_i} is

$$\begin{aligned}
\mathrm{HC}_{p_i} &= h \int_0^{t_1} Q_{l_i}(t)\mathrm{d}t \quad \text{where } i = 1, 2, 3, \ldots, n \\
&= h \int_0^{t_1} [\{(1 - p_i)y - B\} - Dt]\mathrm{d}t \\
&= h \left[\{(1 - p_i)y - B\}t - \frac{Dt^2}{2}\right]_0^{t_1} \\
&= h \left[\{(1 - p_i)y - B\}t_1 - \frac{Dt_1^2}{2}\right]
\end{aligned}$$

$$= h\left[\{(1-p_i)y-B\}\frac{\{(1-p_i)y-B\}}{D} - \frac{D}{2}\left\{\frac{(1-p_i)y-B}{D}\right\}^2\right] \quad \text{since } t_1 = \frac{\{(1-p_i)y-B\}}{D}$$
$$= h\left[\frac{\{(1-p_i)y-B\}^2}{D} - \frac{\{(1-p_i)y-B\}^2}{2D}\right]$$
$$= \frac{h}{2D}[(1-p_i)y-B]^2.$$

Here, the length of all ordering cycles in a shipping cycle are equal and

$$p_1 = p_2 = \cdots = p_n = p.$$

Therefore, the inventory holding cost of perfect quality items per shipping cycle HC_p is

$$\mathrm{HC}_p = \sum_{i=1}^{n}\mathrm{HC}_{p_i} = \mathrm{HC}_{p_1} + \mathrm{HC}_{p_2} + \cdots + \mathrm{HC}_{p_n}$$
$$= \frac{h}{2D}[(1-p_1)y-B]^2 + \frac{h}{2D}[(1-p_2)y-B]^2 + \cdots + \frac{h}{2D}[(1-p_n)y-B]^2$$
$$= \frac{h}{2D}[(1-p)y-B]^2 + \frac{h}{2D}[(1-p)y-B]^2 + \cdots + \frac{h}{2D}[(1-p)y-B]^2$$
$$= \frac{nh}{2D}[(1-p)y-B]^2.$$

Appendix D

The inventory holding cost of imperfect quality items per shipping cycle $\mathrm{HC}_{\mathrm{imp}}$ is

$$\mathrm{HC}_{\mathrm{imp}} = h\left[\sum_{i=1}^{n}p_iyT_i + \sum_{i=1}^{n-1}p_iy\sum_{j=i+1}^{n}T_j + \sum_{i=1}^{n}\frac{p_iy_1^2}{x} + \sum_{i=1}^{n}\frac{p_iy_2^2}{x}\right]$$
$$= h\Bigg[\sum_{i=1}^{n}p_iy\left(\frac{(1-p_i)y+(Dt_2-B)}{D}\right) + \sum_{i=1}^{n-1}p_iy\sum_{j=i+1}^{n}\left(\frac{(1-p_j)y+(Dt_2-B)}{D}\right)$$
$$+ \sum_{i=1}^{n}\frac{p_i}{x}\left(y-\frac{B}{1-p_i}\right)^2 + \sum_{i=1}^{n}\frac{p_i}{x}\left(\frac{B}{1-p_i}\right)^2\Bigg].$$

Since $p_1, p_2, \ldots, p_n$ are independent and identically distributed, we get $p_1 = p_2 = \cdots = p_n = p$ (say) so that

$$\sum_{i=1}^{n}p_iy\left(\frac{(1-p_i)y+(Dt_2-B)}{D}\right) = p_1y\left(\frac{(1-p_1)y+(Dt_2-B)}{D}\right) + p_2y\left(\frac{(1-p_2)y+(Dt_2-B)}{D}\right) + \cdots$$
$$+ p_ny\left(\frac{(1-p_n)y+(Dt_2-B)}{D}\right)$$
$$= py\left(\frac{(1-p)y+(Dt_2-B)}{D}\right) + py\left(\frac{(1-p)y+(Dt_2-B)}{D}\right) + \cdots$$
$$+ py\left(\frac{(1-p)y+(Dt_2-B)}{D}\right)$$
$$= npy\left(\frac{(1-p)y+(Dt_2-B)}{D}\right)$$

and

$$\sum_{i=1}^{n-1}p_iy\sum_{j=i+1}^{n}T_j + \sum_{i=1}^{n}\frac{p_iy_1^2}{x} = \sum_{i=1}^{n-1}p_iy(n-i)\left(\frac{(1-p)y+(Dt_2-B)}{D}\right)$$
$$= \left(\frac{(1-p)y+(Dt_2-B)}{D}\right)y[p_1(n-1) + p_2(n-2) + \cdots + p_{n-1}.1]$$
$$= \left(\frac{(1-p)y+(Dt_2-B)}{D}\right)y[p((n-1) + (n-2) + \cdots + 1)]$$

$$= \frac{n(n-1)}{2D}[(1-p)y + (Dt_2 - B)]yp$$
$$= \frac{n(n-1)p(1-p)y^2}{2D} + \frac{n(n-1)(Dt_2 - B)py}{2D}.$$

Also

$$\begin{aligned}\mathrm{HC_{imp}} &= h\Bigg[npy\left(\frac{(1-p)y + (Dt_2 - B)}{D}\right) + \frac{n(n-1)p(1-p)y^2}{2D} + \frac{n(n-1)(Dt_2 - B)py}{2D} \\ &\quad + \frac{np}{x}\left(y - \frac{B}{1-p}\right)^2 + \frac{np}{x}\left(\frac{B}{1-p}\right)^2\Bigg] \\ &= nh\Bigg[\frac{p(1-p)y^2}{D} + \frac{(Dt_2 - B)py}{D} + \frac{(n-1)p(1-p)y^2}{2D} + \frac{(n-1)(Dt_2 - B)py}{2D} \\ &\quad + \frac{p}{x}\left(y^2 - \frac{2By}{1-p} + \left(\frac{B}{1-p}\right)^2\right) + \frac{p}{x}\left(\frac{B}{1-p}\right)^2\Bigg] \\ &= nh\left[\frac{(n+1)p(1-p)y^2}{2D} + \frac{(n+1)(Dt_2 - B)py}{2D} + \frac{py^2}{x} - \frac{2B}{x}\left(\frac{p}{1-p}\right)y + \frac{2B^2}{x}\left(\frac{p}{(1-p)^2}\right)\right].\end{aligned}$$

Appendix E

Let

$$e_1 = 1 - E(p),$$
$$e_3 = \frac{h}{2}\left[E(1-p)^2 + 2E(p)\frac{D}{x}\right] \quad \text{and} \quad m = E(p).$$

Then,

$$\begin{aligned}E(1-p)^2 &= E(1 - m - p + m)^2 \\ &= E[(1-m)^2 - 2(1-m)(p-m) + (p-m)^2] \\ &= (1-m)^2 - 2(1-m)(E(p) - m) + E[(p-m)^2] \\ &= (1 - E(p))^2 + \mathrm{Var}(p) \\ &= e_1^2 + \mathrm{Var}(p).\end{aligned}$$

Therefore,

$$e_3 = \frac{h}{2}\left[e_1^2 + \mathrm{Var}(p) + 2E(p)\frac{D}{x}\right] > 0.$$

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