Computers and Mathematics with Applications 62 (2011) 577–587

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An EOQ model of homogeneous products while demand is salesmen's initiatives and stock sensitive

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ARTICLE INFO

Article history:
Received 31 March 2011
Received in revised form 16 May 2011
Accepted 17 May 2011

Keywords:
Stock sensitive
Salesmen's initiatives
Control theory
Equilibrium
Effort
Stability

ABSTRACT

The author develops an inventory model to determine the retailer's optimal order quantity for homogeneous products. It is assumed that the amount of display space is limited and the demand of the products is dependent on the display stock level, where a huge stock of one product has a negative effect on the other product. Also, the replenishment rate depends on the level of stock of the items. The objective of the model is to maximize the profit function, considering the effect of inflation and time value of money, by Pontryagin's Maximal Principles. The stability analysis of the concerned dynamical system has been carried out analytically as well as numerically.

1. Introduction

Most researchers and practitioners have recognized salesmen's initiatives and the influence of displayed stock level on customers' demand. According to Levin et al. [1], "large piles of consumer goods displayed in a supermarket will lead the customer to buy more". Silver and Peterson [2] also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. Baker and Urban [3] assumed a power-form inventory-level-dependent demand rate that would decline along with the stock level throughout the entire cycle. Datta and Pal [4] revised the model of Baker and Urban [3] by assuming the demand rate that depleted to a given level of inventory, beyond which it was a constant. By their assumption, not all customers are motivated to purchase goods by a huge stock. Some of the customers may arrive to purchase goods because of its goodwill, good quality or facilities in spite of low stock level. Zhou et al. [5] formulated a decentralized two-echelon supply chain, where the demand of the product is dependent on the inventory level on display. Soni and Shah [6] developed an optimal ordering policy for retailers in a trade credit environment, where demand is partially constant and partially dependent on the stock. Goyal and Chang [7] investigated an ordering–transfer inventory model, when the storage capacity is limited and the demand rate depends on the display stock level. They obtained the retailer's optimal ordering quantity and the number of transfers per order from the warehouse to the display area for maximizing the average profit per unit yield by the retailer. Hsieh et al. [8] developed Datta and Pal's [4] model, allowing partial backlogging. Hsieh and Dye [9] provided some useful properties for finding the optimal replenishment schedule with stock-dependent demand under exponential partial backlogging. Min et al. [10] investigated an EOQ model for perishable items under permissible delay in payments, where demand of products varies linearly with the level of stock. Sajadieh et al. [11] developed an integrated vendor–buyer model for a two-stage supply chain, assuming demand of the products as a positive power function of the displayed inventory. As regards the research articles related to the stock-dependent demand pattern, mention should

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doi:10.1016/j.camwa.2011.05.032

be made of the works of Urban [12], Pal et al. [13], Goh [14], Padmanabhan and Vrat [15], Sarker et al. [16], Datta and Paul [17], Balkhi and Benkherouf [18], Chang [19], Hou and Lin [20], Min and Zhou [21], Ghosh et al. [22], among others.

In recent years, many types of business have been involved in various forms of promotional effort to boost market demand. Goyal and Gunasekaran [23] investigated an integrated production–inventory–marketing model for determining the EPQ (economic production quantity) and EOQ (economic order quantity) for raw materials in a multi-stage production system. They incorporated the effect of different marketing policies such as the price per unit product and the advertisement frequency on the demand of a perishable item. Many studies have focused on the effect of promotions on sales using store or market level data [24–26]. Sun [27] made relation of the customers behavior with different types of promotions and identified that promotions had a strong impact on stronger brands. Based on the work of Divakar et al. [26], Ramanathan and Muyldermans [28] applied structural equation modeling to investigate the impact of promotions and other factors on the sales of soft drinks. Sana and Chaudhuri [29] investigated an inventory model for stock with advertising sensitive demand. Sana [30] developed an interesting multi-item EOQ model for deteriorated and ameliorating items when the time varying demand is influenced by enterprises' initiatives like advertising media and salesmen' effort.

The monetary situation in each countries has changed to such an extent due to large scale inflation, and consequent sharp declines in the purchasing power of money. So, it has not been possible to ignore the effects of inflation and time value of money. Buzacott [31] was the first in this direction who derived expressions for the optimal order quantity, considering inflation and time value of money. In this direction, the works of Biermann and Thomas [32], Brahmbatt [33], Datta and Pal [34], Bose et al. [35], Chung et al. [36], Chen [37], Chung and Lin [38] are worth mentioning, among others.

In this paper, a dynamical system of differential equation of two types of homogeneous products has been considered when the demand rate of each product depends on the salesmen' initiatives and level of stock of the products. The replenishment rates of the products are also dependent on the on-hand inventory. The huge stock of one product decreases the demand of the another product. The stability analysis of the system has been done. Finally, a profit function incorporating inventory cost, selling price, cost of effort has been optimized by Pontryagin's Maximal Principles.

2. Notation

The following notation are considered to develop the model:

Notation:

$X(t)$ — on-hand stock of item 1 at time 't'.
$Y(t)$ — on-hand stock of item 2 at time 't'.
D_x — demand rate of item 1.
D_y — demand rate of item 2.
R_x— replenishment rate of item 1.
R_x— replenishment rate of item 2.
L_x — maximum storage capacity of item 1.
L_y — maximum storage capacity of item 2.
$E(t)$ — joint effort function of the salesmen's initiatives at time 't'.
p_x — purchasing cost per unit item 1.
p_y — purchasing cost per unit item 2.
h_x — inventory cost per unit item 1 per unit time.
h_y — inventory cost per unit item 2 per unit time.
γ — cost per unit effort.
s_x — selling price per unit item 1.
s_y — selling price per unit item 2.
$\delta = (r - i)$ — r and i are rate of interest and inflation per unit currency respectively.

3. Formulation of the model

The author considers a stock-dependent inventory model of two similar products such that the demand of the products depends on the on-hand inventory $(X(t), Y(t))$ and effort $(E(t))$ by advertising or salesmen' initiatives. As the demand of the products are dependent on the level of stock, the replenishment rate varies with the stock level of the products. Here, the demand rates of items 1 and 2 are as follows:

$$D_x = \frac{C_x E X}{l_1 E + l_2 X} - \left(1 - \frac{a_y}{a_y + Y}\right) X, \tag{1}$$

$$D_y = \frac{C_y E Y}{l_3 E + l_4 Y} - \left(1 - \frac{a_x}{a_x + X}\right) Y \tag{2}$$

where C_x and C_y are the positive coefficients of the demand of items 1 and 2 respectively. L_x and L_y are the storage capacities of the items 1 and 2 respectively. It is noticed that $(D_x, D_y) \to (C_x X/l_1 - (1 - \frac{a_y}{a_y+Y})X, C_y Y/l_3 - (1 - \frac{a_x}{a_x+X} Y))$ as $E \to \infty$ for fixed values of (X, Y). The parameters (l_1, l_3) are proportional to the ratios of the stock level to the demand rates at higher level of effort and the parameters (l_2, l_4) are proportional to the ratios of the effort level to the demand rates at higher stock levels. The replenishment rates of items 1 and 2 are dependent on the level of on-hand inventory $(X(t), Y(t))$ at time t those are given below:

$$R_x = r_x(1 - X/L_x)X, \tag{3}$$

$$R_y = r_y(1 - Y/L_y)Y \tag{4}$$

where (r_x, r_y) are positive replenishment coefficients respectively. Then, the governing differential equations are as follows:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} r_x(1 - X/L_x)X - \dfrac{C_xEX}{l_1E + l_2X} + \left(1 - \dfrac{a_y}{a_y + Y}\right)X \\ r_y(1 - Y/L_y)Y - \dfrac{C_yEY}{l_3E + l_4Y} + \left(1 - \dfrac{a_x}{a_x + X}\right)Y \end{pmatrix} = \begin{pmatrix} F_x(X, Y) \\ G_y(X, Y) \end{pmatrix}. \tag{5}$$

For non-zero critical points $(\bar{x}, \bar{y})$, the following conditions are satisfied:

$$r_x(1 - \bar{x}/L_x) = \frac{C_xE}{l_1E + l_2\bar{x}} - \left(1 - \frac{a_y}{a_y + \bar{y}}\right) \tag{6}$$

$$r_y(1 - \bar{y}/L_y) = \frac{C_yE}{l_3E + l_4\bar{y}} - \left(1 - \frac{a_x}{a_x + \bar{x}}\right) \tag{7}$$

solving the Eqs. (6) and (7), we have got the critical points $(\bar{x}, \bar{y})$ of the dynamical system. Now, differentiating $F_x(X, Y)$ and $G_y(X, Y)$ partially with respect to $'X'$ and $'Y'$, we have

$$\frac{\partial}{\partial X}(F_x(X, Y)) = r_x(1 - X/L_x) - \frac{C_xE}{l_1E + l_2X} + \left(1 - \frac{a_y}{a_y + Y}\right) + X\left[-\frac{r_x}{L_x} + \frac{C_xl_2E}{(l_1E + l_2X)^2}\right],$$

$$\frac{\partial}{\partial Y}(F_x(X, Y)) = X\left[\frac{a_y}{(a_y + Y)^2}\right],$$

$$\frac{\partial}{\partial X}(G_y(X, Y)) = Y\left[\frac{a_x}{(a_x + X)^2}\right],$$

$$\frac{\partial}{\partial Y}(G_y(X, Y)) = r_y(1 - Y/L_y) - \frac{C_yE}{l_3E + l_4Y} + \left(1 - \frac{a_x}{a_x + X}\right) + Y\left[-\frac{r_y}{L_y} + \frac{C_yl_4E}{(l_3E + l_4Y)^2}\right].$$

At the non-zero critical point $(\bar{x}, \bar{y})$, the above partial derivatives are as follows:

$$\frac{\partial}{\partial X}(F_x(\bar{x}, \bar{y})) = \bar{x}\left[-\frac{r_x}{L_x} + \frac{C_xl_2E}{(l_1E + l_2\bar{x})^2}\right],$$

$$\frac{\partial}{\partial Y}(F_x(\bar{x}, \bar{Y})) = \bar{x}\left[\frac{a_y}{(a_y + \bar{y})^2}\right],$$

$$\frac{\partial}{\partial X}(G_y(\bar{x}, \bar{y})) = \bar{y}\left[\frac{a_x}{(a_x + \bar{x})^2}\right],$$

$$\frac{\partial}{\partial Y}(G_y(\bar{x}, \bar{y})) = \bar{y}\left[-\frac{r_y}{L_y} + \frac{C_yl_4E}{(l_3E + l_4\bar{y})^2}\right].$$

3.1. Boundedness of the system

Lemma. *All solutions of the system of Eq. (5) which start in R_2^+ are uniformly bounded.*

Proof. Let us consider the function

$$U(X, Y) = X + \frac{1}{\ell}Y \tag{8}$$

where ℓ is a positive constant.
The time derivative of Eq. (8) is

$$\begin{aligned} \dot{U} &= \dot{X} + \frac{1}{\ell}\dot{Y} \\ &= X\left[r_x\left(1 - \frac{X}{L_x}\right) - \frac{C_xE}{l_1E + l_2X} + \left(1 - \frac{a_y}{a_y + Y}\right)\right] + \frac{1}{\ell}Y\left[r_y\left(1 - \frac{Y}{L_y}\right) - \frac{C_yE}{l_3E + l_4Y} + \left(1 - \frac{a_x}{a_x + X}\right)\right]. \end{aligned}$$

For each $s > 0$, we obtain

$$\begin{aligned}\dot{U} + sU &= \dot{X} + \frac{1}{\ell}\dot{Y} \\ &= X\left[r_x\left(1 - \frac{X}{L_x}\right) - \frac{C_x E}{l_1 E + l_2 X} + \left(1 - \frac{a_y}{a_y + Y}\right)\right] \\ &\quad + \frac{1}{\ell} Y\left[r_y\left(1 - \frac{Y}{L_y}\right) - \frac{C_y E}{l_3 E + l_4 Y} + \left(1 - \frac{a_x}{a_x + X}\right)\right] + s\left(X + \frac{1}{\ell}Y\right) \\ &< X\left|\left[r_x - \frac{C_x}{l_1} + 1 + s\right]\right| + \frac{1}{\ell} Y\left|\left[r_y - \frac{C_y}{l_3} + 1 + s\right]\right| \\ &< L_x\left|\left[r_x - \frac{C_x}{l_1} + 1 + s\right]\right| + \frac{1}{\ell} L_y\left|\left[r_y - \frac{C_y}{l_3} + 1 + s\right]\right|.\end{aligned}$$

Therefore,

$$\dot{U} + sU < K$$

where $K/2 = Max[L_x|\{r_x - \frac{C_x}{l_1} + 1 + s\}|, \frac{1}{\ell} L_y|\{r_y - \frac{C_y}{l_3} + 1 + s\}|]$.
Applying the theory of differential inequality, we have

$$0 < U(X, Y) < (K/s)(1 - e^{-st}) + U(X(0), Y(0))e^{-st}. \tag{9}$$

When $t \to \infty$, the above yields $0 < U < K/s$. Therefore, all the solution of Eq. (5) that start in R_2^+ confined to the region R where

$$R = \{(X, Y)\epsilon R_2^+ : U = (K/s) - \epsilon,\ \text{for any } \epsilon > 0\}.$$

Hence the proof. □

3.2. Local stability analysis

We shall now investigate the local behavior of critical points of the dynamical system in Eq. (5). The variational matrix of the system of Eq. (5) is

$$V(\bar{x}, \bar{y}) = \begin{pmatrix} \frac{\partial}{\partial X}(F_x(\bar{x}, \bar{y})) & \frac{\partial}{\partial Y}(F_x(\bar{x}, \bar{y})) \\ \frac{\partial}{\partial X}(G_y(\bar{x}, \bar{y})) & \frac{\partial}{\partial Y}(G_y(\bar{x}, \bar{y})) \end{pmatrix}. \tag{10}$$

The characteristic equation of $V(\bar{x}, \bar{y})$ is $\lambda^2 - \lambda\psi_1(\bar{x}, \bar{y}) + \psi_2(\bar{x}, \bar{y}) = 0$.

Now, $(\bar{x}, \bar{y})$ to be a stable node if both the eigenvalues of the above is negative, i.e.,

$$\psi_1(\bar{x}, \bar{y}) < 0 \quad \text{and} \quad \psi_2(\bar{x}, \bar{y}) > 0 \tag{11}$$

are satisfied, where

$$\begin{aligned}\psi_1(\bar{x}, \bar{y}) &= \frac{\partial}{\partial X}(F_x(\bar{x}, \bar{y})) + \frac{\partial}{\partial Y}(G_y(\bar{x}, \bar{y})) \\ &= \bar{x}\left[-\frac{r_x}{L_x} + \frac{C_x l_2 E}{(l_1 E + l_2 \bar{x})^2}\right] + \bar{y}\left[-\frac{r_y}{L_y} + \frac{C_y l_4 E}{(l_3 E + l_4 \bar{y})^2}\right]\end{aligned}$$

and

$$\begin{aligned}\psi_2(\bar{x}, \bar{y}) &= \left[\frac{\partial}{\partial X}(F_x(\bar{x}, \bar{y}))\right]\left[\frac{\partial}{\partial Y}(G_y(\bar{x}, \bar{y}))\right] - \left[\frac{\partial}{\partial Y}(F_x(\bar{x}, \bar{y}))\right]\left[\frac{\partial}{\partial X}(G_y(\bar{x}, \bar{y}))\right] \\ &= \left[\bar{x}\left\{-\frac{r_x}{L_x} + \frac{C_x l_2 E}{(l_1 E + l_2 \bar{x})^2}\right\}\right]\left[\bar{y}\left\{-\frac{r_y}{L_y} + \frac{C_y l_4 E}{(l_3 E + l_4 \bar{y})^2}\right\}\right] - \bar{x}\bar{y}\left[\frac{a_x}{(a_x + \bar{x})^2}\right]\left[\frac{a_y}{(a_y + \bar{y})^2}\right].\end{aligned}$$

3.3. Global stability analysis

We shall study the global stability of the system of Eq. (5) by considering a suitable Lyapunov function

$$F(X,Y) = [(X-\bar{x}) - \bar{x}\ln(X/\bar{x})] + h[(Y-\bar{y}) - \bar{y}\ln(Y/\bar{y})]$$

where h is a suitable constant to be determined later. $F(\bar{x},\bar{y})$ is zero at the equilibrium point $(\bar{x},\bar{y})$ and is positive for all other values of $(X,Y)\,\epsilon\, R_2^+$. The time derivative of F along the trajectories of Eq. (5) is

$$\begin{aligned}
\dot{F} &= \left(\frac{X-\bar{x}}{X}\right)\dot{X} + h\left(\frac{Y-\bar{x}}{Y}\right)\dot{Y} \\
&= (X-\bar{x})\left[r_x(1-X/L_x) - \frac{C_xE}{l_1E+l_2X} + \left(1-\frac{a_y}{a_y+Y}\right)\right] \\
&\quad + h(Y-\bar{y})\left[r_y(1-Y/L_y) - \frac{C_yE}{l_3E+l_4Y} + \left(1-\frac{a_x}{a_x+X}\right)\right] \\
&= (X-\bar{x})\left[-r_x\left(\frac{X-\bar{x}}{L_x}\right) + C_xE\left\{\frac{l_2(X-\bar{x})}{(l_1E+l_2X)(l_1E+l_2\bar{x})} + \frac{a_y(Y-\bar{y})}{(a_y+Y)(a_y+\bar{y})}\right\}\right] \\
&\quad + h(Y-\bar{y})\left[-r_y\left(\frac{Y-\bar{y}}{L_y}\right) + C_yE\left\{\frac{l_4(Y-\bar{y})}{(l_3E+l_4Y)(l_3E+l_4\bar{y})} + \frac{a_x(X-\bar{x})}{(a_x+X)(a_x+\bar{x})}\right\}\right] \\
&= (X-\bar{x})^2\left[-\frac{r_x}{L_x} + \frac{C_xEl_2}{(l_1E+l_2X)(l_1E+l_2\bar{x})}\right] + (X-\bar{x})(Y-\bar{y})\left[\frac{a_y}{(a_y+Y)(a_y+\bar{y})} + \frac{ha_x}{(a_x+X)(a_x+\bar{x})}\right] \\
&\quad + (Y-\bar{y})^2h\left[-\frac{r_y}{L_y} + \frac{C_yEl_4}{(l_3E+l_4Y)(l_3E+l_4\bar{y})}\right] \\
&= [X-\bar{x}, Y-\bar{y}]^TP[X-\bar{x}, Y-\bar{y}]
\end{aligned}$$

where

$$P = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$a_{11} = \left[-\frac{r_x}{L_x} + \frac{C_xEl_2}{(l_1E+l_2X)(l_1E+l_2\bar{x})}\right],$$

$$a_{12} = \frac{1}{2}\left[\left\{\frac{a_y}{(a_y+Y)(a_y+\bar{y})}\right\} + h\left\{\frac{a_x}{(a_x+X)(a_x+\bar{x})}\right\}\right] = a_{21},$$

$$a_{22} = \left[-\frac{r_y}{L_y} + \frac{C_yEl_4}{(l_3E+l_4Y)(l_3E+l_4\bar{y})}\right]h.$$

The eigenvalues of the characteristic equation of the above matrix are both negative if $a_{11}+a_{22} < 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$ are satisfied. Therefore, the interior equilibrium point $(\bar{x},\bar{y})$ is globally asymptotically stable if the above inequalities hold simultaneously.

3.4. Optimal goal of the policy

The net profit of the project, including inflation and time value of money, is

$$J = \int_0^\infty \pi(X,Y,E)e^{-\delta t}\mathrm{d}t \tag{12}$$

where

$$\begin{aligned}
\pi(X,Y,E) \;=\; &(s_x-p_x)\left\{\frac{C_xEX}{l_1E+l_2X} - \left(1-\frac{a_y}{a_y+Y}\right)X\right\} + (s_y-p_y)\left\{\frac{C_yEY}{l_3E+l_4Y} - \left(1-\frac{a_x}{a_x+X}\right)Y\right\} \\
&- h_xX - h_yY - \gamma E,
\end{aligned}$$

$$\frac{\partial\pi}{\partial X} = (s_x-p_x)\left\{\frac{C_xl_1E^2}{(l_1E+l_2X)^2} - \left(1-\frac{a_y}{a_y+Y}\right)\right\} - (s_y-p_y)\left\{\frac{a_xY}{(a_x+X)^2}\right\} - h_x,$$

$$\frac{\partial\pi}{\partial Y} = (s_y-p_y)\left\{\frac{C_yl_3E^2}{(l_3E+l_4Y)^2} - \left(1-\frac{a_x}{a_x+X}\right)\right\} - (s_x-p_x)\left\{\frac{a_yX}{(a_y+Y)^2}\right\} - h_y,$$

$$\frac{\partial\pi}{\partial E} = (s_x-p_x)\frac{C_xl_2X^2}{(l_1E+l_2X)^2} + (s_y-p_y)\frac{C_yl_4Y^2}{(l_3E+l_4Y)^2} - \gamma.$$

Now our objective is to maximize J subject to the state of Eq. (5), using Pontryagin's Maximum Principles. The control variable $E(t)$ is subject to the constraint $0 \leq E(t) \leq E_{max}$, E_{max} is a feasible upper limit for the harvesting effort. The Hamiltonian of the problem is

$$H = \pi(X, Y, E)e^{-\delta t} + \lambda_1(t)F_x(X, Y, E) + \lambda_2(t)G_y(X, Y, E) \tag{13}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables. The optimal control $E(t)$ which maximizes H, must satisfy the following conditions:

$$\frac{\partial H}{\partial E} = 0$$
$$-\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\partial H}{\partial X}$$
$$-\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \frac{\partial H}{\partial Y}.$$

Now, $\partial H/\partial E = 0$ at $(\bar{x}, \bar{y}, \bar{E})$ gives us

$$\left[(s_x - p_x)\frac{C_x l_2 X^2}{(l_1 E + l_2 X)^2} + (s_y - p_y)\frac{C_y l_4 Y^2}{(l_3 E + l_4 Y)^2} - \gamma\right]e^{-\delta t} - \lambda_1(t)\frac{C_x l_2 \bar{x}^2}{(l_1\bar{E} + l_2\bar{x})^2} - \lambda_2(t)\frac{C_y l_4 \bar{y}^2}{(l_3\bar{E} + l_4\bar{y})^2} = 0. \tag{14}$$

Now, $-\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\partial H}{\partial X}$ at $(\bar{x}, \bar{y}, \bar{E})$ gives us

$$-\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \left[\frac{\partial \pi}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})} e^{-\delta t} + \lambda_1\left[\frac{\partial F_x}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})} + \lambda_2\left[\frac{\partial G_y}{\partial X}\right]_{(\bar{x},\bar{y},\bar{E})}. \tag{15}$$

Similarly,

$$-\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \left[\frac{\partial \pi}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})} e^{-\delta t} + \lambda_1\left[\frac{\partial F_x}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})} + \lambda_2\left[\frac{\partial G_y}{\partial Y}\right]_{(\bar{x},\bar{y},\bar{E})}. \tag{16}$$

From Eq. (15), we have

$$\lambda_2 = -\left[\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} + \left(\frac{\partial \pi}{\partial X}\right)e^{-\delta t} + \lambda_1\frac{\partial F_x}{\partial X}\right]\Big/\left[\frac{\partial G_y}{\partial X}\right]. \tag{17}$$

Substituting the above in Eq. (16), we have

$$\frac{\mathrm{d}^2\lambda_1}{\mathrm{d}t^2} + \frac{\mathrm{d}\lambda_1}{\mathrm{d}t}\left[\left(\frac{\partial G_1}{\partial X}\right) + \left(\frac{\partial G_y}{\partial Y}\right)\right] + \lambda_1\left[\left(\frac{\partial F_x}{\partial X}\right)\left(\frac{\partial G_y}{\partial Y}\right) - \left(\frac{\partial F_x}{\partial Y}\right)\left(\frac{\partial G_y}{\partial X}\right)\right]$$
$$= \left(\delta\left(\frac{\partial \pi}{\partial X}\right) + \left(\frac{\partial G_y}{\partial X}\right)\left(\frac{\partial \pi}{\partial Y}\right) - \left(\frac{\partial G_y}{\partial Y}\right)\left(\frac{\partial \pi}{\partial X}\right)\right)e^{-\delta t}$$
$$= Q_1 e^{-\delta t}$$

$$\text{i.e., } \frac{\mathrm{d}^2\lambda_1}{\mathrm{d}t^2} + \psi_1\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} + \psi_2\lambda_1 = Q_1 e^{-\delta t} \tag{18}$$

$$Q_1 = \left[\left(\delta\left(\frac{\partial \pi}{\partial X}\right) + \left(\frac{\partial G_y}{\partial X}\right)\left(\frac{\partial \pi}{\partial Y}\right) - \left(\frac{\partial G_y}{\partial Y}\right)\left(\frac{\partial \pi}{\partial X}\right)\right)\right]_{(\bar{x},\bar{y},\bar{E})}$$
$$= \left[\delta + \bar{y}\left\{\frac{r_y}{L_y} - \frac{C_y l_4\bar{E}}{(l_3\bar{E} + l_4\bar{y})^2}\right\}\right]\left[(s_x - p_x)\left\{\frac{C_x l_1\bar{E}^2}{(l_1\bar{E} + l_2\bar{x})^2} - \left(1 - \frac{a_y}{a_y + \bar{y}}\right)\right\} - (s_y - p_y)\frac{a_x\bar{y}}{(a_x + \bar{x})^2} - h_x\right]$$
$$+ \frac{a_x\bar{y}}{(a_x + \bar{x})^2}\left[-(s_x - p_x)\frac{a_y\bar{x}}{(a_y + \bar{y})^2} + (s_y - p_y)\left\{\frac{C_y l_3\bar{E}^2}{(l_3\bar{E} + l_4\bar{y})^2} - \left(1 - \frac{a_x}{a_x + \bar{x}}\right)\right\} - h_y\right]$$

and (ψ_1, ψ_2) are as before. The auxiliary equation of Eq. (18) is

$$\mu^2 + \psi_1\mu + \psi_2 = 0. \tag{19}$$

The roots (μ_1, μ_2) of Eq. (19) are positive by virtue of Eq. (11). Therefore, the solution of $\lambda_1(t)$ is

$$\lambda_1(t) = Ae^{\mu_1 t} + Be^{\mu_2 t} + \frac{Q_1}{\delta^2 - \psi_1\delta + \psi_2}e^{-\delta t}.$$

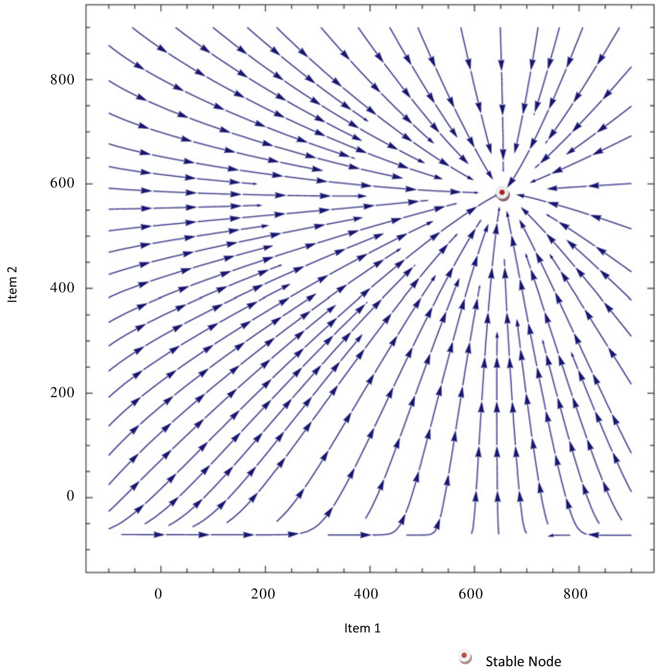

Fig. 1. Phase portrait of item 1 (x) and item 2 (y).

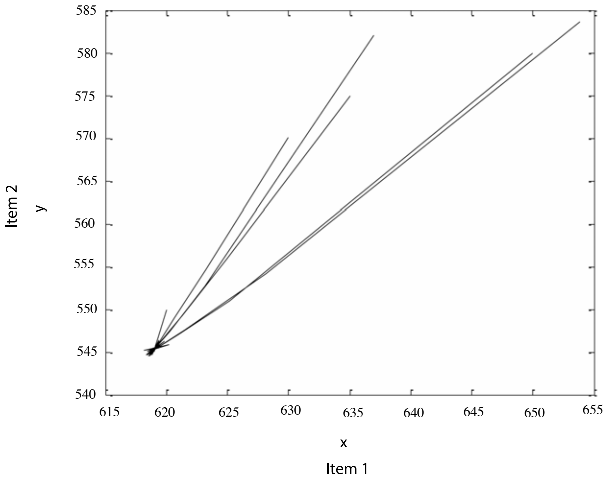

Fig. 2. Global attractor of example 1.

The shadow price $\lambda_1(t)e^{\delta t}$ remains bounded as $t \to \infty$ if and only if $A = 0 = B$ and then

$$\lambda_1(t) = \frac{Q_1}{\delta^2 - \psi_1\delta + \psi_2} e^{-\delta t}. \tag{20}$$

Similarly, we have

$$\lambda_2(t) = \frac{Q_2}{\delta^2 - \psi_1\delta + \psi_2} e^{-\delta t} \tag{21}$$

where

$$\begin{aligned} Q_2 &= \left[\left(\delta\left(\frac{\partial\pi}{\partial Y}\right) + \left(\frac{\partial F_x}{\partial Y}\right)\left(\frac{\partial\pi}{\partial X}\right) - \left(\frac{\partial F_x}{\partial X}\right)\left(\frac{\partial\pi}{\partial Y}\right)\right)\right]_{(\bar{x},\bar{y},\bar{E})} \\ &= \left[\delta + \bar{x}\left\{\frac{r_x}{L_x} - \frac{C_x l_2 \bar{E}}{(l_1\bar{E} + l_2\bar{x})^2}\right\}\right]\left[(s_y - p_y)\left\{\frac{C_y l_3 \bar{E}^2}{(l_3\bar{E} + l_2\bar{y})^2} - \left(1 - \frac{a_x}{a_x + \bar{x}}\right)\right\} - (s_x - p_x)\frac{a_y\bar{x}}{(a_y + \bar{y})^2} - h_y\right] \end{aligned}$$

$$+\frac{a_y\bar{x}}{(a_y+\bar{y})^2}\left[-(s_y-p_y)\frac{a_x\bar{y}}{(a_x+\bar{x})^2}+(s_x-p_x)\left\{\frac{C_xl_1\bar{E}^2}{(l_1\bar{E}+l_2\bar{x})^2}-\left(1-\frac{a_y}{a_y+\bar{y}}\right)\right\}-h_x\right].$$

Substituting $\lambda_1(t)$ and $\lambda_2(t)$ in Eq. (14), we have

$$\left[(s_x-p_x)\frac{C_xl_2\bar{x}^2}{(l_1\bar{E}+l_2\bar{x})^2}+(s_y-p_y)\frac{C_yl_4\bar{y}^2}{(l_3\bar{E}+l_4\bar{y})^2}-\gamma\right]$$
$$=\left[\frac{Q_1}{\delta^2-\psi_1\delta+\psi_2}\right]\frac{C_xl_2\bar{x}^2}{(l_1\bar{E}+l_2\bar{x})^2}+\left[\frac{Q_2}{\delta^2-\psi_1\delta+\psi_2}\right]\frac{C_yl_4\bar{y}^2}{(l_3\bar{E}+l_4\bar{y})^2}. \quad (22)$$

Now, solving Eqs. (6), (7) and (22), we have the optimal equilibrium solution $(\bar{x}, \bar{y}, \bar{E})$. Let us consider a numerical example as follows:

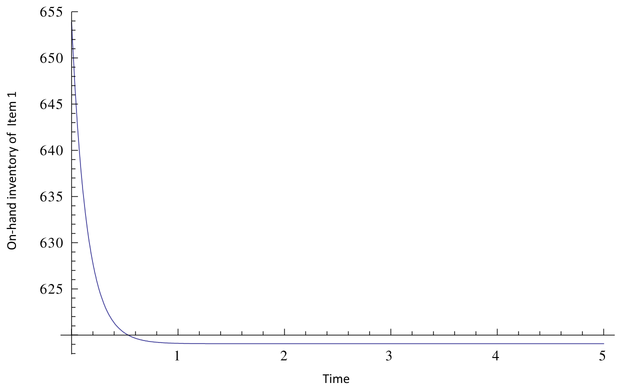

Fig. 3. On-hand inventory of item 1 versus time.

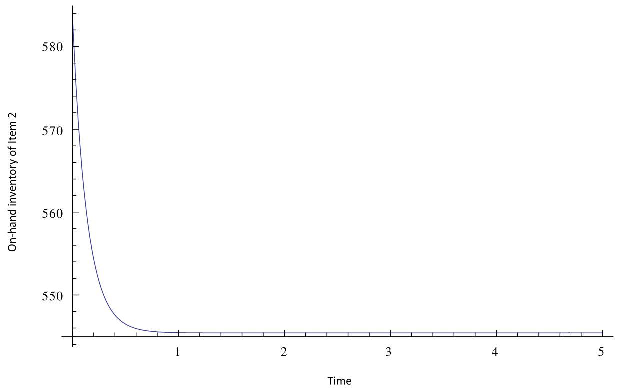

Fig. 4. On-hand inventory of item 2 versus time.

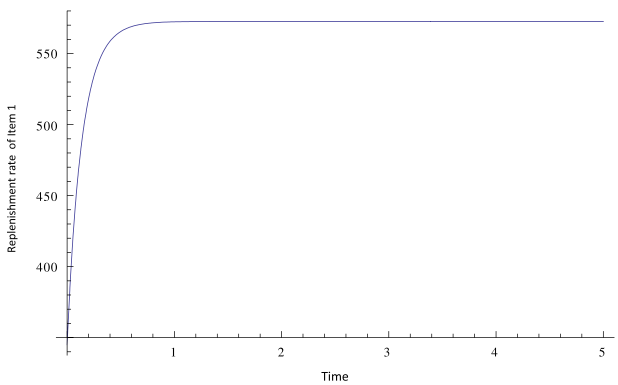

Fig. 5. Replenishment rate of item 1 versus time.

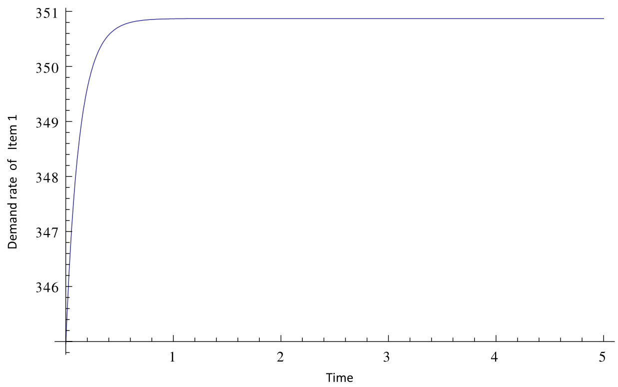

Fig. 6. Demand rate of item 1 versus time.

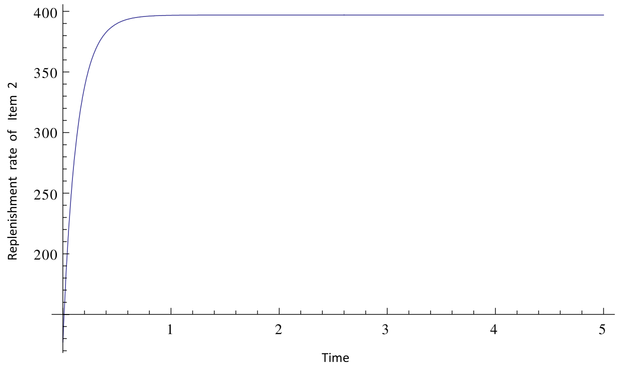

Fig. 7. Replenishment rate of item 2 versus time.

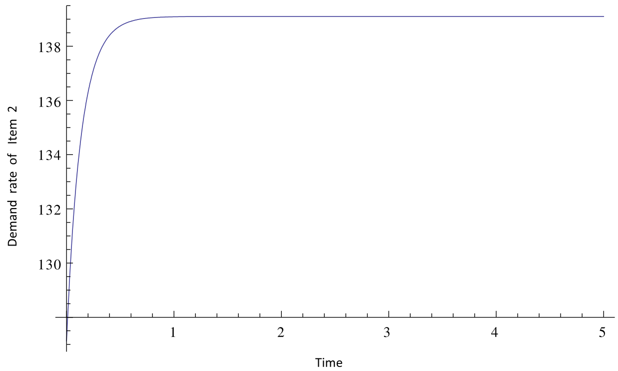

Fig. 8. Demand rate of item 2 versus time.

Example 1. The values of the parameters are considered in appropriate units as follows: $s_x = \$50$, $p_x = \$25$, $s_y = \$60$, $p_y = \$30$, $h_x = \$0.5$, $h_y = \$0.5$, $r_x = 8$, $r_y = 8$, $C_x = \$0.8$, $C_y = \$0.6$, $L_x = 700$ units, $L_y = 600$ units, $\gamma = \$100$, $a_x = 2000$, $a_y = 2500$, $l_1 = 0.3$, $l_2 = 0.2$, $l_3 = 0.2$, $l_4 = 0.3$, $r = 16\%$, $i = 11\%$, $\delta = 0.05$. Then the optimal solution is $(\bar{x} = 653.84, \bar{y} = 583.66, \bar{E} = 160.26)$. This solution is a stable node (see Fig. 1) because the eigenvalues are negative $(-7.44691, -6.89116)$. This is locally as well as globally stable (see Fig. 2). The on-hand inventories, replenishment rates and demand rates of items 1 and 2 are showed in Figs. 3–8 as follows.

4. Conclusion

The salesmen' effort (advertising, promotional effort, etc.) plays an important role to boost sale the items that results in more profit in oligopolistic marketing management. Determination of demands and costs due to salesmen' effort is quite difficult. The author has formulated the new demand functions which are increasing function of common effort (E) and level of stock displayed of the same product. But, the stock displayed of one product decreases the demand of other product those stock level are smaller than the other. Frankly speaking, not all customers are attracted or motivated to purchase goods by the huge stock. Some of the customers purchase goods because of its goodwill, good quality or facilities, and brand images of the product. The replenishment rates of each item depend on the on-hand inventory level of the concerned item. It is also considered the effect of inflation and time value of money on the profit function. The approach in this paper is to concentrate on investment for the purpose of salesmen' initiatives and replenishment rates of the homogeneous products which maximize the profit. The model provides a major new contribution – the effect of salesmen' effort and level of stocks of homogeneous products on demand in a dynamical system – to operations in management practice.

Acknowledgements

The author expresses his gratitude to the editor and referees for useful comments that were very helpful in improving the presentation of the article.

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