Mathematical and Computer Modelling 54 (2011) 35–49

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

# An integrated project of fishery and poultry

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ARTICLE INFO

*Article history:*
Received 6 October 2010
Received in revised form 14 January 2011
Accepted 16 January 2011

*Keywords:*
Bioeconomic
Equilibrium
Effort
Harvesting
Deterioration
Control theory

ABSTRACT

This paper considers a joint project of fishery and poultry in which the growth rates of both species vary with available nutrients and environmental carrying capacities of biomasses. The nutrients for both species are functions of the biomasses of the two species. The harvesting rates of fish and birds of poultry depend nonlinearly on common effort function. The existence of steady states and their stability (local and global) are investigated analytically. The joint profit of the project is maximized, using Pontryagin's maximum principle. The model is also illustrated by suitable numerical examples.



## 1. Introduction

The exploitation of biological renewable resources like fisheries and poultries has received great attention from researchers and practitioners of farm management. Clark [1,2] have studied the issues and techniques associated with bioeconomic exploitation of these resources in detail. Clark [2] has also examined the effects of harvesting single species using the Gause model [3]. Harvesting of two species using the Gause model [3] has been discussed by Chaudhuri [4]. Chaudhuri and SahaRay [5] have considered the problem of exploiting a prey–predator community in which the growth of both the prey and the predator obeys the logistic law of growth. Purohit and Chaudhuri [6] have developed this model further, considering taxation as a control instrument, and they opted for a more realistic catch rate function, where the dynamics of the multispecies are governed by Gompertz' law of growth. Das et al. [7] discussed the problem of non-selective harvesting of a prey–predator system, using a reasonable catch-rate function instead of the usual catch-per-unit-effort hypothesis. The formulation of a realistic model of a multi-species community is quite difficult. Clark [1] has discussed a model of combined non-selective harvesting of two ecologically independent populations, using a logistic law of growth. Multi-species harvesting models have been studied in detail by Chaudhuri [4,8], Mesterton-Gibbons [9], Dai and Tang [10], Jerry and Raussi [11], among others. The harvesting of population species is commonly practiced in fisheries, forestry, and wildlife management. Bioeconomic modeling is concerned with the scientific management of the exploitation of renewable resources. Problems related to the exploitation of multi-species systems are not only interesting but also difficult, as there are theoretical as well as practical difficulties in the determination of an optimal policy for the harvesting of a multi-species system.

In a competitive market, the quality of a product in any business organization is a big issue. Conforming quality of the product being sold in the market is necessary to maintain the brand image of the enterprise; non-conforming quality (deteriorated) items are used for another purpose at a lower marketing value. Deterioration, in general, may be considered as the result of various effects on the stock such as damage, decreasing usefulness for the main purpose, and many more.

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doi:10.1016/j.mcm.2011.01.031

Goyal and Giri [12] have presented a review of the inventory literature for deteriorating items since the early 1990s. Goyal and Gunasekaran [13] developed an integrated production–inventory–marketing model for determining the EPQ (economic production quantity) and EOQ (economic order quantity) for raw materials in a multi-stage production system. This model considered the effect of different marketing policies such as the price per unit product and the advertisement frequency on the demand of a perishable item. Other papers related to this area include those by Wee [14,15], Cardenas-Barron [16], Goyal and Cardenas-Barron [17], Khanra and Chaudhuri [18], Sana and Chaudhuri [19], Mukhopadhyay et al. [20,21], Ghosh and Chaudhuri [22], among others.

According to Moore et al. [23] and Paudel et al. [24], the US produces approximately 8.7 billion broiler chickens annually, which results in 13–26 million metric tons of poultry litter (i.e., excreta, feathers, spilled feed, bedding material, soil and dead birds). It is estimated that nearly 80% of poultry units in the US use antibiotics in the feed [25]. Nandi et al. [26] show that poultry litter contains large amounts of antibiotic-resistant bacteria and resistance genes associated with the use of antibiotics in poultry production. This has raised the concern of environmental dispersal of antibiotic resistance. Poultry litter is generally piled between 1 and 4 m deep and stored in open sheds until it is used in land as a soil amendment. House flies thrive in poultry litter; they participate in the dispersion of antibiotic resistance from poultry houses into the environment [27]. Synanthropic flies have evolved to live in proximity to humans; they have been found to carry a number of different pathogenic microorganisms, including viruses and bacteria, and they can play a major role in the epidemiology of infections in the human body [28–30]. Flies may also play an important role in spreading avian influenza. In Japan, researchers captured flies in proximity to broiler facilities during an outbreak of highly pathogenic avian influenza in Kyoto in 2004. The H5N1 influenza virus, carried by flies in the poultry litter, was found in the chickens of the infected poultry farm [31]. Graham et al. [32] investigated whether large-scale broiler poultry production results in many obstacles to biocontainment. Antibiotic-resistant enterococci and staphylococci have been isolated from poultry litter [33–35]. Furthermore, owing to methods of waste storage at farms, large amounts of fresh and stored poultry litter available outside the poultry houses can also serve as a substrate for the development of fly populations. The fresh poultry litter may be used as a nutrient for fish, which is economically beneficial to the farms as well as for environmental protection.

According to Singholka [36] and Michael [37], prawns feed on chicken feed, or ground fish flesh mixed with cooked broken rice, beef, hog, etc. Probably much of the supplemental food (broken rice, dead poultry, beef, hog, fish processing waste, prawn processing wastes, snails, etc.) added acts as a fertilizer and increases the biological productivity of the pond as a whole rather than acting as a true prawn feed. Certainly it seems that, particularly in macro brachium culture, much of the pelleted food added is eaten by small fish, which themselves form a source of food for the prawns. Taiganides [38] shows that poultry litter is nutrient rich, but there is a great variability in quality at the time of use as fish production input. Fish species play an important role in determining the loading rates of poultry waste when considering its value for fertilization of the pond. The nutrient value of the waste is measured by the rate of nitrogen (dissolved inorganic nitrogen), phosphorous (soluble reactive phosphorous) and dissolved oxygen release. Air-breathing fish, such as clarias catfish, silver-striped catfish and pangasius hypothalmus, can tolerate the highest input levels and they also require extra feed to sustain growth. These fish species are also probably inefficient to take phytoplankton-dominated food web.

Based on the above philosophy, the current study is an attempt to model a combined project of fishery and poultry of birds. It is rational that the deteriorated (non-conforming quality) fish (mainly shrimp and other small fish) are used as a nutrient for the poultry. Conversely, the excreta of birds, dead birds and poultry-processing by-products such as chicken bones, intestines and whole carcasses are used as a nutrient for the fishery, after conversion of poultry litter. Consequently, the nutrients of fishery and poultry are interconnected. Quite often, these are controlled with an outside supply of nutrients. The growth rates of fish and birds in poultry are considered as functions of the available nutrients, volume of on-hand biomass and environmental carrying capacities simultaneously. The firm management also controls the growth rates by breeding species or supplying child species in the firm, if needed.

## 2. Notation

The following notation is used to develop the model.

$X(t)$—biomass of fish at time $t$.
$Y(t)$—biomass of birds (broiler/duck) at time $t$.
$\dot{X}(t)$—the derivative of $X$ with respect to time $t$.
$\dot{Y}(t)$—the derivative of $Y$ with respect to time $t$.
$N_x(t)$—amount of nutrient at time $t$ for fish.
$(\kappa_x, \kappa_y)$—positive constants.
$N_y(t)$—amount of nutrient at time $t$ for broiler.
$(L_x, L_y)$—environmental carrying capacities of $(X, Y)$, respectively.
$(r_x, r_y)$—biotic potential of $(X, Y)$, respectively.
$\theta_x$—deterioration rate which is a fraction of the on-hand biomass of $X(t)$; it is a function of $Y$.
$\theta_y$—deterioration rate which is a fraction of the on-hand biomass of $Y(t)$; it is a function of $X$.
$E(t)$—joint effort function of the project at time $t$.
$(\beta_x, \beta_y)$—amount of waste per unit mass of $(X, Y)$, respectively.
$(\gamma_x, \gamma_y)$—food values per unit mass of nutrients of $(\theta_x X, \theta_y Y)$, respectively.

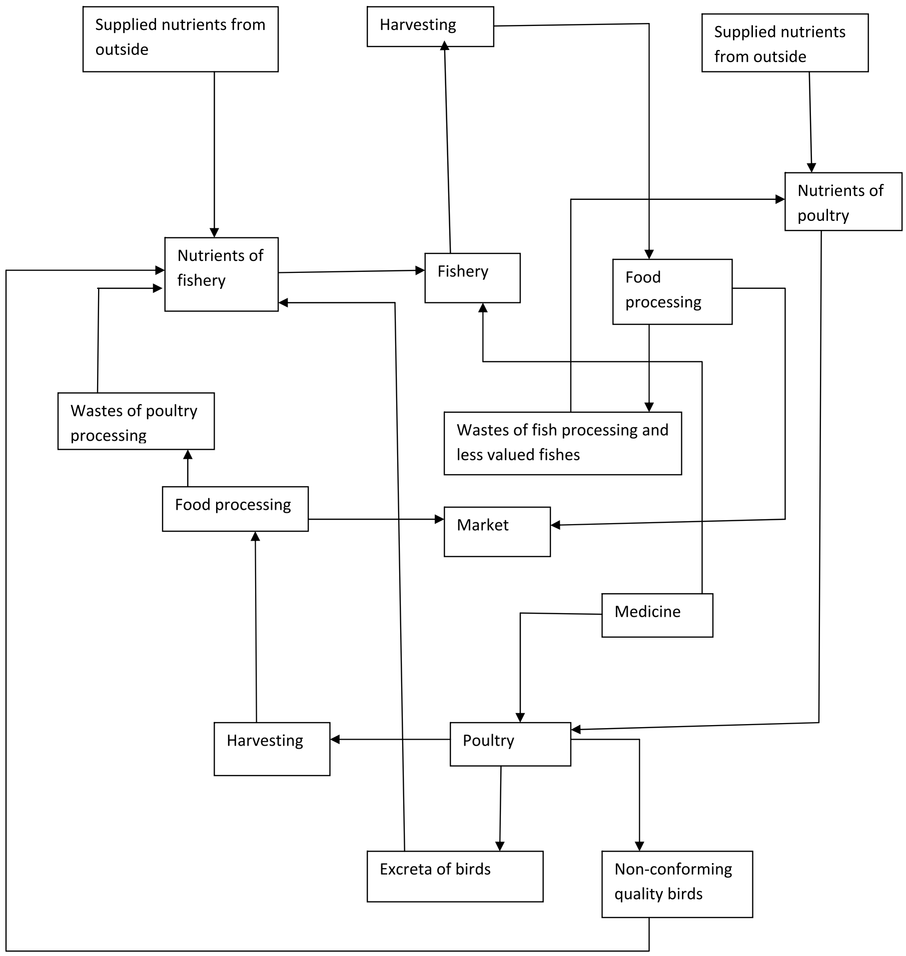


**Fig. 1.** Pictorial representation of the project.

$(N_{01}, N_{02})$—supplied nutrients, from outside, per unit biomass of fish and birds of poultry, respectively.
$(\alpha_x, \alpha_y)$—conversion factors, both positive, of nutrients from waste by-products of fish processing and poultry waste, respectively.
$(\tau_x, \tau_y)$—absorbed nutrients per unit biomass of $(X, Y)$, respectively.
$(C_x, C_y)$—harvesting coefficients of $(X, Y)$.
$(C_1, C_2)$—cost of unit mass of supplied nutrients $(N_{01}, N_{02})$, respectively.
$(C_3, C_4)$—cost of medicine per unit mass of $(X, Y)$, respectively.
$(\xi_x, \xi_y)$—cost of effort per unit biomass of $(X, Y)$, respectively.
$C(X, Y)$—cost per unit effort.
$(p_x, p_y)$—selling price per unit mass of fish and birds of poultry, respectively.
$\delta = (r - i)$—$r$ and $i$ are rates of interest and inflation per unit currency.

## 3. Formulation of the model

The model considers a joint project of fishery and poultry. Quite often, the excreta of birds of poultry, dead birds, poultry-processing by-products and living birds are used as nutrients (food) for fish. Also, deteriorated fish, shrimp and small fish, which have lower market value, are used as nutrients (food) for broilers (see Fig. 1).

The state of nutrients of $(X, Y)$ at time $t$ is as follows:

$$\begin{aligned} N_x(t) &= N_{01}X(t) + \alpha_y\beta_y Y(t) + \gamma_y\theta_y Y(t) - \tau_x X(t) \\ &= N_1 X(t) + \alpha_y\beta_y Y(t) + \gamma_y\theta_y Y(t), \quad \text{where } N_1 = N_{01} - \tau_x \end{aligned} \tag{1}$$

and

$$\begin{aligned} N_y(t) &= N_{02}Y(t) + \alpha_x\beta_x X(t) + \gamma_x\theta_x X(t) - \tau_y Y(t) \\ &= N_2 Y(t) + \alpha_x\beta_x X(t) + \gamma_x\theta_x X(t), \quad \text{where } N_2 = N_{02} - \tau_y. \end{aligned} \tag{2}$$

Here, $\theta_y = \left(1 - \frac{a_x}{a_x+X}\right)$, $\theta_x = \left(1 - \frac{a_y}{a_y+Y}\right)$ are deteriorated fractions of on-hand stock of species $X$ and $Y$, respectively, where $a_x$ and $a_y$ are positive constants. $\theta_y(0 < \theta_y < 1)$ is an increasing function of $X$ because a huge biomass of $X$ may be used as a direct nutrient of species $Y$. Similarly, $\theta_x(0 < \theta_x < 1)$ is an increasing function of species $Y$. $N_{01}$ is supplied nutrient per unit mass of $X$ from outside, $\alpha_y\beta_y Y(t)$ is converted nutrient from poultry waste excluding dead birds, $\gamma_y\theta_y Y$ (including dead and living birds) is a direct nutrient of $X$, and $\tau_x$ is the absorbed nutrient per unit mass of $X$. $N_{02}$ is the supplied nutrient per unit mass of poultry, $(\alpha_x\beta_x + \gamma_x\theta_x)X$ is converted nutrient from fish-processing by-products, deteriorated and less-valued fish, and $\tau_y$ is the absorbed nutrient per unit mass of poultry birds. Substituting the values of $\theta_x$ and $\theta_y$ in Eqs. (1) and (2), we have

$$N_x(t) = N_1X(t) + \delta_yY(t) - \left(\frac{\gamma_y a_x}{a_x + X(t)}\right)Y(t), \quad \text{where } \delta_y = \alpha_y\beta_y + \gamma_y \tag{3}$$

and

$$N_y(t) = N_2Y(t) + \delta_xX(t) - \left(\frac{\gamma_x a_y}{a_y + Y(t)}\right)X(t), \quad \text{where } \delta_x = \alpha_x\beta_x + \gamma_x, \tag{4}$$

respectively. The governing dynamical system of the species $(X, Y)$ is as follows:

$$\begin{aligned}\dot{X}(t) &= r_x\left(\frac{N_x}{\kappa_x + N_x}\right)\left(1 - \frac{X}{L_x}\right)X - h_x - \theta_xX \\ &= r_x\left(1 - \frac{\kappa_x}{\kappa_x + N_x}\right)\left(1 - \frac{X}{L_x}\right)X - h_x - \theta_xX\end{aligned} \tag{5}$$

and

$$\begin{aligned}\dot{Y}(t) &= r_y\left(\frac{N_y}{\kappa_y + N_y}\right)\left(1 - \frac{Y}{L_y}\right)Y - h_y - \theta_yY \\ &= r_y\left(1 - \frac{\kappa_y}{\kappa_y + N_y}\right)\left(1 - \frac{Y}{L_y}\right)Y - h_y - \theta_yY,\end{aligned} \tag{6}$$

where $r_x\left(1 - \frac{\kappa_x}{\kappa_x+N_x}\right)\left(1 - \frac{X}{L_x}\right)X$ is the growth rate of $X$, which depends upon the existing nutrient ($N_x$), environmental carrying capacity and on-hand biomass of $X$, and $r_y\left(1 - \frac{\kappa_y}{\kappa_y+N_y}\right)\left(1 - \frac{Y}{L_y}\right)Y$ is the growth rate of $Y$, which depends upon the existing nutrient ($N_y$), environmental carrying capacity and on-hand biomass of $Y$. When $N_x \to \infty$, i.e., the available nutrient is more than a sufficient amount, then the growth rate tends to $r_x\left(1 - \frac{X}{L_x}\right)$. When $X \to L_x$, i.e., the biomass of $X$ reaches the environmental carrying capacity, then the growth rate of $X$ tends to zero, which is rational in practice. Similarly, when $N_y \to \infty$, the growth rate of $Y$ tends to $r_y\left(1 - \frac{Y}{L_y}\right)Y$ and the growth rate tends to zero when $Y \to L_y$. The harvesting rates $(h_x, h_y)$ of the species $(X, Y)$ are considered as follows:

$$h_x = \frac{C_xEX}{l_1E + l_2X} \tag{7}$$

and

$$h_y = \frac{C_yEY}{l_3E + l_4Y}, \tag{8}$$

where $C_x$ and $C_y$ are the harvesting coefficients of the two species. $L_x$ and $L_y$ are the environmental carrying capacities of the two species. It is noticed that $(h_x, h_y) \to (C_xX/l_1, C_yY/l_3)$ as $E \to \infty$ for fixed values of $(X, Y)$. The parameters $(l_1, l_3)$ are proportional to the ratios of the stock level to the harvesting rates at higher level of effort, and the parameters $(l_2, l_4)$ are proportional to the ratios of the effort level to the harvesting rates at higher stock levels. Although the species are non-interacting directly, their growths are mutually dependent because of the common effort $(E)$. Now, substituting Eqs. (7) and (8) in Eqs. (5) and (6), we have

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} = \begin{pmatrix}G_1(X, Y)\\ G_2(X, Y)\end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}G_1(X, Y) &= r_x\left(1 - \frac{\kappa_x}{\kappa_x + N_x}\right)\left(1 - \frac{X}{L_x}\right)X - h_x - \theta_xX \\ &= X\left[r_x\left(1 - \frac{\kappa_x}{\kappa_x + N_1X + \delta_yY - \gamma_y\left(\frac{a_x}{a_x+X}\right)Y}\right)\left(1 - \frac{X}{L_x}\right) - \frac{C_xE}{l_1E + l_2X} - \left(1 - \frac{a_y}{a_y + Y}\right)\right]\end{aligned}$$

and

$$G_2(X,Y) = r_y\left(1-\frac{\kappa_y}{\kappa_y+N_y}\right)\left(1-\frac{Y}{L_y}\right)Y - h_y - \theta_y Y$$
$$= Y\left[r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_x\left(\frac{a_y}{a_y+Y}\right)X}\right)\left(1-\frac{Y}{L_y}\right)-\frac{C_yE}{l_3E+l_4Y}-\left(1-\frac{a_x}{a_x+X}\right)\right].$$

For non-zero critical points $(\bar{x},\bar{y})$, the following conditions are satisfied:

$$r_x\left(1-\frac{\kappa_x}{\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y\left(\frac{a_x}{a_x+\bar{x}}\right)\bar{y}}\right)\left(1-\frac{\bar{x}}{L_x}\right)=\frac{C_xE}{l_1E+l_2\bar{x}}+\left(1-\frac{a_y}{a_y+\bar{y}}\right) \tag{10}$$

$$r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x\left(\frac{a_y}{a_y+\bar{y}}\right)\bar{x}}\right)\left(1-\frac{\bar{y}}{L_y}\right)=\frac{C_yE}{l_3E+l_4\bar{y}}+\left(1-\frac{a_x}{a_x+\bar{x}}\right). \tag{11}$$

Solving Eqs. (10) and (11), we have the critical points of the dynamical system. Now, differentiating $G_1(X,Y)$ and $G_2(X,Y)$ partially with respect to $X$ and $Y$, we have

$$\frac{\partial}{\partial X}(G_1(X,Y)) = r_x\left(1-\frac{\kappa_x}{\kappa_x+N_1X+\delta_yY-\gamma_y\left(\frac{a_x}{a_x+X}\right)Y}\right)\left(1-\frac{X}{L_x}\right)-\frac{C_xE}{l_1E+l_2X}-\left(1-\frac{a_y}{a_y+Y}\right)$$
$$+X\left[r_x\left\{\frac{\kappa_x\left(N_1+\frac{\gamma_ya_xY}{(a_x+X)^2}\right)(1-X/L_x)}{(\kappa_x+N_1X+\delta_yY-\gamma_ya_xY/(a_x+X))^2}-\frac{1}{L_x}\left(1-\frac{\kappa_x}{\kappa_x+N_1X+\delta_yY-\gamma_ya_xY/(a_x+X)}\right)\right\}+\frac{C_xl_2E}{(l_1E+l_2X)^2}\right]$$

$$\frac{\partial}{\partial Y}(G_1(X,Y)) = X\left[r_x\kappa_x\left(1-\frac{X}{L_x}\right)\left\{\frac{\delta_y-\gamma_ya_x/(a_x+X)}{\left(\kappa_x+N_1X+\delta_yY-\gamma_y\left(\frac{a_x}{a_x+X}\right)Y\right)^2}\right\}-\frac{a_y}{(a_y+Y)^2}\right],$$

$$\frac{\partial}{\partial X}(G_2(X,Y)) = Y\left[r_y\kappa_y\left(1-\frac{Y}{L_y}\right)\left\{\frac{\delta_x-\gamma_xa_y/(a_y+Y)}{\left(\kappa_y+N_2Y+\delta_xX-\gamma_x\left(\frac{a_y}{a_y+Y}\right)X\right)^2}\right\}-\frac{a_x}{(a_x+X)^2}\right],$$

$$\frac{\partial}{\partial Y}(G_2(X,Y)) = r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_x\left(\frac{a_y}{a_y+Y}\right)X}\right)\left(1-\frac{Y}{L_y}\right)-\frac{C_yE}{l_3E+l_4Y}-\left(1-\frac{a_x}{a_x+X}\right)$$
$$+Y\left[r_y\left\{\frac{\kappa_y\left(N_2+\frac{\gamma_xa_yX}{(a_y+Y)^2}\right)(1-Y/L_y)}{(\kappa_y+N_2Y+\delta_xX-\gamma_xa_yX/(a_y+Y))^2}-\frac{1}{L_y}\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_xa_yX/(a_y+Y)}\right)\right\}+\frac{C_yl_4E}{(l_3E+l_4Y)^2}\right].$$

At the non-zero critical point $(\bar{x},\bar{y})$, the above partial derivatives are as follows:

$$\frac{\partial}{\partial X}(G_1(\bar{x},\bar{y})) = \bar{x}\left[r_x\left\{\frac{\kappa_x\left(N_1+\frac{\gamma_ya_x\bar{y}}{(a_x+\bar{x})^2}\right)(1-\bar{x}/L_x)}{(\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_ya_x\bar{y}/(a_x+\bar{x}))^2}-\frac{1}{L_x}\left(1-\frac{\kappa_x}{\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_ya_x\bar{y}/(a_x+\bar{x})}\right)\right\}+\frac{C_xl_2E}{(l_1E+l_2\bar{x})^2}\right],$$

$$\frac{\partial}{\partial Y}(G_1(\bar{x},\bar{y})) = \bar{x}\left[r_x\kappa_x\left(1-\frac{\bar{x}}{L_x}\right)\left\{\frac{\delta_y-\gamma_ya_x/(a_x+\bar{x})}{\left(\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y\left(\frac{a_x}{a_x+\bar{x}}\right)\bar{y}\right)^2}\right\}-\frac{a_y}{(a_y+\bar{y})^2}\right],$$

$$\frac{\partial}{\partial X}(G_2(\bar{x},\bar{y})) = \bar{y}\left[r_y\kappa_y\left(1-\frac{\bar{y}}{L_y}\right)\left\{\frac{\delta_x-\gamma_x a_y/(a_y+\bar{y})}{\left(\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x\left(\frac{a_y}{a_y+\bar{y}}\right)\bar{x}\right)^2}\right\}-\frac{a_x}{(a_x+\bar{x})^2}\right],$$

$$\frac{\partial}{\partial Y}(G_2(\bar{x},\bar{y})) = \bar{y}\left[r_y\left\{\frac{\kappa_y(N_2+\frac{\gamma_x a_y\bar{x}}{(a_y+\bar{y})^2})(1-\bar{y}/L_y)}{(\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y}))^2} - \frac{1}{L_y}\left(1-\frac{\kappa_y}{\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y})}\right)\right\}+\frac{C_y l_4 E}{(l_3E+l_4\bar{y})^2}\right].$$

### 3.1. Equilibria of the system

The possible steady states of the dynamical system (9) are the solutions, $(\bar{x},\bar{y})$, of Eqs. (10) and (11).

### 3.2. Boundedness of the system

**Lemma.** *All solutions of the dynamical system (9) which start in $R_2^+$ are uniformly bounded.*

**Proof.** Let us consider the function

$$U(X,Y) = X + \frac{1}{\ell}Y. \tag{12}$$

The time derivative of Eq. (12) is

$$\begin{aligned}\dot{U} &= \dot{X}+\frac{1}{\ell}\dot{Y}\\ &= X\left[r_x\left(1-\frac{\kappa_x}{\kappa_x+N_1X+\delta_yY-\gamma_y a_xY/(a_x+X)}\right)\left(1-\frac{X}{L_x}\right)-\frac{C_xE}{l_1E+l_2X}-\left(1-\frac{a_y}{a_y+Y}\right)\right]+\frac{1}{\ell}Y\\ &\quad\times\left[r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_x a_yX/(a_y+Y)}\right)\left(1-\frac{Y}{L_y}\right)-\frac{C_yE}{l_3E+l_4Y}-\left(1-\frac{a_x}{a_x+X}\right)\right].\end{aligned}$$

For each $s > 0$, we obtain

$$\begin{aligned}\dot{U}+sU &= \dot{X}+\frac{1}{\ell}\dot{Y}+s\left(X+\frac{1}{\ell}Y\right)\\ &= X\left[r_x\left(1-\frac{\kappa_x}{\kappa_x+N_1X+\delta_yY-\gamma_y a_xY/(a_x+X)}\right)\left(1-\frac{X}{L_x}\right)\right.\\ &\quad\left.-\frac{C_xE}{l_1E+l_2X}-\left(1-\frac{a_y}{a_y+Y}\right)\right]+\frac{1}{\ell}Y\left[r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_x a_yX/(a_y+Y)}\right)\right.\\ &\quad\left.\times\left(1-\frac{Y}{L_y}\right)-\frac{C_yE}{l_3E+l_4Y}-\left(1-\frac{a_x}{a_x+X}\right)\right]+s\left(X+\frac{1}{\ell}Y\right)\\ &< X\left[r_x-\frac{C_xE}{l_1E+l_2X}+a_y\right]+\frac{1}{\ell}Y\left[r_y-\frac{C_yE}{l_3E+l_4Y}+a_x\right]+s\left(X+\frac{1}{\ell}Y\right)\\ &= \frac{X}{l_1E+l_2X}[(r_x+s)(l_1E+l_2X)-C_xE+a_y(l_1E+l_2X)]\\ &\quad+\frac{Y}{\ell(l_3E+l_4Y)}[(r_y+s)(l_3E+l_4Y)-C_yE+a_x(l_3E+l_4Y)]\\ &< \frac{1}{l_1E}[(r_x+s+a_y)l_2X^2+\{(r_x+s+a_y)l_1-C_x\}EX]+\frac{1}{\ell l_3E}[(r_y+s+a_x)l_4Y^2+\{(r_y+s+a_x)l_3-C_y\}EY]\end{aligned}$$

$$
\begin{aligned}
&< \left|\frac{(r_x+s+a_y)l_2}{l_1E}\right|\left(X+\frac{(r_x+s+a_y)l_1E-C_xE}{2(r_x+s+a_y)l_2}\right)^2+\left|\frac{(r_y+s+a_x)l_4}{\ell l_3E}\right|\left(Y+\frac{(r_y+s+a_x)l_3E-C_yE}{2(r_y+s+a_x)l_4}\right)^2\\
&< \left|\frac{(r_x+s+a_y)l_2}{l_1E}\right|(L_x+A_x)^2+\left|\frac{(r_y+s+a_x)l_4}{\ell l_3E}\right|(L_y+A_y)^2\\
&< K, \quad \text{since } 0\le X\le L_x \text{ and } 0\le Y\le L_y,
\end{aligned}
$$

where

$$
\begin{aligned}
\sqrt{K/2} &= \mathrm{Max}\left[\sqrt{\left|\frac{(r_x+s+a_y)l_2}{l_1E}\right|}|L_x+A_x|, \sqrt{\left|\frac{(r_y+s+a_x)l_4}{\ell l_3E}\right|}|Y+A_y|\right],\\
A_x &= \frac{(r_x+s+a_y)l_1E-C_xE}{2(r_x+s+a_y)l_2},\\
A_y &= \frac{(r_y+s+a_x)l_3E-C_yE}{2(r_y+s+a_x)l_4}.
\end{aligned}
$$

Applying the theory of differential inequality, we have

$$0 < U(X,Y) < (K/s)(1-\mathrm{e}^{-st})+U(X(0),Y(0))\mathrm{e}^{-st}. \tag{13}$$

When $t\to\infty$, the above yields $0<U<K/s$. Therefore, all solutions of (9) that start in $R_2^+$ are confined to the region $R$, where

$$R=\{(X,Y)\epsilon R_2^+ : U=(K/s)-\epsilon,\ \text{for any } \epsilon>0\}.$$

Hence the proof. □

### 3.3. Local stability analysis

We shall now investigate the local behaviour of critical points of the dynamical system (9). The variational matrix of the system is

$$V(\bar{x},\bar{y})=\begin{pmatrix}\frac{\partial}{\partial X}(G_1(\bar{x},\bar{y})) & \frac{\partial}{\partial Y}(G_1(\bar{x},\bar{y}))\\ \frac{\partial}{\partial X}(G_2(\bar{x},\bar{y})) & \frac{\partial}{\partial Y}(G_2(\bar{x},\bar{y}))\end{pmatrix}. \tag{14}$$

The characteristic equation of $V(\bar{x},\bar{y})$ is $\lambda^2-\lambda\left(\frac{\partial G_1(\bar{x},\bar{y})}{\partial X}+\frac{\partial G_2(\bar{x},\bar{y})}{\partial Y}\right)+\left(\frac{\partial G_1(\bar{x},\bar{y})}{\partial X}\right)\left(\frac{\partial G_2(\bar{x},\bar{y})}{\partial Y}\right)-\left(\frac{\partial G_1(\bar{x},\bar{y})}{\partial Y}\right)\left(\frac{\partial G_2(\bar{x},\bar{y})}{\partial X}\right)=0$, or, $\lambda^2-\psi_1(\bar{x},\bar{y})\lambda+\psi_2(\bar{x},\bar{y})=0$, where

$$
\begin{aligned}
\psi_1(\bar{x},\bar{y}) &= \frac{\partial}{\partial X}(G_1(\bar{x},\bar{y}))+\frac{\partial}{\partial Y}(G_2(\bar{x},\bar{y}))\\
&= \bar{x}\Bigg[r_x\Bigg\{\frac{\kappa_x\left(N_1+\frac{\gamma_y a_x\bar{y}}{(a_x+\bar{x})^2}\right)(1-\bar{x}/L_x)}{(\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y a_x\bar{y}/(a_x+\bar{x}))^2}-\frac{1}{L_x}\left(1-\frac{\kappa_x}{\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y a_x\bar{y}/(a_x+\bar{x})}\right)\Bigg\}\\
&\quad +\frac{C_xl_2E}{(l_1E+l_2\bar{x})^2}\Bigg]+\bar{y}\Bigg[r_y\Bigg\{\frac{\kappa_y(N_2+\frac{\gamma_x a_y\bar{x}}{(a_y+\bar{y})^2})(1-\bar{y}/L_y)}{(\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y}))^2}\\
&\quad -\frac{1}{L_y}\left(1-\frac{\kappa_y}{\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y})}\right)\Bigg\}+\frac{C_yl_4E}{(l_3E+l_4\bar{y})^2}\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_2(\bar{x},\bar{y}) &= \bar{x}\bar{y}\Bigg[r_x\Bigg\{\frac{\kappa_x\left(N_1+\frac{\gamma_y a_x\bar{y}}{(a_x+\bar{x})^2}\right)(1-\bar{x}/L_x)}{(\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y a_x\bar{y}/(a_x+\bar{x}))^2}\\
&\quad -\frac{1}{L_x}\left(1-\frac{\kappa_x}{\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y a_x\bar{y}/(a_x+\bar{x})}\right)\Bigg\}+\frac{C_xl_2E}{(l_1E+l_2\bar{x})^2}\Bigg]
\end{aligned}
$$

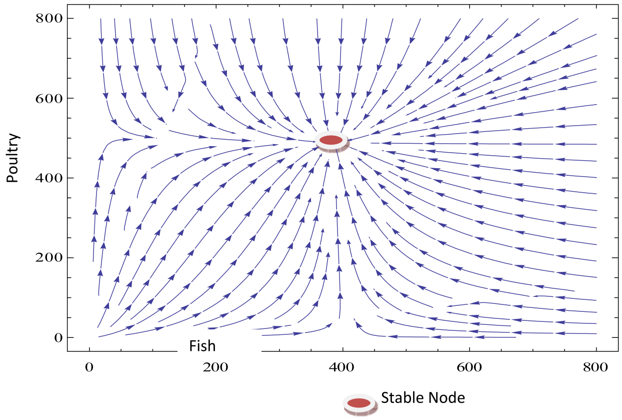


**Fig. 2.** Phase portrait of Example 1.

$$\times\left[r_y\left\{\frac{\kappa_y\left(N_2+\frac{\gamma_x a_y\bar{x}}{(a_y+\bar{y})^2}\right)(1-\bar{y}/L_y)}{(\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y}))^2}\right.\right.$$
$$\left.\left.-\frac{1}{L_y}\left(1-\frac{\kappa_y}{\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x a_y\bar{x}/(a_y+\bar{y})}\right)\right\}+\frac{C_y l_4 E}{(l_3E+l_4\bar{y})^2}\right]$$
$$-\bar{x}\bar{y}\left[r_x\kappa_x\left(1-\frac{\bar{x}}{L_x}\right)\left\{\frac{\delta_y-\gamma_y a_x/(a_x+\bar{x})}{\left(\kappa_x+N_1\bar{x}+\delta_y\bar{y}-\gamma_y\left(\frac{a_x}{a_x+\bar{x}}\right)\bar{y}\right)^2}-\frac{a_y}{(a_y+\bar{y})^2}\right\}\right]$$
$$\times\left[r_y\kappa_y\left(1-\frac{\bar{y}}{L_y}\right)\left\{\frac{\delta_x-\gamma_x a_y/(a_y+\bar{y})}{\left(\kappa_y+N_2\bar{y}+\delta_x\bar{x}-\gamma_x\left(\frac{a_y}{a_y+\bar{y}}\right)\bar{x}\right)^2}\right\}-\frac{a_x}{(a_x+\bar{x})^2}\right].$$

Now, $(\bar{x},\bar{y})$ is a stable node if both the eigenvalues of the above are negative, i.e.,

$$\psi_1(\bar{x},\bar{y})<0 \quad \text{and} \quad \psi_2(\bar{x},\bar{y})>0 \tag{15}$$

are satisfied.

Let us consider the following numerical example.

**Example 1.** The values of the parameters in appropriate units are considered as follows: $\kappa_x=4$, $\kappa_y=4$, $N_{01}=1.5$, $N_{02}=1.5$, $r_x=10.0$, $r_y=12.0$, $C_x=0.8$, $C_y=0.6$, $L_x=500$, $L_y=550$, $a_x=20.0$, $a_y=30.0$, $\alpha_x=0.2$, $\alpha_y=0.5$, $\beta_x=0.3$, $\beta_y=0.6$, $\gamma_x=1.0$, $\gamma_y=1.0$, $\tau_x=6.5$, $\tau_y=7.2$, $E=100.0$, $l_1=0.2$, $l_2=0.1$, $l_3=0.3$, $l_4=0.25$. Then, the critical point $(384.85, 488.46)$ is a locally stable node (see Fig. 2) because the eigenvalues $(-10.36, -06.83)$ are negative.

### 3.4. Global stability analysis

We shall study the global stability of system (9) by considering a suitable Lyapunov function

$$F(X,Y)=[(X-\bar{x})-\bar{x}\ln(X/\bar{x})]+h[(Y-\bar{y})-\bar{y}\ln(Y/\bar{y})],$$

where $h$ is a suitable constant. $F(\bar{x},\bar{y})$ is zero at the equilibrium point $(\bar{x},\bar{y})$ and is positive for all other values of $(X,Y)\epsilon R_2^+$. The time derivative of $F$ along the trajectories of Eq. (9) is

$$\dot{F}=\left(\frac{X-\bar{x}}{X}\right)\dot{X}+h\left(\frac{Y-\bar{y}}{Y}\right)\dot{Y}$$
$$=(X-\bar{x})\left[r_x\left(1-\frac{\kappa_x}{\kappa_x+N_1X+\delta_yY-\gamma_y\left(\frac{a_x}{a_x+X}\right)Y}\right)\left(1-\frac{X}{L_x}\right)-\frac{C_xE}{l_1E+l_2X}-\left(1-\frac{a_y}{a_y+Y}\right)\right]$$
$$+h(Y-\bar{y})\left[r_y\left(1-\frac{\kappa_y}{\kappa_y+N_2Y+\delta_xX-\gamma_x\left(\frac{a_y}{a_y+Y}\right)X}\right)\left(1-\frac{Y}{L_y}\right)-\frac{C_yE}{l_3E+l_4Y}-\left(1-\frac{a_x}{a_x+X}\right)\right]$$
$$=[X-\bar{x},Y-\bar{y}]^TP[X-\bar{x},Y-\bar{y}],$$

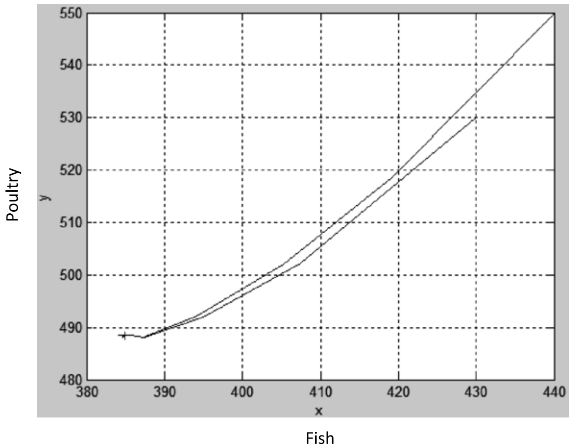


**Fig. 3.** Global attractor of Example 1.

where

$$P = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= \frac{r_x \kappa_x}{A(X,Y)A(\bar{x},\bar{y})}\left(N_1 - \frac{\gamma_y a_x \bar{y}}{D(X)D(\bar{x})}\right) - \frac{r_x}{L_x} \\
&\quad + \frac{\kappa_x r_x}{L_x A(X,Y)A(\bar{x},\bar{y})}\left(\kappa_x + (\delta_y - \gamma_y)\bar{y} + \frac{\gamma_y a_x}{D(X)D(\bar{x})}((a_x + \bar{x})\bar{y} + X)\right) + \frac{C_x E l_2}{C(X)C(\bar{x})} \\
a_{12} &= \frac{1}{2}\left[\frac{\kappa_x r_x}{A(X,Y)A(\bar{x},\bar{y})}\left(\delta_y - \gamma_y + \frac{\gamma_y a_x (a_x + \bar{x})}{D(X)D(\bar{x})}\right) + \frac{\kappa_x r_x}{L_x A(X,Y)A(\bar{x},\bar{y})}\left(-(\delta_y - \gamma_y)\bar{x} - \frac{\gamma_y a_x (a_x + 1)\bar{x}}{D(X)D(\bar{x})}\right)\right. \\
&\quad - \frac{a_y}{D(Y)D(\bar{y})} + h\frac{\kappa_y r_y}{B(X,Y)B(\bar{x},\bar{y})}\left(\delta_x - \gamma_x + \frac{\gamma_x a_y (a_y + \bar{y})}{D(Y)D(\bar{y})}\right) \\
&\quad \left. + \frac{\kappa_y r_y}{L_y B(X,Y)B(\bar{x},\bar{y})}\left(-(\delta_x - \gamma_x)\bar{y} - \frac{\gamma_x a_y (a_y + 1)\bar{y}}{D(Y)D(\bar{y})}\right) - \frac{a_x}{D(X)D(\bar{x})}\right] = a_{21}, \\
a_{22} &= h\left[\frac{r_y \kappa_y}{B(X,Y)B(\bar{x},\bar{y})}\left(N_2 - \frac{\gamma_x a_y \bar{x}}{D(Y)D(\bar{y})}\right) - \frac{r_y}{L_y}\right. \\
&\quad \left. + \frac{\kappa_y r_y}{L_y B(X,Y)B(\bar{x},\bar{y})}\left(\kappa_y + (\delta_x - \gamma_x)\bar{x} + \frac{\gamma_x a_y}{D(Y)D(\bar{y})}((a_y + \bar{y})\bar{x} + Y)\right) + \frac{C_y E l_4}{C(Y)C(\bar{y})}\right],
\end{aligned}$$

$$A(X,Y) = \kappa_x + N_1 X + (\delta_y - \gamma_y)Y + \frac{\gamma_y a_x Y}{a_x + X},$$

$$B(X,Y) = \kappa_y + N_2 Y + (\delta_x - \gamma_x)X + \frac{\gamma_x a_y X}{a_y + Y},$$

$$C(X) = l_1 E + l_2 X,$$

$$C(Y) = l_3 E + l_4 Y,$$

$$D(X) = a_x + X,$$

$$D(Y) = a_y + Y.$$

The eigenvalues of the characteristic equation of the above matrix are both negative if $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$ are satisfied. Therefore, the interior equilibrium point $(\bar{x}, \bar{y})$ is globally asymptotically stable if the above inequalities hold simultaneously. The critical point (384.85, 488.46) of Example 1 is globally stable (see Fig. 3), as the above inequalities hold.

*3.5. Bionomic equilibrium*

The biological equilibrium is given by (9). An economic equilibrium exists when the total revenue earned by selling the harvested biomass equals the total system cost. Here, we consider a reasonable cost per unit effort as

$$Ç(X, Y) = \xi_x X + \xi_y Y,$$

where $(\xi_x, \xi_y)$ are the costs per unit biomass of $(X, Y)$ per unit effort for harvesting and $(C_3, C_4)$ are the medicinal costs per unit biomass of $(X, Y)$, which is applied just after harvesting. Then, the revenue at any time is given by

$$\begin{aligned}\pi(X, Y, E) &= \text{Income from sales} - \text{Cost of supplied nutrients} \\ &\quad - \text{Cost of medicine applied just after harvesting} - \text{Cost of effort} \\ &= \frac{p_x C_x E X}{l_1 E + l_2 X} + \frac{p_y C_y E Y}{l_3 E + l_4 Y} - C_1 N_{01} X - C_2 N_{02} Y \\ &\quad - C_3\left(1 - \frac{C_x E}{l_1 E + l_2 X}\right) X - C_4\left(1 - \frac{C_y E}{l_3 E + l_4 Y}\right) Y - (\xi_x X + \xi_y Y) E \\ &= \frac{(p_x - C_3) C_x E X}{l_1 E + l_2 X} + \frac{(p_y - C_4) C_y E Y}{l_3 E + l_4 Y} - (C_1 N_{01} + C_3) X - (C_2 N_{02} + C_4) Y - (\xi_x X + \xi_y Y) E. \end{aligned} \tag{16}$$

Now, differentiating $\pi(X, Y, E)$ with respect to $X$ and $Y$, we have

$$\frac{\partial \pi}{\partial X} = \frac{(p_x - C_3) C_x l_1 E^2}{(l_1 E + l_2 X)^2} - (C_1 N_{01} + C_3) - \xi_x E$$

and

$$\frac{\partial \pi}{\partial Y} = \frac{(p_y - C_4) C_y l_3 E^2}{(l_3 E + l_4 Y)^2} - (C_2 N_{02} + C_4) - \xi_y E,$$

respectively. For biological equilibrium, $\dot{X} = 0 = \dot{Y}$ is attained at $(\bar{x}, \bar{y})$ under the constraints given in Eqs. (10) and (11), simultaneously. Now, substituting $(\bar{x}, \bar{y})$ from Eqs. (10) and (11), we have

$$\bar{E} = \frac{l_2 \bar{x}\left[\frac{r_x N_x}{\kappa_x + N_x}(1 - \bar{x}/L_x) - \left(1 - \frac{a_y}{a_y + \bar{y}}\right)\right]}{\left[C_x - l_1\left\{\frac{r_x N_x}{\kappa_x + N_x}(1 - \bar{x}/L_x) - \left(1 - \frac{a_y}{a_y + \bar{y}}\right)\right\}\right]} \tag{17}$$

$$= \frac{l_4 \bar{y}\left[\frac{r_y N_y}{\kappa_y + N_y}(1 - \bar{y}/L_y) - \left(1 - \frac{a_x}{a_x + \bar{x}}\right)\right]}{\left[C_y - l_3\left\{\frac{r_y N_y}{\kappa_y + N_y}(1 - \bar{y}/L_y) - \left(1 - \frac{a_x}{a_x + \bar{x}}\right)\right\}\right]}. \tag{18}$$

In bioeconomic equilibrium, $\pi(\bar{x}, \bar{y}, \bar{E}) = 0$ implies that

$$\pi(\bar{x}, \bar{y}, \bar{E}) = \frac{(p_x - C_3) C_x \bar{E} \bar{x}}{l_1 E + l_2 \bar{x}} + \frac{(p_y - C_4) C_y \bar{E} \bar{y}}{l_3 \bar{E} + l_4 \bar{y}} - (C_1 N_{01} + C_3)\bar{x} - (C_2 N_{02} + C_4)\bar{y} - (\xi_x \bar{x} + \xi_y \bar{y})\bar{E} = 0. \tag{19}$$

Solving Eqs. (17)–(19), we have the bioeconomic equilibrium solutions $(\bar{x}, \bar{y}, \bar{E})$.

Let us consider the following numerical example.

**Example 2.** The values of the parameters in appropriate units are considered as follows: $\kappa_x = 4$, $\kappa_y = 4$, $N_{01} = 1.5$, $N_{02} = 1.5$, $r_x = 10.0$, $r_y = 12.0$, $C_x = 0.8$, $C_y = 0.6$, $L_x = 25$, $L_y = 20$, $a_x = 3.0$, $a_y = 2.5$, $\alpha_x = 0.2$, $\alpha_y = 0.5$, $\beta_x = 0.3$, $\beta_y = 0.6$, $\gamma_x = 1.0$, $\gamma_y = 1.0$, $\tau_x = 6.5$, $\tau_y = 7.2$, $p_x = 100$, $p_y = 75$, $C_1 = 3.0$, $C_2 = 2.0$, $C_3 = 2.0$, $C_4 = 3.0$, $\xi_x = 0.5$, $\xi_y = 1.0$, $l_1 = 0.2$, $l_2 = 0.1$, $l_3 = 0.3$, $l_4 = 0.25$. Then the optimal solution ($\bar{x} = 22.66$, $\bar{y} = 18.54$, $\bar{E} = 0.29$) is a locally stable node (see Fig. 4) because the eigenvalues $(-11.79, -09.43)$ are negative.

*3.6. Optimal harvesting policy*

The net profit of the project, including inflation and time value of money, is

$$J = \int_0^\infty \pi(X, Y, E) e^{-\delta t} dt. \tag{20}$$

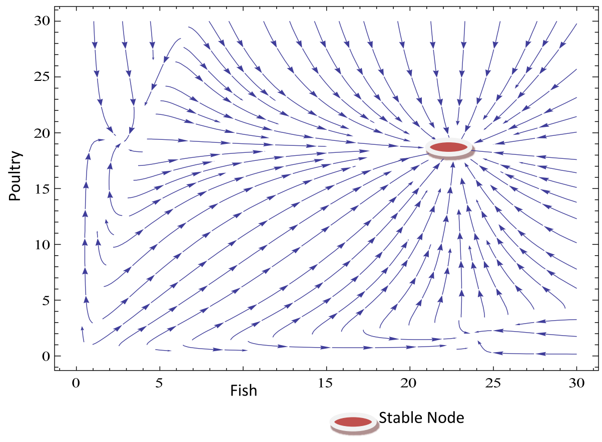


**Fig. 4.** Phase portrait of Example 2.

Now, our objective is to maximize $J$ subject to system (9), using Pontryagin's maximum principle. The control variable $E(t)$ is subject to the constraint $0 \le E(t) \le E_{\max}$; $E_{\max}$ is a feasible upper limit for the harvesting effort. The Hamiltonian of the problem is

$$H = \left[\frac{(p_x - C_3)C_x EX}{l_1 E + l_2 X} + \frac{(p_y - C_4)C_y EY}{l_3 E + l_4 Y} - (C_1 N_{01} + C_3)X - (C_2 N_{02} + C_4)Y - (\xi_x X + \xi_y Y)E\right] e^{-\delta t} + \lambda_1(t) G_1(X, Y, E) + \lambda_2(t) G_2(X, Y, E), \tag{21}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables. The optimal control $E(t)$, which maximizes $H$, must satisfy the following conditions:

$$\frac{\partial H}{\partial E} = 0$$

$$-\frac{d\lambda_1}{dt} = \frac{\partial H}{\partial X}$$

$$-\frac{d\lambda_2}{dt} = \frac{\partial H}{\partial Y}.$$

Now, $\partial H/\partial E = 0$ at $(\bar{x}, \bar{y}, \bar{E})$ gives us

$$\left[\frac{(p_x - C_3)C_x l_2 (\bar{x})^2}{(l_1\bar{E} + l_2\bar{x})^2} + \frac{(p_y - C_4)C_y l_4 (\bar{y})^2}{(l_3\bar{E} + l_4\bar{y})^2} - (\xi_x\bar{x} + \xi_y\bar{y})\right] e^{-\delta t} - \lambda_1 \frac{C_x l_2 \bar{x}^2}{(l_1\bar{E} + l_2\bar{x})^2} - \lambda_2 \frac{C_y l_4 \bar{y}^2}{(l_3\bar{E} + l_4\bar{y})^2} = 0. \tag{22}$$

Now, $-\frac{d\lambda_1}{dt} = \frac{\partial H}{\partial X}$ at $(\bar{x}, \bar{y}, \bar{E})$ gives us

$$-\frac{d\lambda_1}{dt} = \left[\frac{\partial \pi}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})} e^{-\delta t} + \lambda_1 \left[\frac{\partial G_1}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})} + \lambda_2 \left[\frac{\partial G_2}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})}. \tag{23}$$

Similarly,

$$-\frac{d\lambda_2}{dt} = \left[\frac{\partial \pi}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})} e^{-\delta t} + \lambda_1 \left[\frac{\partial G_1}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})} + \lambda_2 \left[\frac{\partial G_2}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})}. \tag{24}$$

From Eq. (23), we have

$$\lambda_2 = -\left[\frac{d\lambda_1}{dt} + \left(\frac{\partial \pi}{\partial X}\right) e^{-\delta t} + \lambda_1 \frac{\partial G_1}{\partial X}\right] \Big/ \left[\frac{\partial G_2}{\partial X}\right]. \tag{25}$$

Substituting the above in Eq. (24), we have

$$\frac{d^2\lambda_1}{dt^2} + \frac{d\lambda_1}{dt}\left[\left(\frac{\partial G_1}{\partial X}\right) + \left(\frac{\partial G_2}{\partial Y}\right)\right] + \lambda_1\left[\left(\frac{\partial G_1}{\partial X}\right)\left(\frac{\partial G_2}{\partial Y}\right) - \left(\frac{\partial G_1}{\partial Y}\right)\left(\frac{\partial G_2}{\partial X}\right)\right]$$

$$= \left(\delta\left(\frac{\partial \pi}{\partial X}\right) + \left(\frac{\partial G_2}{\partial X}\right)\left(\frac{\partial \pi}{\partial Y}\right) - \left(\frac{\partial G_2}{\partial Y}\right)\left(\frac{\partial \pi}{\partial X}\right)\right) e^{-\delta t}$$

$$= Q_1 e^{-\delta t}$$

$$\text{i.e.,} \quad \frac{d^2\lambda_1}{dt^2} + \psi_1 \frac{d\lambda_1}{dt} + \psi_2 \lambda_1 = Q_1 e^{-\delta t}, \tag{26}$$

where

$$Q_1 = \left[\delta\left(\frac{\partial\pi}{\partial X}\right) + \left(\frac{\partial G_2}{\partial X}\right)\left(\frac{\partial\pi}{\partial Y}\right) - \left(\frac{\partial G_2}{\partial Y}\right)\left(\frac{\partial\pi}{\partial X}\right)\right]_{(\bar{x},\bar{y},\bar{E})}$$

$$= \left[\delta - \bar{y}\left[r_y\left\{\frac{\kappa_y\left(N_2 + \frac{\gamma_x a_y \bar{x}}{(a_y+\bar{y})^2}\right)(1-\bar{y}/L_y)}{(\kappa_y + N_2\bar{y} + \delta_x\bar{x} - \gamma_x a_y\bar{x}/(a_y+\bar{y}))^2} - \frac{1}{L_y}\left(1 - \frac{\kappa_y}{\kappa_y + N_2\bar{y} + \delta_x\bar{x} - \gamma_x a_y\bar{x}/(a_y+\bar{y})}\right)\right\} + \frac{C_y l_4 E}{(l_3E + l_4\bar{y})^2}\right]\right]$$

$$\times\left[\frac{(p_x - C_3)C_x l_1\bar{E}^2}{(l_1E + l_2\bar{x})^2} - (C_1N_{01} + C_3) - \xi_x\bar{E}\right] + \left[\frac{(p_y - C_4)C_y l_3\bar{E}^2}{(l_3E + l_4\bar{y})^2} - (C_2N_{02} + C_4) - \xi_y\bar{E}\right]$$

$$\times\bar{y}\left[r_y\kappa_y\left(1 - \frac{\bar{y}}{L_y}\right)\left\{\frac{\delta_x - \gamma_x a_y/(a_y+\bar{y})}{\left(\kappa_y + N_2\bar{y} + \delta_x\bar{x} - \gamma_x\left(\frac{a_y}{a_y+\bar{y}}\right)\bar{x}\right)^2}\right\} - \frac{a_x}{(a_x+\bar{x})^2}\right]$$

and $(\psi_1, \psi_2)$ are as before. The auxiliary equation of Eq. (26) is

$$\mu^2 + \psi_1\mu + \psi_2 = 0. \tag{27}$$

The roots $(\mu_1, \mu_2)$ of Eq. (27) are positive by virtue of Eq. (15). Therefore, the solution of $\lambda_1(t)$ is

$$\lambda_1(t) = Ae^{\mu_1 t} + Be^{\mu_2 t} + \frac{Q_1}{\delta^2 - \psi_1\delta + \psi_2}e^{-\delta t}.$$

The shadow price $\lambda_1(t)e^{\delta t}$ remains bounded as $t \to \infty$ if and only if $A = 0 = B$, and then

$$\lambda_1(t) = \frac{Q_1}{\delta^2 - \psi_1\delta + \psi_2}e^{-\delta t}. \tag{28}$$

Similarly, we have

$$\lambda_2(t) = \frac{Q_2}{\delta^2 - \psi_1\delta + \psi_2}e^{-\delta t}, \tag{29}$$

where

$$Q_2 = \left[\left(\delta\left(\frac{\partial\pi}{\partial Y}\right) + \left(\frac{\partial G_1}{\partial Y}\right)\left(\frac{\partial\pi}{\partial X}\right) - \left(\frac{\partial G_1}{\partial X}\right)\left(\frac{\partial\pi}{\partial Y}\right)\right)\right]_{(\bar{x},\bar{y},\bar{E})}$$

$$= \left[\delta - \left[\bar{x}\left[r_x\left\{\frac{\kappa_x(N_1 + \frac{\gamma_y a_x\bar{y}}{(a_x+\bar{x})^2})(1-\bar{x}/L_x)}{(\kappa_x + N_1\bar{x} + \delta_y\bar{y} - \gamma_y a_x\bar{y}/(a_x+\bar{x}))^2} - \frac{1}{L_x}\left(1 - \frac{\kappa_x}{\kappa_x + N_1\bar{x} + \delta_y\bar{y} - \gamma_y a_x\bar{y}/(a_x+\bar{x})}\right)\right\} + \frac{C_x l_2\bar{E}}{(l_1\bar{E} + l_2\bar{x})^2}\right]\right]\right]$$

$$\times\left[\frac{(p_y - C_4)C_y l_3\bar{E}^2}{(l_3\bar{E} + l_4\bar{y})^2} - (C_2N_{02} + C_4) - \xi_y\bar{E}\right] + \left[\frac{(p_x - C_3)C_x l_1\bar{E}^2}{(l_1\bar{E} + l_2\bar{x})^2} - (C_1N_{01} + C_3) - \xi_x\bar{E}\right]$$

$$\times\bar{x}\left[r_x\kappa_x\left(1 - \frac{\bar{x}}{L_x}\right)\left\{\frac{\delta_y - \gamma_y a_x/(a_x+\bar{x})}{\left(\kappa_x + N_1\bar{x} + \delta_y\bar{y} - \gamma_y\left(\frac{a_x}{a_x+\bar{x}}\right)\bar{y}\right)^2}\right\} - \frac{a_y}{(a_y+\bar{y})^2}\right].$$

Substituting $\lambda_1(t)$ and $\lambda_2(t)$ in Eq. (22), we have

$$\frac{(p_x - C_3)C_x l_2(\bar{x})^2}{(l_1\bar{E} + l_2\bar{x})^2} + \frac{(p_y - C_4)C_y l_4(\bar{y})^2}{(l_3\bar{E} + l_4\bar{y})^2} = (\xi_x\bar{x} + \xi_y\bar{y}) + \left[\frac{Q_1}{\delta^2 - \psi_1\delta + \psi_2}\right]\frac{C_x l_2\bar{x}^2}{(l_1\bar{E} + l_2\bar{x})^2} + \left[\frac{Q_2}{\delta^2 - \psi_1\delta + \psi_2}\right]\frac{C_y l_4\bar{y}^2}{(l_3\bar{E} + l_4\bar{y})^2}. \tag{30}$$

Now, solving Eqs. (17), (18) and (30), we have the optimal equilibrium solution $(\bar{x}, \bar{y}, \bar{E})$. Let us consider the following numerical example.

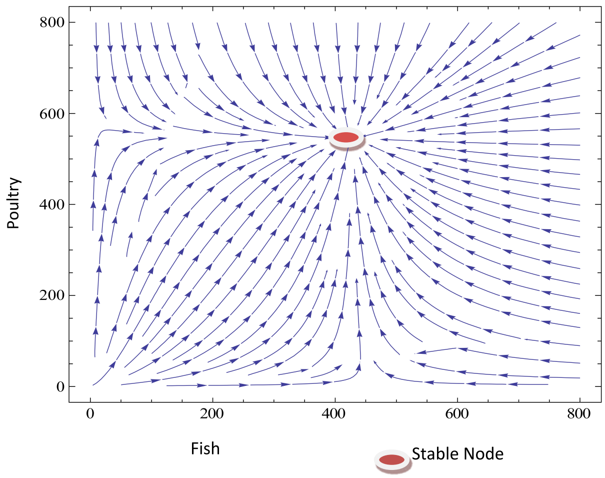


**Fig. 5.** Phase portrait of Example 3.

**Example 3.** The values of the parameters in appropriate units are considered as follows: $\kappa_x = 4$, $\kappa_y = 4$, $N_{01} = 1.5$, $N_{02} = 1.5$, $r_x = 10.0$, $r_y = 12.0$, $C_x = 0.8$, $C_y = 0.6$, $L_x = 500$, $L_y = 600$, $a_x = 20$, $a_y = 30$, $\alpha_x = 0.2$, $\alpha_y = 0.5$, $\beta_x = 0.3$, $\beta_y = 0.6$, $\gamma_x = 1.0$, $\gamma_y = 1.0$, $\tau_x = 6.5$, $\tau_y = 7.2$, $p_x = 100$, $p_y = 75$, $C_1 = 3.0$, $C_2 = 2.0$, $C_3 = 2.0$, $C_4 = 3.0$, $\xi_x = 0.5$, $\xi_y = 1.0$, $l_1 = 0.2$, $l_2 = 0.1$, $l_3 = 0.3$, $l_4 = 0.25$. Then the optimal solution ($\bar{x} = 426.22$, $\bar{y} = 545.62$, $\bar{E} = 32.67$) is a locally stable node as well as a globally stable node (see Figs. 5 and 6) because the eigenvalues ($-10.81$, $-08.09$) are negative.

## 4. Conclusion

The integration of poultry and fishery can increase the overall production intensity and provide economies on land, labour and water requirements for both poultry and fish. For example, the waste products of 1500 poultry can produce 10 metric tons (tonnes) of fish in one hectare of static water fish ponds without other feeds or fertilizers [39]. Poultry waste/litter contains more nutrients than other livestock wastes. Typically, it contains less moisture, fibre and compounds such as fish pond fertilizers. Much of the nutrient content of feed supplied to poultry is voided as excretory or faecal waste. These nutrients can be used as pond fertilizer to stimulate the production of natural food organisms such as phytoplankton and detritus. A variety of carp, shrimp and tilapia (*Oreochroms* Sp.) can grow rapidly on such natural feeds alone. Poultry litter can be used fresh, or after processing, to enhance natural food production in sunlit tropical ponds. Stable and high water temperature and sunlight ensure year-round growth of fish and their natural feeds. The tropics, where average temperatures remain above 25 °C, are ideal for culturing fish using poultry waste as a nutrient, although this practice also occurs in subtropical and lower-temperature (>20 °C) climates during suitable periods of the year. Poultry processing by-products such as chicken bones, intestines and whole carcasses have greater value as direct feeds and are normally used for higher-value fish species. In most poultry and fishery systems, poultry litter raised on balanced feeds gives the most nutrient-rich waste and produces the fish, but systems are frequently suboptimal, resulting in inefficient waste or space use. Poultry manure is used either directly on-site, through the siting of poultry houses over ponds, or after collection, storage and transport to the site of fish culture. Construction of a poultry house over a pond allows one to drop the waste directly in, saving labour costs. In the peri-urban, flood-prone land often used, the cost to fill land for poultry housing, and the opportunity cost of land itself, are reduced. The timely availability of replacement stock, veterinary support, and market demand may be critical to maintaining both poultry, and their waste, production. Higher loadings of waste necessitate water exchange or mechanical aeration to maintain dissolved oxygen. Overloading of poultry waste can also be avoided by housing poultry over concrete or earthen floors rather than directly over ponds and regular manual or mechanical collection and addition. This option may reduce construction costs considerably, and it also enables farmers to sell manure that is surplus to their requirements. The present model is a combined project of fishery and poultry. The motivation behind the concept is the use of deteriorated fish (mainly shrimp, small fish, low-valued fish, etc.) as the nutrient for poultry, and after conversion of poultry litter, the excreta of birds, dead birds and by-products of poultry processing can be used as a nutrient of the fishery.

From the analysis of the model, the following conclusions can be drawn.

1. The growth rates of fish and birds in poultry are considered as functions of available nutrients, environmental carrying capacity and volume of on-hand biomass simultaneously. Both species (fishery and poultry) are governed by the logistic law of growth, which is new in this literature.
2. The existence of local stability and global stability validates the model, and the phase portrait also shows that it is a stable node. This confirms the fact that the nutrients of fishery and poultry are interconnected.

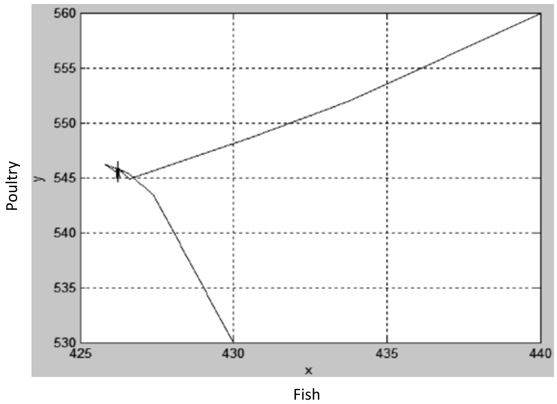


**Fig. 6.** Global attractor of Example 3.

3. The bionomic(biological as well as economic) equilibria of the exploited system has been established through a numerical example.
4. The optimal harvest policy is analyzed by invoking Pontryagin's maximal principle, subject to the state equations and the control constraints. The optimal equilibrium solution is obtained by a suitable and realistic numerical example.

The present model is unique in many ways. The unique features of the model are outlined below.

(i) A relationship between the fishery and poultry is created through nutrients and common effort in the present article.
(ii) The functions of growth rates of both species are new in the recent research works in this field.
(iii) The concept of deterioration has been introduced to the dynamical systems.
(iv) The concepts of inflation and time value of money of cost and profit parameters are considered in the present article.

Future extension of the present model can be done in many ways.

- The effect of time delay in the combined harvesting may be investigated in the present article. In general, delay differential equations build much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable, resulting in a fluctuation in the populations of the species.
- The age structure of both species with and without time delays with mature populations of harvesting is also an important characteristic to be considered.

## Acknowledgements

The author greatly appreciates the editors and referees for their valuable and constructive suggestions to improve the article.

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