CUPhysics/02/2007

A novel approach to study realistic navigations on networks

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We consider navigation or search schemes on networks which are realistic in the sense that not all search chains can be completed. We show that the quantity $\mu = \rho/s_d$, where s_d is the average dynamic shortest distance and ρ the success rate of completion of a search, is a consistent measure for the quality of a search strategy. Taking the example of realistic searches on scale-free networks, we find that μ scales with the system size N as $N^{-\delta}$, where δ decreases as the searching strategy is improved. This measure is also shown to be sensitive to the distintinguishing characteristics of networks. In this new approach, a dynamic small world (DSW) effect is said to exist when $\delta \approx 0$. We show that such a DSW indeed exists in social networks in which the linking probability is dependent on social distances.

PACS numbers: 89.75.Hc, 89.70.+c, 89.75.Fb

I. INTRODUCTION

During the last few years there has been a lot of activity in the study of networks [1, 2] once it was realised that networks of diverse nature exhibit many common features in their underlying structure. The most important property that appeared to be commonly occurring in such networks is the small world property. This means that if any two nodes in the network is separated by an average number of s steps, then $s \propto \ln(N)$, where N is the total number of nodes in the network. In some networks, even slower variation (i.e., sub-logarithmic) scaling has been observed [3].

The first indication that networks have small world behaviour emerged from an experimental study by Milgram [4], in which it was shown that any two persons (in the USA) can be connected by an average number of six steps. Following the tremendous interest in the study of networks, a new experiment has been done to verify this property in real social networks [5, 6]. Some studies which involve simulations on real networks [7, 8, 9] have been made also. Parallely, the question of navigation on small world networks has been addressed theoretically in many model networks [10, 11, 12, 13, 14, 15, 16, 17, 18].

It must be noted that it is not necessary that a navigation or searching on a small world network would show the small world property, i.e., the dynamic paths s_d may not scale as $\ln(N)$. This is because searching is done using local information only while the average shortest distances are calculated using the global knowledge of the network. This was explicitly shown by Kleinberg [10] in a theoretical study where nodes where placed on a two dimensional Euclidean space. Each node here has connections to its nearest neighbours as well as to neighbours at a distance l with probability $P(l) \propto l^{-\alpha}$. Although the network is globally a small world for a range of values of α, navigation on such networks using greedy algorithm showed a small world behaviour only at $\alpha = 2$. In general the path length showed a sublinear power law increase with N.

Recently, the scale-free property of networks has also been found in many real-world networks which precisely means that the degree distribution follows a behaviour $P(k) \propto k^{-\gamma}$, with the value of γ between 2 and 3 for most networks. Navigation on such networks has also been theoretically studied yielding the result that rather than logarithmic, there is a power-law variation of the path lengths with N as in the non-scale free Euclidean network (except for particular algorithms or values of γ) [11, 12]. This confirms once again that for navigation in a small world network, it is not necessary that the path lengths will reflect the small world behaviour.

Obviously, a lot depends on the algorithm also, but any realistic algorithm is basically local. For Euclidean networks, the greedy algorithm appears to be the most popular one from the findings of the original Milgram experiment as well as that of [5, 6]. Here, once the source and the target nodes are chosen, the strategy is to connect to a neighbour closest to the target. In scale-free Barabási-Albert (BA) networks [19], the two extreme strategies of a random search and highest degree search have been used as well as a preferential search scheme based on the degree [12]. Strategies involving other properties like betweenness centrality etc. have also been investigated [17].

A relevant question in this context is how to test the quality of the search strategy. Noting that the path lengths in all the studies show a behaviour $s_d \propto N^{\tau_1}$, one may expect that a lower value of τ_1 indicates a better search strategy.

In realistic searches as in Milgram's experiments and that described in [6], the search may not be successful or complete always. (In fact for graphs which are not fully connected, there is always a finite possibility that a connected path does not exist, even with global knowledge [20, 21].) Here, apart from the search length, the success rate is also an important factor. Indeed, most works referring to Milgram's experiments ignore the fact that very few chains were completed in the initial experiment (3 out of 60) and better success rate (35% or more) was achieved only after randomness in the search was compromised in the sense that the choice of the source persons was made with some bias [4]. Many of the theoretical models do not consider the probability of termination at all and the interest is to find out whatever the search length is.

We argue that for realistic searches, where termination of search paths is a possibility, one must consider both the success rate as well as the path lengths to adjudge the quality of a search strategy.

We have generated random scale free networks and used a tunable preferential search strategy which can be extended from a random search (RS) to a highest degree search (HDS) scheme. In general we find that ρ, the success rate also follows a power law behaviour such that $\rho \propto N^{-\tau_2}$. On a scale-free network, it is expected that a high degree search will improve the search quality. However, we have shown that a comparison of search strategies based on the behaviour of path lengths alone may lead to an entirely different conclusion. On the other hand, we find that the ratio of the success rate to the path length, $\mu = \rho/s_d$ has a power law behaviour with N, i.e., $\mu \propto N^{-\delta}$ where δ decreases as one conducts a higher degree search (section II). We therefore claim that μ is a reliable measure to test the quality of a realistic search on a network involving both the success rate and the shortest path lengths in an effective manner.

Calculation of μ on simple networks suggests that δ can take up values between zero and one (section III). Corresponding to $\delta = 0$ we have the best searchability; this we call a “Dynamical small world” (DSW) effect. In section IV, we have shown that μ is indeed sensitive to the distinguishing features of networks using the examples of different kinds of scale-free networks with the same degree exponent γ. Finally, in section V we have considered a social network in which the nodes possess a “similarity” factor and the the linking probability of two nodes depends parametrically on it. Here, we have shown that it is possible to obtain DSW (i.e., δ assumes values very close to zero) for a finite range of values of the parameter.

II. SEARCHING ON RANDOM SCALE FREE NETWORKS

In this section we describe the searching procedure on random scale-free (RSF) networks, which have no other features other than being scale-free. Random scale-free networks are generated by assigning the degree to each node following a scale-free distribution $P(k) \propto k^{-\gamma}$, allowing k to vary from k_{min} to k_{max}. k_{max} is taken to be $N^{1/\gamma}$ while k_{min} is allowed to vary from 1 to higher values. The links are then established randomly between the nodes with the given distribution. Links are assigned in steps; we start with the node with the highest degree, in the next step the node with the second highest degree and so on. This method has been previously used to generate random scale-free networks [22].

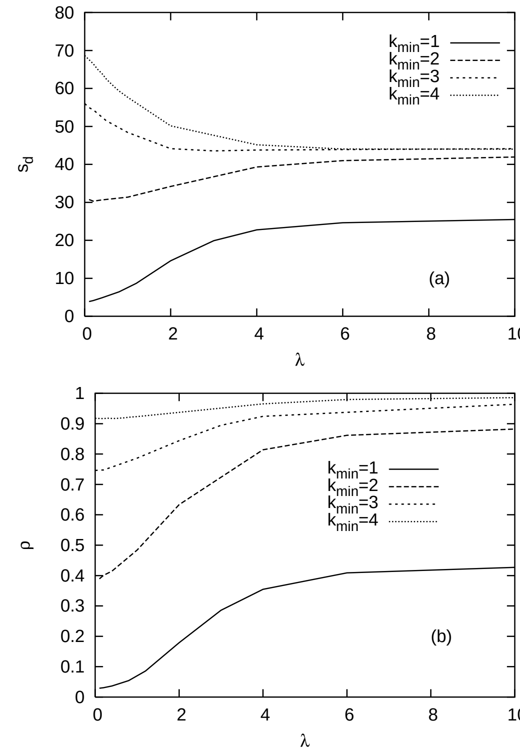

FIG. 1: The path lengths (a) and success rates (b) are shown against λ for a random scale-free network for values of $k_{min} = 1, 2, 3$ and 4.

The search strategy is like this :
A source node and a target node are selected randomly. The source node will send the signal to one of its neighbouring nodes provided that node has not already taken part in the search. This is in tune with Milgram-like experiments.

If one of its neighbour happens to be the target itself, the message will be send to the target. If not, then the *ith* neighbour will receive the message with a probability Π_i, where

$$\Pi_i \propto k_i^{\lambda}, \qquad (1)$$

and k_i is the degree of the ith node. This continues and if it happens that the message cannot be passed any more, the chain will remain incomplete. We take the average dynamic shortest path as the average of all completed chain lengths. Obviously $\lambda = 0$ corresponds to a random search while $\lambda \to \infty$ gives rise to the highest degree search. $\lambda = 0, 1, \infty$ were considered for the BA network in [12], although the algorithm was slightly different.

We first report the results for s_d and ρ for $\gamma = 2$ for a network of $N = 1000$ nodes with different values of k_{min}. The path length against λ shows a very interesting result. Normally one would expect that the path length will reduce with higher degree searches. While this happens for

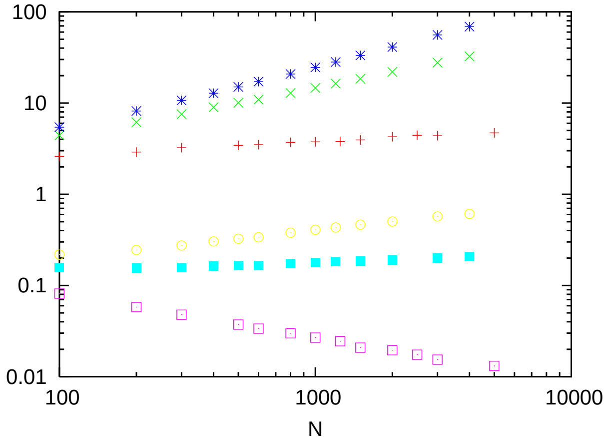

FIG. 2: The path lengths s_d (set of three curves in the top) and the success rates (set of three curves in the bottom) are shown against system size N for $k_{min} = 1$ for five different values of $\lambda = 6.0,\ 2.0$ and 0.0 (from top to bottom).

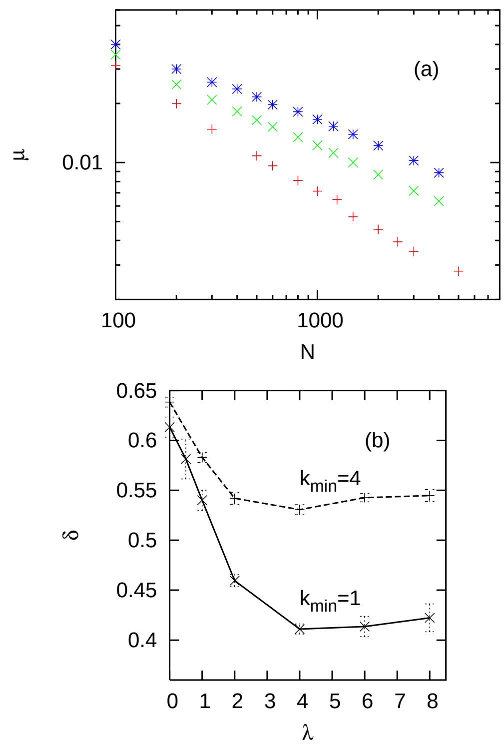

FIG. 3: (a) The ratio $\mu = \rho/s_d$ against N shown for $\lambda =$ 6.0, 2.0 and 0.0 (from top to bottom) for a RSF network with $k_{min} = 1$. In (b) The corresponding exponents δ are shown against λ for $k_{min} = 1$ and 4. For both cases, there is a monotonic decrease of δ with λ.

$k_{min} = 3$ or 4, for smaller values of k_{min}, this is not true. Here, on the contrary, the path length increases with λ (Fig. 1a). For all values of k_{min}, we notice that the path lengths saturate to a constant value which corresponds to the highest degree search. This occurs for finite values of $\lambda \sim 8$. The fraction of successful attempts ρ shows a uniform behaviour with λ; for all values of k_{min}, it increases with λ (Fig. 1b) as expected.

The reason why for small k_{min} there is a difference in behaviour of s_d is quite obvious. When k_{min} is small and a random search is carried on, very few chains can be completed, but if completed, these are essentially small in length as the connectivity on an average is small. On making a higher degree search, more chains can be completed but at the expense of increased path lengths.

The static small world effect is quantatively measured by the behaviour of the average shortest path lengths s (global) with N. We notice that the dynamic path lengths s_d varies with λ and in all cases it shows a power law increase with N (except for $\lambda = 0$ and $k_{min=1}$ where it has a logarithmoc increase). However, for k_{min} small, the path length against N shows that it varies as τ_1 with τ_1 increasing with a higher degree search, which is counter intuitive (Fig. 2). So clearly the path length alone is not a good measure of the searchability. The success rate, on the other hand, shows a change in behaviour as λ is varied. The exponent τ_2 shows a decrease with λ; for small values of λ, τ_2 is positive and it goes on to assume negative values for higher λ.

However, when we study the behaviour of the ratio $\mu = \rho/s_d$ as a function of N, we find a consistent behaviour $\mu \propto N^{-\delta}$, where δ decreases as λ is increased (Fig. 3a).

For higher values of k_{min}, path length and success rates show expected behaviour with N; both improve with λ. δ also decreases with λ monotonically as in the case of $k_{min} = 1$ (Fig 3b). Hence one can regard μ as a measure to adjudge the quality of the search strategy in general. For higher values of k_{min} (which implies higher connectivity), variation of δ with λ is not that appreciable showing that for networks with larger connectivity, the sensitivity to degree-based algorithm is not remarkable.

III. CALCULATION OF μ AND δ IN SOME SIMPLE CASES

The searchability factor μ and hence δ can be calculated in some simple cases. Although the examples may seem too idealised, the results help us to understand the bounds of δ.

1. Search on a regular network with nearest neighbour interactions only. The simplest case is a one dimensional chain (Fig. 4a). On any Euclidean network, a greedy allgorithm, i.e., passing the message to a node closest to the target is meaningful. Here as the links are between nearest neighbours only, the search path length is equal to l where l is the distance separating them. In general $l = pN$ where $0 \le p \le 1$. The success rate is hundred

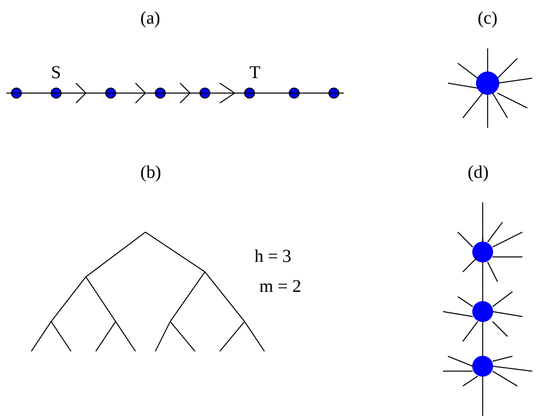

FIG. 4: Simple networks for which μ can be calculated. In (a), a regular network in one dimension is shown, S and T are the source and target nodes respectively. The unique path from S to T is shown. In (b), a tree network with number of levels $h = 3$ and branching ratio $m = 2$ is shown. In (c) a gel or star network and in (d), a chain of such gels (or hubs) have been shown.

percent here such that

$$\mu_{reg} = 1/pN \tag{2}$$

and $\delta = 1$ in this case.

Note that in this case, since the message cannot be passed to the same node more than once, the search must take a unique path, it has to be directed towards the target in steps of 1 (shown by the arrows in Fig 4a). Thus random search is not conceivable here. If the message can be passed to the same node more than once, one can use a random search and the situation is analogous to a random walker. The path length will now be proportional to l^2 and the corresponding $\delta = 2$. This is in fact the worst case scenario: however, it does not belong to the class of navigation considered here where messages can be received only once.

2. Search on a tree with uniform branching ratio: Let us imagine a tree with branching ratio m (Fig. 4b) and with h levels. Let us consider the case of sending a message from the top (zeroeth level) to a node at the bottom (hth) level. The probability that a message is sent to this node at the hth level from the top is $(1/m)^h$. Noting that the number of nodes $N \approx m^{h+1}$,

$$\mu_{tree} \approx N^{-1} \ln(m)/\ln(N), \tag{3}$$

implying $\delta = 1$ here. More generalised cases where search from arbitrary sources and targets are considered may give different values of δ. For example, if one considers search paths of length 1 only, then the success rate is $\sim 1/m$ giving $\delta = 0$. Larger paths will have a different δ and δ in principle can vary from 0 to 2. If one takes all the possibilities, i.e., all possible pairs of source and targets, then the average δ can be estimated from the following expression:

$$\langle \mu \rangle = \frac{\sum_{n=1}^{2h} \mu_n f_n}{\sum_{n=1}^{2h} f_n} \tag{4}$$

where μ_n is the ratio ρ/s_d for two nodes separated by n edges and f_n the number of such pairs. The terms in the numerator decrease with n; for $h >> 1$, $f_1 \approx m^{h+1}$, $\mu_1 = 1/m$, while $f_{2h} \approx m^{2h}$ and $\mu_{2h} = \frac{1}{2hm^{2h}}$. Thus the leading order term in the numerator is $O(N)$ while the leading order term in the denominator is $O(N^2)$ giving $\delta = 1$ for very large trees.

3. Search on a gel - Here we consider a network in the form of a gel or a star (Fig. 4c) where there is a central node to which all other nodes (having only one link) are attached. Again path lengths are of the order 1 and success is a surity. Hence $\delta = 0$ here.

4. A chain of hubs or gels:

We consider a idealised chain of hubs or gels, it contains l hubs with degree k each (Fig. 4d); the total number of nodes $N = kl$. Now a search path length will be of the order of l, say αl. If we consider a random search, the probability of completing the search is $(1/k)^{\alpha l}$. Or,

$$\mu_{hubs} = \frac{1}{k}^{\alpha N/k} / \alpha \frac{N}{k}. \tag{5}$$

The thermodynamic limit $N \to \infty$ may correspond to two different cases. The first is when l is small but N/k is finite. The search probability is small (but non-zero) and μ does not vary with N (as the ration N/k remains finite) giving $\delta = 0$.

On the other hand if l is large and k finite, $\ln(\mu) \approx \alpha N/k \ln(k)$ such that μ goes exponentially to zero rather than as a power law.

The above discussion shows that when $\mu \sim N^{-\delta}$, the value of δ for small world networks lies between 0 and 1 when the search is done randomly. Obviously, for a more efficient algorithm δ is lesser than that for the random search and thus it can be said that for searches in which repetition of messengers is not allowed, $0 \leq \delta \leq 1$. A dynamic small world in this perspective is one in which $\delta = 0$.

In this section, the simple networks which have been considered have no loops and hence the paths from one node to another is unique. Thus the important quantity determining μ is essentially the success rate. In the trivial cases like the linear chain and the gel, even the success rate is deterministic. On more complicated networks, both s_d and ρ will have to be computed probabilistically.

IV. SENSITIVITY OF μ TO THE DISTINGUISHING CHARACTERISTICS OF MODEL NETWORKS

In this section we show that our measure of searchability μ is sensitive to the distinguishing features of different networks. We have considered networks which are equivalent as far as the degree distribution is concerened in the sense they are all scale-free with the same exponent but differ in certain other aspects. We have selected scale-free networks with the same exponent $\gamma = 3$ and used a HDS algorithm which is meaningful in a scale-free network. In the two subsection we give a brief desciption of the models we have considered and the results obtained.

A. Models

We have considered three different types of scale free models:

1. The Barabási Albert (BA) network:

The BA network is grown using a preferential attachment scheme. Here, we have started with a single node and nodes are attached to the network one by one. An incoming node gets attached to the ith existing node with the probability

$$\mathcal{P}_{BA} \propto k_i, \tag{6}$$

where k_i is the degree of the ith node at that time. This network has a has a tree structure as we allow only one link to an incoming node.

2. The random scale free (RSF) network:

This network is the same as that described in section II. We have taken $\gamma = 3$ here and kept $k_{min} = 1$ such that it is comparable to the BA network.

3. Scale free assortative network (SFA)

The third network which we have considered is a scale-free model with tunable assortativity [23]. This scale-free assortative (SFA) network is generated by modifying slightly the method of generating the RSF network. To each node, the degree is assigned from a scale-free distribution as in RSF. Let k_i and k_j be the degrees assigned to nodes i and j. While establishing the links of the ith node, attachment to the jth node will be definitely made if $k_i = k_j$. Otherwise they will be linked with a probability

$$\mathcal{P}_{SFA} \propto |k_i - k_j|^{-\sigma}. \tag{7}$$

Here σ controls the assortativity factor. For $\sigma = 0$, there is no assortativity and for $\sigma > 0$ (< 0), the network is assortative (disassortative).

It may be noted that both the RSF and the BA networks have zero assortativity [23].

B. Results

We employ the HDS algorithm, i.e, a node always passes on the information to the neighbour with the highest degree. In the BA network, for system sizes considered here, s_d shows a power law increase with N with an exponent $\tau_1 < 0.1$ which becomes smaller with N showing that it approaches a logarithmic increase with N for

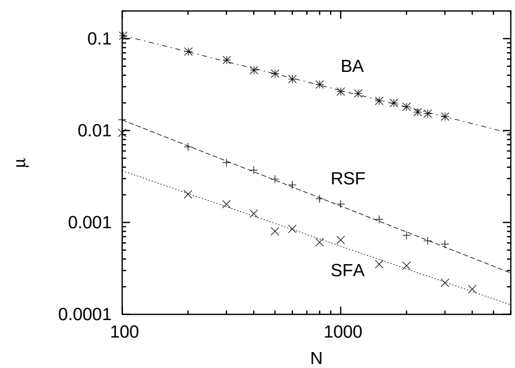

FIG. 5: The behaviour of μ against N for three different scale-free networks (BA - Barabási-Albert, RSF - Random scale-free and SFA - Scale free assortative) with identical degree distribution $P(k) \propto k^{-\gamma}$ ($\gamma = 3$) is shown. The straight lines correspond to the best fits.

large N. This is to be expected as it has a tree structure. (In [12] it was shown that even with loops, the highest degree search scheme in the BA network results in a logarithmic increase of the path lengths with N.) ρ follows a power law decay with N such that $\mu \propto N^{-0.6}$.

For the RSF network, we again apply HDS and find that in this case $\mu \propto N^{-0.93}$, although the path length varies logarithmically with N. (It may be mentioned here that for the RSF, as γ is increased from 2 to 3, the searchability becomes less efficient in terms of μ and also becomes less sensitive to the factor λ.)

Thus in terms of simply the path lengths the RSF and BA networks are equally searchable which is counterintuitive. On the other hand μ gives a more realistic result.

In the third network, even when it is weakly assortative ($\sigma = 0.1$), we find that δ shows a deviation from the random network value. Here $\delta \simeq 0.87$ which means that it is more searchable than the RSF but less than the BA network with the same algorithm. The behaviour of μ with N for different networks is shown in Fig. 5.

As already mentioned, in the BA as well as in the RSF networks, the path length shows a logarithmic increase with N but since δ is appreciably greater than zero, neither of these two networks is an ideal dynamic small world. The random scale-free network has a worse searchability because of the sharper decay of the success rate. In the SFA network, there is a tendency of clustering of high degree nodes thereby allowing loops. Hence the path length does not scale logarithmically with N. However, one only needs to know the behaviour of μ to comment on the searchability and here it shows a slower decay with N compared to the RSF network although the magnitude of μ is lesser.

V. DYNAMIC SMALL WORLD EFFECT IN A MODEL SOCIAL NETWORK

Till now we have discussed search strategies on network which are basically characterised by their degree distribution. Social networks, which are found to be scale-free in many cases, have some additional features. It has been found that in a society, people have different characteristics and the links depend on these bonds. These characteristics can be related to profession, hobbies, geographical locations etc.

We have considered a very simplified picture with one such characteristic which we call the similarity factor ξ of the individuals varying between 0 and 1 randomly. In a scale-free network with given distribution $P(k) = k^{-\gamma}$, we first assign the degrees as well as the similarity factors to the nodes. The bonding between two nodes, subject to this distribution is now made according to

$$\mathcal{P}_{i,j} \propto |\xi_i - \xi_j|^{-\alpha}. \tag{8}$$

Thus we will have a scale-free network of degree exponent γ in which similar nodes will try to link up for $\alpha > 0$. The similarity factors ξ_i can be looked upon as coordinates in a one dimensional lattice in which case the network is scale-free network as well as Euclidean.

In the earlier cases only degree based strategies have been considered as the networks did not have any other characteristic feature. But as shown in [6] such a strategy is rather artificial for social networks. Here we have considered two different strategies; one is the standard HDS while the other one is a greedy algorithm based on the similarity factor. In the latter, a node sends the signal to a neighbour which is most similar to the destination node. We call this the highest similarity search (HSS). We investigate how the searching strategies are dependent on α based on the value of δ.

We consider networks with $\gamma = 2$ and $k_{min} = 2$ here and vary α to obtain networks varying on the basis of similarity based bondage between the nodes for the same degree exponent. We have shown in section IV that the measure μ is able to distinguish between networks of same degree exponent which differ in other characteristics and hence it is meaningful to employ this measure to investigate the searchability based on different algorithms here.

Again we find that μ shows a power law decay with N for both strategies in general. However, for the highest degree search, the value of δ shows marginal dependence on α and the network is moderately searchable with this strategy as δ lies between 0.5 and 0.6. Using the HSS on the other hand we find an appreciable variation of δ with α. In fact, we find that for $1 < \alpha < 2$, the network is highly searchable as δ is $O(0.1)$ here. For $\alpha = 1.5$, δ becomes lesser for higher N values indicating that it might go to zero in the thermodynamic limit giving rise to a truly dynamical small world effect.

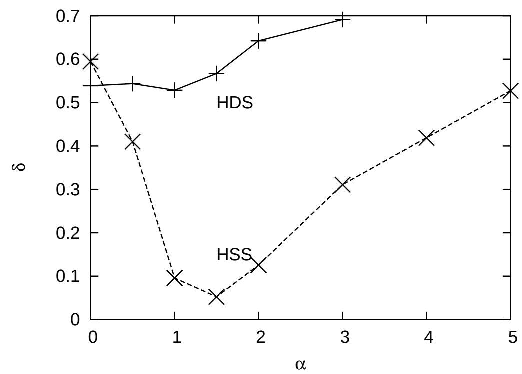

FIG. 6: The values of δ against α shows a minimum where the network is most searchable for the HSS. For the HDS, the values of δ are generally higher and also less sensitive to α.

It is true that not all social networks are scale-free. However, our studies lead to the important result that *even for scale-free networks*, if there is a social distance dependent linking scheme, the degree based search is rather inefficient (compared to the one based on the social characteristic). Thus we conclude that for any social network, the degree based search will be less useful.

Here we have considered networks of size $N \leq 10000$ with a cutoff in the degree distribution at $N^{1/\gamma}$ ($\gamma = 2$). In order to simulate the friendship or acquintance networks, where in principle, the whole population of the earth is involved, it is better to use a degree distribution with an exponential cutoff or simply exponential decay to make the study more realistic [24].

VI. SUMMARY AND CONCLUSIONS

In this work we have proposed a new method to study the searchability on networks where the search may not be complete always as transmission of signals cannot be repeated to the same node. In real searches, failure to complete a chain may be due to various other reasons [6]. The motivation comes from Milgram-like experiments of searches where the success rate is really low when the source nodes are randomly selected. We first show that only the study of search path lengths as a function of the system size is not enough to determine the searchability of a network. Rather, the ratio of the success rate to the path length, μ, is a more reasonable measure. We have established this by executing searches on scale-free networks but the significance of μ is by no means limited to scale-free networks only.

We have shown that in general μ behaves as $N^{-\delta}$ with δ showing a decrease for a better algorithm. The value of δ is indicated to be between 0 and 1. The zero value signifies what we have termed a dynamic small world effect.

Extending the study to networks in which social distances matter, we have shown that algorithms based on social distances rather than on degree tend to be more successful in general. This is again consistent with the findings of [6]. Here we have introduced a parameter α which governs the similarity dependent linking probability between two nodes. We find that μ, the searchability, is maximum at $\alpha \approx 1.5$, where δ is very close to zero. This corresponds to a network where the similarity factor controls the linkings to a moderate extent. This seems realistic; in a society where only very similar people share links (i.e., α has very large values) the searchability cannot be high. On the other hand, for low values of α, the network is almost random, where the searchability is again not very high.

Our studies are motivated by the finding of [4, 5, 6] and [9]. We have obtained reasonable qualitative agreement with the experimental observations (specifically with that of [6]) as already mentioned. Further comparison is not feasible for various reasons. In real experiments, several reasons for termination of the search exist giving rise to a lower success rate. Also, the main emphasis of the present paper is the scaling behaviour of the relevant quantities with the system size N, which has not been studied in these experiments.

Acknowledgement: Financial support from CSIR grant no. 3(1029)/05-EMR-II is acknowledged. The author thanks K. Bharadwaj and K. Basu Hajra for discussions.

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